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# On the homomorphisms between scalar generalized Verma modules

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Abstract

In this article, we study the homomorphisms between scalar generalized Verma modules. We conjecture that any homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms. The purpose of this article is to confirm the conjecture for some parabolic subalgebras under the assumption that the infinitesimal characters are regular.

## Introduction

We study the homomorphisms between generalized Verma modules, which are induced from one-dimensional representations (such generalized Verma modules are called scalar, cf. [Boe85]).

Classification of the homomorphisms between scalar generalized Verma modules is equivalent to that of equivariant differential operators between the spaces of sections of homogeneous line bundles on generalized flag manifolds (cf. [CS90, Dob88, Hua93, Jak85, Kos75]).

In [Ver68], Verma constructed homomorphisms between Verma modules associated with root reflections. Bernstein, I. M. Gelfand, and S. I. Gelfand proved that all the non-trivial homomorphisms between Verma modules are compositions of homomorphisms constructed by Verma [BGG71].

Later, Lepowsky studied the generalized Verma modules. In particular, Lepowsky [Lep75b] constructed a class of non-trivial homomorphisms between scalar generalized Verma modules associated to the parabolic subalgebras which are the complexifications of the minimal parabolic subalgebras of real reductive Lie algebras.

In [Mat06], elementary homomorphisms between scalar generalized Verma modules are introduced. They can be regarded as a generalization of homomorphisms introduced by Verma and Lepowsky.

We propose a conjecture on the classification of the homomorphisms between scalar generalized Verma modules, which can be regarded as a generalization of the above-mentioned result of Bernstein *et al.*

Conjecture A. All the non-trivial homomorphisms between scalar generalized Verma modules are compositions of elementary homomorphisms.

Soergel's result [Soe90, Theorem 11] implies that Conjecture A is reduced to the integral infinitesimal character case.

The purpose of this article is to confirm the conjecture for some parabolic subalgebras under the assumption that the infinitesimal characters are regular. In order to explain our results, we

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introduce some notations. Let $\mathfrak{g}$ be a complex reductive Lie algebra and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We denote by $\Delta$ (respectively $W$) the root system (respectively the Weyl group) with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix a basis $\Pi$ of $\Delta$. For $\Theta \subsetneq \Pi$, we put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall \alpha \in \Theta \alpha(H) = 0\}$ and $\Sigma_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Delta\} - \{0\}$. We denote by $\mathfrak{p}_\Theta$ the standard parabolic subalgebra corresponding to $\Theta$ and by $\mathfrak{l}_\Theta$ its Levi subalgebra containing $\mathfrak{h}$. We consider the Weyl group for parabolic subalgebra $W(\Theta) = \{w \in\mid w\Theta = \Theta\}$.

We call $\Theta$ normal if any two parabolic subalgebras with the Levi part $\mathfrak{l}_\Theta$ are conjugate under an inner automorphism of $\mathfrak{g}$. If $\Theta$ is normal, we call $\mathfrak{p}_\Theta$ normal. For example, complexified minimal parabolic subalgebras of real simple Lie algebras except $\mathfrak{su}(p,q)\,(p-1>q>0)$, $\mathfrak{so}^*(4n+2)$, $\mathfrak{e}_{6(-14)}$ are normal. Roughly speaking, if $\Theta$ is normal, the reflection $\sigma_\gamma$ on $\mathfrak{a}_\Theta$ with respect to $\gamma \in \Sigma_\Theta$ can be regarded as an involution of the Weyl group for $(\mathfrak{g}, \mathfrak{h})$. A normal subset $\Theta$ of $\Pi$ is called strictly normal if $\sigma_\gamma$ is a Duflo involution of some Weyl group (see Definition 4.2.1). If $\Theta$ is strictly normal, there exists an elementary homomorphism with respect to $\sigma_\gamma$ for each $\gamma \in \Sigma_\Theta$.

Let $\mathfrak{p}_\Theta$ be a complexified minimal parabolic subalgebra of a real simple Lie algebra and assume $\mathfrak{p}_\Theta$ is normal but is not strictly normal. Then, $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of $\mathfrak{so}(2n+1-q,q)\,(n>q\geqslant 1)$, or $\mathfrak{sp}(n,n)\,(n\geqslant 1)$.

The main result of this article is the following theorem.

Theorem B (Theorem 5.1.3). *If $\Theta$ is strictly normal, then each non-trivial homomorphism between scalar generalized Verma modules induced from $\mathfrak{p}_\Theta$ with regular integral infinitesimal character is a composition of elementary homomorphisms.*

The idea presented here seems not useful for confirming the conjecture in the general case. However, we may confirm the conjecture for some other parabolic subalgebras. For example, via case-by-case consideration, we can prove the following result.

Theorem C. *If $\Theta$ is normal and $\mathfrak{g}$ is an exceptional Lie algebra, then each non-trivial homomorphism between scalar generalized Verma modules induced from $\mathfrak{p}_\Theta$ with a regular integral infinitesimal character is composed of elementary homomorphisms.*

We shall give a proof of Theorem C in a subsequent paper.

This article consists of five sections.

We fix notations and introduce some fundamental material in § 1.

