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Spacetime Killing tensors in general relativity

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In this thesis we investigate higher order symmetries in general relativity. The emphasis is mainly on valence two spacetime Killing tensors (K-tensors), which yield quadratic constants of motion and enable for example separation of the Hamilton-Jacobi equation in most Petrov type D vacuum solutions. A further symmetry, namely curvature collineations are also discussed, with reference to non-expanding and twist-free vacuum type-N gravitational fields.

We begin in Chapter 1 with a description of the Lie algebra structure of K-tensors and Killing-Yano tensors. In Chapter 2 we introduce the Newman-Penrose formalism to rewrite the K-tensor equations:

$$(1) \qquad g^{\mu(\nu}\partial_\mu S^{\rho\sigma)} - S^{\mu(\nu}\partial_\mu g^{\rho\sigma)} = \quad , \quad S^{[\mu\nu]} = 0 \; ,$$

in null tetrad form. Then employing the classification scheme for a second rank symmetric tensor, Plebański [4], $S^{\mu\nu}$ can be classified in a similar way to the Ricci tensor. This results in fifteen canonical classes. Taking one class at a time, the K-tensor equations can be integrated together with the Einstein field equations (vacuum or non-vacuum) and Bianchi identities to yield in theory the form of both the K-tensor and the metric.

The Newman-Penrose form of the Killing-Yano equations:

$$(2) \qquad Y^{\mu(\nu;\rho)} = 0 \; , \quad Y^{(\mu\nu)} = 0 \; ,$$

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are given in Chapter 3 and all the spacetimes satisfying these equations are found by classifying $Y^{\mu\nu}$ in a similar way to the electromagnetic bivector field.

The vacuum results found in the previous two chapters can also be generalized to Einstein-Maxwell fields by considering the supplementary conditions:

$$S^{\rho}{}_{(\mu}F_{\nu)\rho} = 0 \ , \quad Y^{\rho}{}_{[\mu}F_{\nu]\rho} = 0 \ ,$$

together with the original equations (1) and (2) respectively. This results in constants of motion along charged particle orbits.

In Chapter 4 we discuss the variable separation of the Hamilton-Jacobi equation:

(3) $$g^{\mu\nu}S,_{\mu}S,_{\nu}-\lambda = 0 \ ,$$

with a view to investigating the relationship between (3) and K-tensors of valence two. In particular we consider the condition for which the Hamilton-Jacobi equation separates for geodesics in an n-dimensional Riemannian or pseudo-Riemannian manifold when the Jacobi-action separates with respect to x^1 as:

$$S(x^{\mu}) = S_1(x^1) + S_2(x^2, \ldots, x^n) \ .$$

This problem is tantamount to considering the integrability condition:

(4) $$V_{\mu}R^{\mu}{}_{\nu\rho\sigma} = 0 \ ,$$

where V is a non-null hypersurface orthogonal Killing vector. Equation (4) arises in various other places in general relativity, in particular as the integrability conditions of the equations:

$$L_V g = \phi g \ ; \quad \phi \in \mathbb{R} \ .$$

In this chapter equation (4) is examined and the components of the Riemann tensor for the spacetimes which admit non-zero solutions V_{μ} of this equation are given.

Chapter 5 contains a discussion of curvature collineations. A spacetime is said to admit a curvature collineation if there is a vector field

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ξ which satisfies:

(5) $$L_\xi R^\mu_{\lambda\rho\sigma} = 0 .$$

A necessary condition for a motion to be curvature collineations is that:

$$h_{\mu\nu} R^\mu_{\lambda\rho\sigma} + h_{\mu\lambda} R^\mu_{\nu\rho\sigma} = 0 ,$$

where

$$h_{\mu\nu} := L_\xi g_{\mu\nu} = 2\xi_{(\mu;\nu)} .$$

For non flat vacuum spacetimes Collinson [1] has shown that:

$$h_{\mu\nu} = \phi g_{\mu\nu} + \alpha l_\mu l_\nu ,$$

where $\alpha = 0$ for all space times except type N, in which case l is the repeated principle null congruence of the Weyl tensor. The curvature collineation equations are solved in this chapter for two families of Petrov type N plane-fronted gravitational wave solutions of Einstein's vacuum field equations.

Finally in Chapter 6 the K-tensor equations are solved for the plane-fronted gravitational waves with parallel rays. This leads to a number of interesting spacetimes, some which have a non-separable Hamilton-Jacobi equation. Solutions are also found which admit non trivial curvature collineations as well as irreducible K-tensors and in some cases conformal K-tensors.

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