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SCHWARZ LEMMA FOR HOLOMORPHIC MAPPINGS IN THE UNIT BALL

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Abstract. In this note, we establish a Schwarz–Pick type inequality for holomorphic mappings between unit balls $\mathbf{B}_n$ and $\mathbf{B}_m$ in corresponding complex spaces. We also prove a Schwarz–Pick type inequality for pluri-harmonic functions.

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1. Introduction. The Fréchet derivative of a holomorphic mapping $f : \Omega \to \mathbf{C}^m$, where $\Omega \subset \mathbf{C}^n$, is defined to be the unique linear map $A = f'(z) : \mathbf{C}^n \to \mathbf{C}^m$ such that $f(z+h) = f(z) + f'(z)h + O(|h|^2)$. The norm of such a map is defined by

$$\|A\| = \sup_{|z|=1} |Az|. \tag{1}$$

The classical Schwarz lemma states that $|f(z)| \leqslant |z|$ for every holomorphic mapping of the unit disk $\mathbf{B}_1 \subset \mathbf{C}$ into itself satisfying the condition $f(0) = 0$. This inequality implies the following inequality for the derivative

$$|g'(z)| \leqslant \frac{1 - |g(z)|^2}{1 - |z|^2} \tag{2}$$

for every holomorphic mapping g of the unit disk into itself. On the other hand, if n, m are two positive integers and $\mathbf{B}_n \subset \mathbf{C}^n$ is the unit ball, then every holomorphic mapping $f : \mathbf{B}_n \to \mathbf{B}_m$, with $f(0) = 0$ satisfies the inequality $|f(z)| \leqslant |z|$, but the counterpart of (2) in the space does not hold provided that $m \geqslant 2$ (see e.g. [**4**] and corresponding sharp inequality (5) below). However, it holds for $m = 1$, while for $m \geqslant 2$, it holds in its weaker form namely $1 - |g(z)|^2$ should be replaced by $\sqrt{1 - |g(z)|^2}$. This is proved in Theorem 2.1, which is the main result of this paper. By using the case $m = 1$, in Theorem 2.3, we prove a Schwarz–Pick type inequality for pluri-harmonic functions, which extends a corresponding result for real harmonic functions [**3**].

1.1. Automorphisms of the unit ball. Let P_a be the orthogonal projection of $\mathbf{C}^n$ onto the subspace $[a]$ generated by a, and let

$$Q = Q_a = I - P_a$$

be the projection onto the orthogonal complement of $[a]$. To be quite explicit, $P_0 = 0$ and $P = P_a(z) = \frac{\langle z,a\rangle a}{\langle a,a\rangle}$. Put $s_a = (1 - |a|^2)^{1/2}$ and define

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}.$$

If $\Omega = \{z \in \mathbf{C}^n : \langle z, a\rangle \neq 1\}$, then φ_a is holomorphic in Ω. It is clear that $\overline{\mathbf{B}_n} \subset \Omega$ for $|a| < 1$.

PROPOSITION 1.1 ([**6**, Theorem 2.2.2]). *If φ_a is defined as above, then*

(*a*) $\varphi_a(0) = a$ *and* $\varphi_a(a) = 0$*;*
(*b*) $\varphi_a'(0) = -s^2 P_a - sQ_a$;
(*c*) $\varphi_a'(a) = -\frac{1}{s^2} P_a - \frac{1}{s} Q_a$;
(*d*) φ_a *is an involution:* $\varphi_a(\varphi_a(z)) = z$*;*
(*e*) φ_a *is a biholomorphism of the closed unit ball onto itself.*

2. The main results.

THEOREM 2.1 (The main theorem). *If f is a holomorphic mapping of the unit ball $\mathbf{B}_n \subset \mathbf{C}^n$ into $\mathbf{B}_m \subset \mathbf{C}^m$, then for $m \geqslant 2$*

$$\|f'(z)\| \leqslant \frac{\sqrt{1 - |f(z)|^2}}{1 - |z|^2}, \quad z \in \mathbf{B}_n \tag{3}$$

and for $m = 1$, we have that

$$\|f'(z)\| \leqslant \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbf{B}_n. \tag{4}$$

The inequalities (3) *and* (4) *are sharp for $z = 0$. In other words, we have the following sharp inequalities*

$$\|f'(0)\| \leqslant \begin{cases} \sqrt{1 - |f(0)|^2}, & \text{if } m \geqslant 2; \\ 1 - |f(0)|^2, & \text{if } m = 1. \end{cases} \tag{5}$$

We need the following lemma.

LEMMA 2.2. *Let n be a positive integer and let $M = M_a = -s^2 P_a - sQ_a$ and $N = N_a = -\frac{1}{s^2} P_a - \frac{1}{s} Q_a$, where $s = \sqrt{1 - |a|^2}$. Then,*

$$\|M\| = \begin{cases} \sqrt{1 - |a|^2}, & \text{if } n \geqslant 2; \\ 1 - |a|^2, & \text{if } n = 1 \end{cases}$$

and

$$\|N\| = \frac{1}{1 - |a|^2}.$$

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Proof of Lemma 2.2. If $a=0$, then $M=N=-I$, where I is the identity. So the claim of the lemma follows easily.