In § 2, we explain how to reduce the problem to the integral infinitesimal character case. We also show that we can associate an element of $W(\Theta)$ to a homomorphism between generalized Verma modules with regular infinitesimal characters.

In § 3, we introduce the notion of normal parabolic subalgebras and describe the classification. We prove that the Bruhat ordering on $W(\Theta)$ coincides with the restriction of that of $W$ to $W(\Theta)$ for each normal $\Theta$.

In § 4, we introduce the notion of an elementary homomorphism and describe related notions and results.

In § 5, we introduce the notion of strictly normal parabolic subalgebras and describe the classification. We also prove Theorem B.

## 1. Notations and preliminaries

### 1.1 General notations

In this article, we use the following notations and conventions.

As usual we denote the complex number field, the real number field, the ring of (rational) integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. The expression $(1/2)\mathbb{N}$ means the set $\{n/2 \mid n \in \mathbb{N}\}$, and $1/2+\mathbb{N}$ means the set $\{1/2+n \mid n \in \mathbb{N}\}$. We denote by $\emptyset$ the empty set. For any (non-commutative) $\mathbb{C}$-algebra $R$, 'ideal' means '2-sided ideal', '$R$-module' means 'left $R$-module', and sometimes we denote by 0 (respectively 1) the trivial $R$-module $\{0\}$ (respectively $\mathbb{C}$). Often, we identify a (small) category and the set of its objects. Hereafter 'dim' means the dimension as a complex vector space, and '$\otimes$' (respectively Hom) means the tensor product over $\mathbb{C}$ (respectively the space of $\mathbb{C}$-linear mappings), unless we specify otherwise. For a complex vector space $V$, we denote by $V^*$ the dual vector space. For $a, b \in \mathbb{C}$, '$a \leqslant b$' means that $a, b \in \mathbb{R}$ and $a \leqslant b$. We denote by $A - B$ the set theoretical difference. 'card $A$' denotes the cardinality of a set $A$.

### 1.2 Notations for reductive Lie algebras

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by $\Delta$ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system $\Delta^+$ and let $\Pi$ be the set of simple roots. Let $W$ be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle\, , \rangle$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. For $w \in W$, we denote by $\ell(w)$ the length of $w$ as usual. We also denote the inner product on $\mathfrak{h}^*$ which is induced from the above form by the same symbols $\langle\, , \rangle$. For $\alpha \in \Delta$, we denote by $s_\alpha$ the reflection in $W$ with respect to $\alpha$. We denote by $w_0$ the longest element of $W$. For $\alpha \in \Delta$, we define the coroot $\alpha^\vee$ by $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$, as usual. We denote by $\Delta^\vee$ the dual root system $\{\alpha^\vee \mid \alpha \in \Delta\}$. We call $\lambda \in \mathfrak{h}^*$ is dominant (respectively anti-dominant) if $\langle\lambda, \alpha^\vee\rangle$ is not a negative (respectively positive) integer, for each $\alpha \in \Delta^+$. (Often, 'dominant' here is called 'integrally dominant'.) We call $\lambda \in \mathfrak{h}^*$ regular if $\langle\lambda, \alpha\rangle \neq 0$, for each $\alpha \in \Delta$. We denote by $\mathsf{P}$ the integral weight lattice, namely $\mathsf{P} = \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$. If $\lambda \in \mathfrak{h}^*$ is contained in $\mathsf{P}$, we call $\lambda$ an integral weight. We define $\rho \in \mathsf{P}$ by $\rho = (1/2)\sum_{\alpha\in\Delta^+}\alpha$. Put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}\, [H, X] = \alpha(H)X\}$, $\mathfrak{u} = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. We denote by $\mathsf{Q}$ the root lattice, namely $\mathbb{Z}$-linear span of $\Delta$. We also denote by $\mathsf{Q}^+$ the linear combination of $\Pi$ with non-negative integral coefficients. For $\lambda \in \mathfrak{h}^*$, we denote by $W_\lambda$ the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in \mathsf{Q}\}.$$

We denote by $\Delta_\lambda$ the set of integral roots:

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}\}.$$

It is well known that $W_\lambda$ is the Weyl group for $\Delta_\lambda$. We put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. This is a positive system of $\Delta_\lambda$. We denote by $\Pi_\lambda$ the set of simple roots for $\Delta_\lambda^+$ and denote by $S_\lambda$ (respectively $S$) the set of reflection corresponding to the elements in $\Pi_\lambda$ (respectively $\Pi$). So, $(W_\lambda, S_\lambda)$ and $(W, S)$ are Coxeter systems. We denote by $\mathsf{Q}_\lambda$ the integral root lattice, namely $\mathsf{Q}_\lambda = \mathbb{Z}\Delta_\lambda^+$, and put $\mathsf{Q}_\lambda^+ = \mathbb{N}\Pi_\lambda$.

Next, we fix notations for a parabolic subalgebra (which contains $\mathfrak{b}$). Hereafter, through this article we fix an arbitrary subset $\Theta$ of $\Pi$. Let $\langle\Theta\rangle$ be the set of the elements of $\Delta$ which are written by linear combinations of elements of $\Theta$ over $\mathbb{Z}$. Put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall\alpha \in \Theta\, \alpha(H) = 0\}$, $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha\in\langle\Theta\rangle}\mathfrak{g}_\alpha$, $\mathfrak{n}_\Theta = \sum_{\alpha\in\Delta^+-\langle\Theta\rangle}\mathfrak{g}_\alpha$, $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$. Then $\mathfrak{p}_\Theta$ is a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$. Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $\Theta \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_\Theta$. We denote by $W_\Theta$ the Weyl group for $(\mathfrak{l}_\Theta, \mathfrak{h})$. The Weyl group $W_\Theta$ is identified with a subgroup of $W$ generated by $\{s_\alpha \mid \alpha \in \Theta\}$. We denote by $w_\Theta$ the longest element of $W_\Theta$. Using the invariant non-degenerate bilinear form $\langle\, , \rangle$, we regard ${\mathfrak{a}_\Theta}^*$ as a subspace of $\mathfrak{h}^*$.

Put $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta\rho)$ and $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta\rho)$. Then, $\rho^\Theta \in {\mathfrak{a}_\Theta}^*$.
For $\Theta \subsetneq \Pi$, we define 'the restricted root system' as follows:

$$\Sigma_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Delta\} - \{0\},$$
$$\Sigma_\Theta^+ = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Delta^+\} - \{0\}.$$

Unfortunately, in general, $\Sigma_\Theta$ does not satisfy the axioms of the root systems.

Define

$$\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta \quad \langle\lambda, \alpha^\vee\rangle \in \{1, 2, \ldots\}\},$$
$${}^\circ\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta \quad \langle\lambda, \alpha^\vee\rangle = 1\}.$$

We easily see

$${}^\circ\mathsf{P}_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in \mathsf{P}_\Theta^{++}$, we denote by $\sigma_\Theta(\mu)$ the irreducible finite-dimensional $\mathfrak{l}_\Theta$-representation whose highest weight is $\mu$. Let $E_\Theta(\mu)$ be the representation space of $\sigma_\Theta(\mu)$. We define a left action of $\mathfrak{n}_\Theta$ on $E_\Theta(\mu)$ by $X \cdot v = 0$ for all $X \in \mathfrak{n}_\Theta$ and $v \in E_\Theta(\mu)$. So, we regard $E_\Theta(\mu)$ as a $U(\mathfrak{p}_\Theta)$-module.

For $\mu \in \mathsf{P}_\Theta^{++}$, we define a generalized Verma module [Lep77a] as follows:

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all $\lambda \in \mathfrak{h}^*$, we write $M(\lambda) = M_\emptyset(\lambda)$; $M(\lambda)$ is called a Verma module. For $\mu \in \mathsf{P}_\Theta^{++}$, $M_\Theta(\mu)$ is a quotient module of $M(\mu)$. Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$-module with the highest weight $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in \mathsf{P}_\Theta^{++}$, the canonical projection of $M(\mu)$ to $L(\mu)$ is factored by $M_\Theta(\mu)$.

We have $\dim E_\Theta(\mu - \rho) = 1$ if and only if $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$. If $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$, we call $M_\Theta(\mu)$ a scalar generalized Verma module.

Finally, we fix notations for infinitesimal characters. We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We denote by $\chi_\lambda$ the image of $\lambda \in \mathfrak{h}^*$ under the Harish-Chandra isomorphism from $W\backslash\mathfrak{h}^*$ to $\mathrm{Hom}(Z(\mathfrak{g}), \mathbb{C})$. It is well known that $Z(\mathfrak{g})$ acts on $M(\lambda)$ by $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ for all $\lambda \in \mathfrak{h}^*$. We denote by $\mathbf{Z}_\lambda$ the kernel of $\chi_\lambda$ in $Z(\mathfrak{g})$. Let $M$ be a $U(\mathfrak{g})$-module and $\lambda \in \mathfrak{h}^*$. We say that $M$ has an infinitesimal character $\lambda$ if and only if $Z(\mathfrak{g})$ acts on $M$ by $\chi_\lambda$. For example, a generalized Verma module $M_\Theta(\mu)$ has an infinitesimal character $\mu$.

## 2. Formulation of the problem

We retain the notation of § 1. In particular, $\Theta$ is a proper subset of $\Pi$.

### 2.1 Basic results of Lepowsky

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

Theorem 2.1.1 [Lep76]. *Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$. Then it is true that:*

(1) $\dim \mathrm{Hom}_{U(\mathfrak{g})}(M_\Theta(\mu), M_\Theta(\nu)) \leqslant 1$;

(2) *any non-zero homomorphism of $M_\Theta(\mu)$ to $M_\Theta(\nu)$ is injective.*

Hence, the classification problem of homomorphisms between generalized Verma modules is reduced to the following problem.

*Problem* 1. Let $\mu, \nu \in {}^{\circ}\mathsf{P}_{\Theta}^{++}$. When is $M_{\Theta}(\mu) \subseteq M_{\Theta}(\nu)$?

## 2.2 Reduction to the integral infinitesimal character setting

Since both $\nu \in W\mu$ and $\nu - \mu \in Q^{+}$ are necessary conditions for $M_{\Theta}(\mu) \subseteq M_{\Theta}(\nu)$, we can reformulate our problem as follows.