Assume now that $a\neq 0$. We have

$$s^2P_az+sQ_az=sz+(s^2-s)\frac{\langle z,a\rangle\, a}{\langle a,a\rangle}.$$

So

$$|Mz|^2=|s^2P_az+sQ_az|^2=\left|sz+(s^2-s)\frac{\langle z,a\rangle\, a}{\langle a,a\rangle}\right|^2$$
$$=s^2|z|^2+(s^4-s^2)\frac{|\langle z,a\rangle|^2}{\langle a,a\rangle}\leqslant s^2|z|^2.$$

For $z\perp a$, the previous inequality becomes an equality. It follows that $\|M\|=\sqrt{1-|a|^2}$. The case $n=1$ is trivial and in this case $Q_a\equiv 0$. Further, we establish the norm of the operator

$$N=N_a=-\frac{1}{s^2}P_a-\frac{1}{s}Q_a.$$

Similarly as above, we obtain

$$|Nz|^2=\frac{1}{s^2}|z|^2+\left(\frac{1}{s^4}-\frac{1}{s^2}\right)\frac{|\langle z,a\rangle|^2}{|a|^2}.$$

By using Cauchy–Schwarz inequality and choosing $z=a/|a|$, we obtain that $\|N\|=\frac{1}{1-|a|^2}$. □

Proof of Theorem 2.1. Let φ_a be an involutive automorphism of the unit ball $\mathbf{B}_n$ onto itself such that $\varphi_a(0)=a$ and let $b=f(a)$. Let φ_b be an involutive automorphism of the unit ball $\mathbf{B}_m$ onto itself such that $\varphi_b(b)=0$ and let $g=\varphi_b^{-1}\circ f\circ\varphi_a^{-1}$. Then $f=\varphi_b\circ g\circ\varphi_a$ and so in view of Proposition 1.1, we have

$$f'(a)=\varphi_b'(0)g'(0)\varphi_a'(a)=M_bg'(0)N_a.$$

Since g maps the unit ball into itself and satisfies $g(0)=0$, by [**6**, Theorem 8.1.2], it follows that $|g'(0)|\leqslant 1$. Since $\|A\cdot B\|\leqslant\|A\|\|B\|$, according to Lemma 2.2, we obtain (3).

The sharpness for $z=0$ is proved by the following example. Let $t\in(0,\pi/2)$ and define $f_t(z,w)=(z\sin t,\cos t)$. Then $f_t:\mathbf{B}_2\to\mathbf{B}_2$. Moreover, $\|f_t'(0)\|=\sin t$ and $|f_t(0)|=\cos t$. So $\|f_t'(0)\|=\sqrt{1-|f_t(0)|^2}$. □

THEOREM 2.3. *Let f be a pluri-harmonic function of the unit ball $\mathbf{B}_n$ into $(-1,1)$. Then the following sharp inequality holds:*

$$|\nabla f(z)|\leqslant\frac{4}{\pi}\frac{1-|f(z)|^2}{1-|z|^2},\quad z\in\mathbf{B}_n. \tag{6}$$

For Schwarz–Pick type estimates of arbitrary order partial derivatives for bounded pluri-harmonic mappings defined in the unit polydisk, we refer to the recent paper [**1**].

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Assume that $m = 1$ and let a be a holomorphic function of the unit ball $\mathbf{B}_n$ into $\mathbf{C}^m = \mathbf{C}$. Since a' is $\mathbf{C}$ linear, we regard $a'(z)$ as a vector $(a_{z_1}, \dots a_{z_n})$ from the space $\mathbf{C}^n$ and we will denote it by ∇a.

Proof of Theorem 2.3. Let h be the pluri-harmonic conjugate of f, i.e. assume that $a = f + ih$ is a holomorphic mapping. Then a maps the unit ball $\mathbf{B}_n$ into the vertical strip $S = \{w : -1 < \Re w < 1\}$.

Let

$$g(z) = \frac{2i}{\pi} \log \frac{1+z}{1-z}.$$

Then g is a conformal mapping of the unit disk $\mathbf{U}$ onto the strip S. Hence, $b(z) = g^{-1}(a(z))$ is a holomorphic mapping of the unit ball onto the unit disk $\mathbf{U}$. Then we have that

$$a(z) = \frac{2i}{\pi} \log \frac{1+b(z)}{1-b(z)}. \tag{7}$$

By (4), we have

$$|b'(z)| \leqslant \frac{1-|b(z)|^2}{1-|z|^2}.$$

On the other hand,

$$a'(z) = \frac{4i}{\pi} \frac{b'(z)}{1-b(z)^2}.$$

Since a is holomorphic, we obtain that

$$|a'(z)| = \sqrt{\sum_{k=1}^n |a_{z_k}|^2} = \sqrt{\sum_{k=1}^n f_{x_k}^2 + \sum_{k=1}^n f_{y_k}^2}$$

and

$$|\nabla f| = |(f_{x_1}, f_{y_1}, \dots, f_{x_n}, f_{y_n})| = \sqrt{\sum_{k=1}^n f_{x_k}^2 + \sum_{k=1}^n f_{y_k}^2} = |a'(z)|.$$