*Problem* 2. Let $\lambda \in {}^{\circ}\mathsf{P}_{\Theta}^{++}$ be dominant. Let $x, y \in W_{\lambda}$ be such that $x\lambda, y\lambda \in {}^{\circ}\mathsf{P}_{\Theta}^{++}$. When is $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$?

We fix $\lambda \in {}^{\circ}\mathsf{P}_{\Theta}^{++}$. Then, we can construct a suralgebra $\mathfrak{g}'$ of $\mathfrak{h}$ such that the corresponding Coxeter system is $(W_{\lambda}, \Phi_{\lambda})$. Since $\Theta \subsetneq \Pi_{\lambda}$ holds, we can construct the corresponding parabolic subalgebra $\mathfrak{p}'_{\Theta}$ of $\mathfrak{g}'$. For $\mu \in \mathsf{P}_{\Theta}^{++}$, we denote by $M'_{\Theta}(\mu)$ the corresponding generalized Verma module of $\mathfrak{g}'$. We consider the category $\mathcal{O}$ in the sense of [BGG71] corresponding to our particular choice of positive root system. More precisely, we denote by $\mathcal{O}$ (respectively $\mathcal{O}'$) 'the category $\mathcal{O}$' for $\mathfrak{g}$ (respectively $\mathfrak{g}'$). We denote by $\mathcal{O}_{\lambda}$ (respectively $\mathcal{O}'_{\lambda}$) the full subcategory of $\mathcal{O}$ (respectively $\mathcal{O}'$) consisting of the objects whose irreducible constituents have highest weights in $\{w\lambda \mid w \in W_{\lambda}\}$. Soergel's celebrated theorem [Soe90, Theorem 11] says that there is a category equivalence between $\mathcal{O}_{\lambda}$ and $\mathcal{O}'_{\lambda}$. Under the equivalence a Verma module, $M(x\lambda)\,(x \in W_{\lambda})$ corresponds to $M'(x\lambda)$. We easily see $M_{\Theta}(x\lambda) \cong M(x\lambda)/\sum_{\alpha\in\Theta} M(s_{\alpha}x\lambda)$ and the embedding $M(s_{\alpha}x\lambda) \subseteq M(x\lambda)$ is unique up to scalar multiplication for each $\alpha \in \Theta$. So, we easily see $M_{\Theta}(x\lambda)$ corresponds to $M'_{\Theta}(x\lambda)$ under Soegel's category equivalence. Hence we have the following lemma as a corollary of Soergel's theorem.

Lemma 2.2.1. *Let $\lambda \in \mathfrak{h}^{*}$ be dominant. Let $x, y \in W_{\lambda}$ be such that $x\lambda, y\lambda \in {}^{\circ}\mathsf{P}_{\Theta}^{++}$. Then, the following two conditions are equivalent:*

(1) $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$;

(2) $M'_{\Theta}(x\lambda) \subseteq M'_{\Theta}(y\lambda)$.

This lemma tells us that we may reduce Problem 2 to the case that $\lambda$ is integral.

We discuss another application of Soergel's theorem. We denote by $\mathfrak{g}^{\vee}$ the reductive Lie algebra corresponding to the coroot system $\Delta^{\vee}$. We regard a Cartan subalgebra $\mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{g}^{\vee}$. We attach $\vee$ to the notion with respect to $\mathfrak{g}^{\vee}$ corresponding to that of $\mathfrak{g}$. Then we have the canonical isomorphism $(W, S) \cong (W^{\vee}, S^{\vee})$ of the Coxeter systems. So, we identify them. For $\Theta \subsetneq \Pi$, we put $\Theta^{\vee} = \{\alpha^{\vee} \mid \alpha \in \Theta\} \subsetneq \Pi^{\vee}$. We put ${}^{\circ}\mathsf{P}_{\Theta^{\vee}}^{\vee++} = \{\lambda \in \mathfrak{h}^{*} \mid \langle \lambda, \alpha \rangle = 1\,(\alpha \in \Theta)\}$. For $\lambda \in {}^{\circ}\mathsf{P}_{\Theta^{\vee}}^{\vee++}$, we consider a scalar generalized Verma module $M_{\Theta^{\vee}}^{\vee}(\lambda)$ of $\mathfrak{g}^{\vee}$. The following result is an immediate consequence of Soergel's theorem.

Theorem 2.2.2. *Let $\lambda \in \mathsf{P}$ and $\mu \in \mathsf{P}^{\vee}$ be dominant regular. Let $x, y \in W = W^{\vee}$. We assume that $x\lambda, y\lambda \in {}^{\circ}\mathsf{P}_{\Theta}^{++}$ and $x\mu, y\mu \in {}^{\circ}\mathsf{P}_{\Theta^{\vee}}^{\vee++}$. Then, $M_{\Theta}(x\lambda) \subseteq M_{\Theta}(y\lambda)$ if and only if $M_{\Theta^{\vee}}^{\vee}(x\mu) \subseteq M_{\Theta^{\vee}}^{\vee}(y\mu)$.*

Hence, we may reduce Problem 1 for simple Lie algebras of the type $\mathrm{C}_n$ to that for simple Lie algebras of the type $\mathrm{B}_n$.

## 2.3 Comparison of $\tau$-invariants

We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$

Then, we easily see the following lemma.

Lemma 2.3.1. *We have:*

(a) $W(\Theta) = \{w \in W \mid w\alpha^\vee \in \Theta^\vee \text{ for all } \alpha \in \Theta\}$;
(b) $W(\Theta) = \{w \in W \mid w\rho_\Theta = \rho_\Theta, w\Theta \subseteq \Delta^+\}$;
(c) $w_\Theta w = ww_\Theta$ *for all* $w \in W(\Theta)$;
(d) $W(\Theta)$ *preserves* $\mathfrak{a}_\Theta^*$;
(e) $W(\Theta) \subseteq W_{\rho_\Theta}$.

In this section, we prove the following proposition.

Proposition 2.3.2. *Let* $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$ *be regular. Let* $x \in W_\lambda$ *be such that* $x\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. *Moreover, we assume that* $M_\Theta(x\lambda) \subseteq M_\Theta(\lambda)$. *Then, we have* $x \in W(\Theta)$.

First, we prove the following lemma.

Lemma 2.3.3. *Let* $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$ *be regular and let* $w \in W_\lambda$ *be such that* $w\lambda$ *is dominant. Then, we have* $w\Theta \subsetneq \Pi_\lambda$.

*Proof.* Assume that there is some $\alpha \in \Theta$ such that $w\alpha \notin \Pi_\lambda$. Then $w\alpha^\vee \notin \Pi_\lambda^\vee$. Here, we remark that $\Pi_\lambda^\vee$ is a basis of the positive coroot system $(\Delta^+)^\vee$. So, there exists some $\beta, \gamma \in \Delta^+$ such that $w\alpha^\vee = \beta^\vee + \gamma^\vee$. Since $w\lambda$ is dominant and regular, we have $\langle w\lambda, \beta^\vee\rangle \geqslant 1$ and $\langle w\lambda, \gamma^\vee\rangle \geqslant 1$. Also, $2 \leqslant \langle w\lambda, \beta^\vee + \gamma^\vee\rangle = \langle w\lambda, w\alpha^\vee\rangle = \langle \lambda, \alpha^\vee\rangle$. On the other hand, $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$ implies $\langle \lambda, \alpha^\vee\rangle = 1$. This is a contradiction. □

*Proof of Proposition 2.3.2.* From Lemma 2.2.1, we may reduce the proposition to the case that $\lambda$ is integral. Put $\Theta_1 = w\Theta$ and $\Theta_2 = wx^{-1}\Theta$. From Lemma 2.3.1, we have $\Theta_1 \subseteq \Pi$ and $\Theta_2 \subseteq \Pi$. Since $w_0 w_{\Theta_i}\Theta_i = -w_0\Theta_i$ holds for $i = 1, 2$, we have $w_0 w_{\Theta_i}\Theta_i \subseteq \Pi$. We put $I_1 = \mathrm{Ann}_{U(\mathfrak{g})}(M_\Theta(\lambda))$ and $I_2 = \mathrm{Ann}_{U(\mathfrak{g})}(M_\Theta(x\lambda))$.

From [BJ77, Corollary 4.10], we have $I_1 = \mathrm{Ann}_{U(\mathfrak{g})}(M_{-w_0\Theta_1}(w_0 w_{\Theta_1} w\lambda))$ and $I_2 = \mathrm{Ann}_{U(\mathfrak{g})}(M_{-w_0\Theta_2}(w_0 w_{\Theta_2} w\lambda))$. Since $\langle w_0 w_{\Theta_i} w\lambda, \alpha^\vee\rangle < 0$ for all $\alpha \in \Delta^+ - w_0 w_{\Theta_i}\langle\Theta_i\rangle$, then $M_{w_0 w_{\Theta_i}\Theta_i}(w_0 w_{\Theta_i} w\lambda)$ is irreducible. Hence, $I_1$ and $I_2$ are primitive ideals of the same Gelfand–Kirillov dimension. The $\tau$-invariant of $I_1$ (respectively $I_2$) is $-w_0\Theta_1$ (respectively $-w_0\Theta_2$). On the other hand, $M_\Theta(x\lambda) \subseteq M_\Theta(\lambda)$ implies $I_1 \subseteq I_2$. Hence, we have $I_1 = I_2$ from [BK76, 3.6.Korollar]. Comparing the $\tau$-invariants, we have $-w_0\Theta_1 = -w_0\Theta_2$. Hence, $w\Theta = \Theta_1 = \Theta_2 = wx^{-1}\Theta$. This implies $x \in W(\Theta)$. □

## 3. Some results on Bruhat orderings

### 3.1 Quasi-subsystems

Let $(W_i, S_i)\,(i = 1, 2)$ be finite Coxeter systems. We denote by $\ell_i(w)$ the length of a reduced expression of $w \in W_i$ with respect to $S_i$. We also denote by $\leqslant_i$ the Bruhat ordering for $(W_i, S_i)$.