We will find the best possible constant C such that

$$|\nabla f(z)| \leqslant C \frac{1-|f(z)|^2}{1-|z|^2}.$$

Observe first that

$$|a'(z)| \leqslant \frac{4}{\pi} \frac{1-|b(z)|^2}{|1-b(z)^2|} \frac{1}{1-|z|^2}$$

and find the optimal constant C such that

$$\frac{4}{\pi} \frac{1-|b(z)|^2}{|1-b(z)^2|} \frac{1}{1-|z|^2} \leqslant C \frac{1-|\Re a(z)|^2}{1-|z|^2}$$

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or what is the same, in view of (7),

$$\frac{4\pi}{(\pi^2 - 4|\arg\frac{1+b}{1-b}|^2)}\frac{1-|b|^2}{|1-b^2|} \leqslant C\,, |b| < 1\,.$$

Let $\omega = \frac{1+b}{1-b} = re^{it}$. Then, $-\pi/2 \leqslant t \leqslant \pi/2$ and

$$1-|b|^2 = \frac{4r\cos t}{r^2+2r\cos t+1},$$

$$|1-b^2| = \frac{4r|e^{it}|}{r^2+2r\cos t+1},$$

and hence the last inequality with the constant $C = 4/\pi$ follows from

$$\frac{|\cos t|}{1-\frac{4}{\pi^2}t^2} \leqslant 1$$

which holds for $t \in (-\pi/2, \pi/2)$. This yields (6).

To show that the inequality (6) is sharp, take the pluri-harmonic function

$$f(z) = \frac{2}{\pi}\arctan\frac{2y_1}{1-x_1^2-y_1^2}.$$

It is easy to see that

$$|\nabla f(0)| = \frac{4}{\pi} = \frac{4}{\pi}\frac{1-|f(0)|^2}{1-0^2}.$$

□

By using a scaling argument, we obtain

COROLLARY 2.4. *If f is a holomorphic mapping of the unit ball $\mathbf{B}_n \subset \mathbf{C}^n$ into $\mathbf{C}^m$, then*

$$\|f'(z)\| \leqslant \begin{cases} \frac{\sqrt{\|f\|^2-|f(z)|^2}}{1-|z|^2}, \quad z \in \mathbf{B}_n, & \text{if } m \geqslant 2; \\ \frac{\|f\|^2-|f(z)|^2}{\|f\|(1-|z|^2)}, & \text{if } m = 1. \end{cases} \tag{8}$$

Here, $\|f\| := \sup_z |f(z)|$.

Proof. Let $g(z) = f(z)/\|f\|$. Then, $\|g'(z)\| = \frac{|f(z)|}{\|f\|}$ and so

$$\frac{|f(z)|}{\|f\|} \leqslant \frac{\sqrt{1-|g(z)|^2}}{1-|z|^2} = \frac{\sqrt{1-\frac{|f(z)|^2}{\|f\|^2}}}{1-|z|^2}.$$

This implies (8) for $m \geqslant 2$. Similarly, we prove the case $m = 1$. □

In order to state a new corollary of the main result, recall the definition of the Bloch space $\mathcal{B}$ of holomorphic mappings of the unit ball $\mathbf{B}_n$ into $\mathbf{C}^m$. We say that $f \in \mathcal{B}$ provided its semi-norm satisfies $\|f\|_{\mathcal{B}} = \sup_{|z|<1}(1-|z|^2)|f'(z)| < \infty$. Let $\mathcal{B}_1$ be

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the unit ball of $\mathcal{B}$. Let $\mathfrak{B}$ be the set of bounded holomorphic mappings between $\mathbf{B}_n$ and $\mathbf{C}^m$, i.e. of mappings satisfying the inequality $\|f\| = \sup_z |f(z)| < \infty$.

COROLLARY 2.5. *The inclusion operator* $\mathcal{I} : f \mapsto f$ *between* $\mathfrak{B}$ *and* $\mathcal{B}$ *has norm equal to* 1.

Proof. It is clear from (8) that $|\mathcal{I}| \leqslant 1$. We prove the equality statement. Assume as we may that $n = m = 2$. Let $f_0(z, w) = (z, 0)$. Then,

$$\|f_0\|_{\mathcal{B}} = \sup_{|z|^2+|w|^2\leqslant 1} (1 - |z|^2 - |w|^2)\|f_0'(z, w)\| = 1,$$

and

$$\|f_0\| = \sup_{|z|^2+|w|^2\leqslant 1} \sqrt{|z|^2} = 1.$$

This finishes the proof. □

REMARK 2.6. The inclusion operator is a restriction of a Bergman projection $\mathcal{P}_\alpha$, $\alpha > -1$, between $L^\infty(B_n)$ and $\mathcal{B}$ whose norm is greater than 1. See corresponding results for the plane [**5**] and for the several dimensional space [**2**].

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