Definition 3.1.1. We say that $(W_2, S_2)$ is a quasi-subsystem of $(W_1, S_1)$ if the following, (Q1) and (Q2), hold:

(Q1) $W_2$ is a subgroup of $W_1$;
(Q2) for any reduced expression $w = s_1 \cdots s_k$ of $w \in W_2$ in $(W_2, S_2)$, we have $\ell_1(w) = \ell_1(s_1) + \cdots + \ell_1(s_k)$.

The following lemma is easy.

Lemma 3.1.2. *Assume that* $(W_2, S_2)$ *is a quasi-subsystem of* $(W_1, S_1)$. *Then,* $x \leqslant_2 y$ *implies* $x \leqslant_1 y$ *for all* $x, y \in W_2$.

We have the following lemma.

Lemma 3.1.3. *Assume that* $(W_2, S_2)$ *is a quasi-subsystem of* $(W_1, S_1)$. *Moreover, we assume the following condition* (C).

(C) *For any* $x, y \in W_2$ *and* $s \in S_2$ *such that* $x \leqslant_1 y$, $\ell_1(sy) < \ell_1(y)$, *and* $\ell_1(x) < \ell_1(sx)$, *we have* $sx \leqslant_1 y$.

*Then,* $x \leqslant_1 y$ *implies* $x \leqslant_2 y$ *for all* $x, y \in W_2$.

*Proof.* Let $x, y \in W_2$ be such that $x \leqslant_1 y$. We show $x \leqslant_2 y$ by a double induction with respect to $\ell_2(y)$ and $\ell_2(y) - \ell_2(x)$.

Obviously we may assume $\ell_2(y) > 0$. So, we choose some $s \in S_2$ such that $\ell_2(sy) < \ell_2(y)$.

First, we assume that $\ell_2(sx) < \ell_2(x)$. We fix reduced expressions of $s$, $sx$, and $sy$ in $(W_1, S_1)$ as follows:

$$\begin{aligned} s &= s_1 \cdots s_k \quad (s_1, \ldots, s_k \in S_1), \\ sx &= t_1 \cdots t_h \quad (t_1, \ldots, t_h \in S_1), \\ sy &= r_1 \cdots r_n \quad (r_1, \ldots, r_n \in S_1). \end{aligned}$$

From (Q2), we easily see that $s_m \cdots s_k t_1 \cdots t_h$ and $s_m \cdots s_k r_1 \cdots r_n$ are reduced expressions for all $1 \leqslant m \leqslant k$. Applying [Deo77, Theorem 1.1], we have $s_m \cdots s_k t_1 \cdots t_h \leqslant_1 s_m \cdots s_k r_1 \cdots r_n$ by the induction on $m$. So, we have $sx \leqslant_1 sy$. Since $\ell_2(sy) < \ell_2(y)$, the induction hypothesis implies that $sx \leqslant_2 sy$. Again, applying [Deo77, Theorem 1.1], we have $x \leqslant_2 y$.

Next, we assume that $\ell_2(sx) > \ell_2(x)$. From (Q2), we have $\ell_1(sx) > \ell_1(x)$. So, we have $sx \leqslant_1 y$ from (C). Since $\ell_2(y) - \ell_2(sx) < \ell_2(y) - \ell_2(x)$, we have $sx \leqslant_2 y$ from the induction hypothesis. Since $x \leqslant_2 sx$, we have $x \leqslant_2 y$. □

### 3.2 Θ-useful roots

In this subsection, we use the notation in § 1.

Following Knapp [Kna75], Howlett [How80], and Lusztig [Lus76], we consider useful roots for our purpose.

Hereafter, we fix a subset Θ of Π. For $\alpha \in \Delta$, we put

$$\begin{aligned} \Delta(\alpha) &= \{\beta \in \Delta \mid \exists c \in \mathbb{R} \quad \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\}, \\ \Delta^+(\alpha) &= \Delta(\alpha) \cap \Delta^+, \\ U_\alpha &= \mathbb{C}\Theta + \mathbb{C}\alpha \subseteq \mathfrak{h}^*. \end{aligned}$$

Then $(U_\alpha, \Delta(\alpha), \langle\,,\,\rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle\,,\,\rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. $\alpha|_{\mathfrak{a}_\Theta} = 0$ if and only if $\Theta = \Pi(\alpha)$. For $\alpha \in \Delta^+$, we denote by $W_\Theta(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$. We denote by $w^\alpha$ the longest element of $W_\Theta(\alpha)$. We set

$$\sigma_\alpha = w^\alpha w_\Theta.$$

$\alpha|_{\mathfrak{a}_\Theta} = 0$ if and only if $\sigma_\alpha = 1$.

Definition 3.2.1. (1) We call $\alpha \in \Delta$ Θ-*useful* if the order of $\sigma_\alpha$ is two. We denote by ${}^u\Delta_\Theta$ the set of the useful Θ-roots. We also put ${}^u\Delta_\Theta^+ = {}^u\Delta_\Theta \cap \Delta^+$.

(2) If $\alpha|_{\mathfrak{a}_\Theta} \neq 0$, then $\Pi(\alpha)$ is written as $\Theta \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call $\alpha$ $\Theta$-*reduced.* We put

$$^{ru}\Delta_\Theta^+ = \{\alpha \in {}^u\Delta_\Theta^+ \mid \alpha \text{ is } \Theta\text{-reduced}\}.$$

If $\alpha \in \Delta$ is orthogonal to all the elements in $\Theta$, then we can easily see $\alpha$ is $\Theta$-reduced and $s_\alpha = \sigma_\alpha$. We easily see the following lemma.

Lemma 3.2.2. *Let $\alpha \in \Delta^+$ be $\Theta$-reduced. We denote by $\Delta(\alpha)_0$ be the irreducible component of $\Delta(\alpha)$ containing $\alpha$. We put $\Pi(\alpha)_0 = \Pi(\alpha) \cap \Delta(\alpha)_0$.*

(1) *If $\Delta(\alpha)_0$ is not of the type ADE, then we have $\alpha \in {}^{ur}\Delta_\Theta^+$.*

(2) *If $\Delta(\alpha)_0$ is of the type $\mathrm{D}_{2n}$ $(n \geqslant 2)$, $\mathrm{E}_7$, or $\mathrm{E}_8$, then we have $\alpha \in {}^{ur}\Delta_\Theta^+$.*

(3) *If $\Delta(\alpha)_0$ is of the type $\mathrm{A}_{2n}$ $(n \geqslant 1)$, then we have $\alpha \notin {}^{ur}\Delta_\Theta^+$.*

(4) *We assume that $\Delta(\alpha)_0$ is of the type $\mathrm{A}_{2n+1}$ $(n \geqslant 0)$. We number the elements of $\Pi(\alpha)_0$ as follows:*

$$\Pi(\alpha)_0 = \{\beta_1, \ldots, \beta_{2n+1}\}.$$

*We choose the above numbering so that $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leqslant i \leqslant 2n$. Then $\alpha \in {}^{ur}\Delta_\Theta^+$ if and only if $\alpha = \beta_n$.*

(5) *We assume that $\Delta(\alpha)_0$ is of the type $\mathrm{D}_{2n+1}$ $(n \geqslant 2)$. We number the elements of $\Pi(\alpha)_0$ as follows:*

$$\Pi(\alpha)_0 = \{\beta_1, \ldots, \beta_{2n+1}\}.$$

*We choose the above numbering so that $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leqslant i \leqslant 2n-1$ and $\langle \beta_{2n-1}, \beta_{2n+1}^\vee \rangle = -1$. Then $\alpha \in {}^{ur}\Delta_\Theta^+$ if and only if $\alpha \notin \{\beta_{2n}, \beta_{2n+1}\}$.*

(6) *We assume that $\Delta(\alpha)_0$ is of the type $\mathrm{E}_6$. We number the elements of $\Pi(\alpha)_0$ as follows:*

$$\Pi(\alpha)_0 = \{\beta_1, \ldots, \beta_6\}.$$

*We choose the above numbering so that $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leqslant i \leqslant 4$ and $\langle \beta_3, \beta_6^\vee \rangle = -1$. Then $\alpha \in {}^{ur}\Delta_\Theta^+$ if and only if $\alpha \in \{\beta_3, \beta_6\}$.*

For $\alpha \in {}^{ru}\Delta_\Theta$, we put

$$V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha \rangle = 0\},$$
$$\hat{\alpha} = \alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^*.$$

We easily see the following lemma.

Lemma 3.2.3. *Let $\alpha \in {}^{ru}\Delta_\Theta^+$. Then, we have that:*

(1) *$\sigma_\alpha$ preserves $\mathfrak{a}_\Theta^*$;*

(2) *$\sigma_\alpha \in W(\Theta)$, and, in particular, $\sigma_\alpha \rho_\Theta = \rho_\Theta$;*

(3) *$\sigma_\alpha \hat{\alpha} = -\hat{\alpha}$;*

(4) *$\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$ is the reflection with respect to $V_\alpha$.*

We denote by $W(\Theta)'$ the subgroup of $W$ generated by $\{\sigma_\alpha \mid \alpha \in {}^{ru}\Delta_\Theta^+\}$. We put ${}^u\Sigma_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^* \mid \alpha \in {}^u\Delta_\Theta\}$; ${}^u\Sigma_\Theta$ is a (not necessarily reduced) root system. We also put ${}^{ru}\Sigma_\Theta^+ = \{\alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^* \mid \alpha \in {}^{ru}\Delta_\Theta^+\}$ and ${}^{ru}\Sigma_\Theta = {}^{ru}\Sigma_\Theta^+ \cup -{}^{ru}\Sigma_\Theta^+$; ${}^{ru}\Sigma_\Theta$ is a reduced root system and ${}^{ru}\Sigma_\Theta^+$ is a positive system. We denote by ${}^u\Pi_\Theta$ the simple system for ${}^{ru}\Sigma_\Theta^+$; ${}^u\Pi_\Theta$ is also a basis of ${}^u\Sigma_\Theta$. For $\alpha \in {}^{ru}\Delta_\Theta^+$, $\sigma_\alpha$ depends only on $\alpha|_{\mathfrak{a}_\Theta}$. So, sometimes we write $\sigma_{\alpha|_{\mathfrak{a}_\Theta}}$ for $\sigma_\alpha$. We put $S(\Theta) = \{\sigma_\gamma \mid \gamma \in {}^u\Pi_\Theta\}$.

Theorem 3.2.4 (Howlett [How80, Theorem 6], Lusztig [Lus76, § 5]).

(1) *We have* $W(\Theta)' \subseteq W(\Theta)$.

(2) *For* $\alpha \in {}^u\Delta_\Theta^+$, $\sigma_\alpha(\mathfrak{a}_\Theta^*) = \mathfrak{a}_\Theta^*$. *Moreover,* $\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$ *is the reflection with respect to* $\alpha|_{\mathfrak{a}_\Theta}$ *and* $\sigma_\alpha \rho_\Theta = \rho_\Theta$.

(3) *We define* $\iota : W(\Theta)' \to GL(\mathfrak{a}_\Theta^*)$ *by* $\iota(x) = x|_{\mathfrak{a}_\Theta^*}$. *Then* $\iota$ *is an injective group homomorphism.*

(4) *We have that* $\iota(W(\Theta)')$ *is the reflection group for the root system* ${}^{ru}\Sigma_\Theta$. *Hence* $(W(\Theta)', S(\Theta))$ *is a Coxeter system.*

We denote by $\leqslant_\Theta$ the Bruhat ordering for $(W(\Theta)', S(\Theta))$.

## 3.3 Normal parabolic subalgebras

Definition 3.3.1. We call $\Theta \subsetneq \Pi$ normal if $\Pi - \Theta \subseteq {}^u\Delta_\Theta^+$. We call a standard parabolic subalgebra $\mathfrak{p}_\Theta$ normal if $\Theta$ is normal. A parabolic subalgebra is called normal if it is conjugate to a normal standard parabolic subalgebra by an inner automorphism.

We describe the list of the normal parabolic subalgebras of classical Lie algebras.

(1) Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ (the case of $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ is similar) and let $k$ be a positive integer dividing $n$. We consider the following parabolic subalgebras:

$\mathfrak{p}(A_{n-1,k})$: a parabolic subalgebra of $\mathfrak{g}$ whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k,\mathbb{C})}^{n/k}.$$

(2) Let $\mathfrak{g}$ be a complex simple Lie algebra of the type $X_n$. Here, $X$ means one of $B$, $C$, and $D$. Let $k$ and $\ell$ be positive integers such that $k$ divides $n-\ell$. If $X = D$, then we assume that $\ell \neq 1$.

We consider the following parabolic subalgebras:

$\mathfrak{p}(X_{n,k,\ell})$: a parabolic subalgebra of $\mathfrak{g}$ whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k,\mathbb{C})}^{(n-\ell)/k} \oplus X_\ell.$$

Here, $X_\ell$ denotes the complex simple Lie algebra of the type $X_\ell$. Namely $B_\ell = \mathfrak{so}(2\ell+1,\mathbb{C})$, $C_n = \mathfrak{sp}(\ell,\mathbb{C})$, and $D_n = \mathfrak{so}(2\ell,\mathbb{C})$. Also, $X_0$ means the zero Lie algebra.

From Lemma 3.2.2, we easily see the following proposition.

Proposition 3.3.2. (1) *We have that* $\mathfrak{p}(A_{n-1,k})$ *is normal. Conversely any normal parabolic subalgebra is conjugate to* $\mathfrak{p}(A_{n,k})$ *for some* $k$.

(2) *We have that* $\mathfrak{p}(X_{n,k,\ell})$ *is normal, unless* $X = D$, $\ell = 0$, *and* $k$ *is an odd number greater than* 1. *Any normal parabolic subalgebra is conjugate to one of such* $\mathfrak{p}(X_{n,k,\ell})$ *by an inner automorphism.*

For exceptional simple Lie algebras, we have the following results. If $\Theta$ is the empty set, it is obviously normal. So, we consider $\emptyset \neq \Theta \subsetneq \Pi$.

We number the simple roots $\Pi = \{\alpha_1, \ldots\}$ of exceptional simple Lie algebras $\mathrm{F}_4$, $\mathrm{E}_6$, $\mathrm{E}_7$, $\mathrm{E}_8$ as follows:

$$1 - 2 \Leftarrow 3 - 4\,, \qquad \begin{array}{ccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 \\ & & & & | & & & & \\ & & & & 6 & & & & \end{array}$$

$$\begin{array}{ccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 6 & - & 7 \\ & & & & & & | & & & & \\ & & & & & & 5 & & & & \end{array}$$

$$\begin{array}{ccccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 7 & - & 8. \\ & & & & & & & & | & & & & \\ & & & & & & & & 6 & & & & \end{array}$$

Let $X$ be $F$ or $E$. We denote by $X_{r,i_1\cdots i_k}$ $\Theta = \Pi - \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$, where $\Pi$ is the above-mentioned numbered basis for the exceptional simple Lie algebra $\mathfrak{g}$ of the type $X_r$.

Proposition 3.3.3.

(1) *Assume that $\mathfrak{g}$ is of type $\mathrm{G}_2$. Then any subset of $\Pi$ is normal.*
(2) *Assume that $\mathfrak{g}$ is of type $\mathrm{F}_4$. If* $\operatorname{card}\Theta = 3$, $\Theta \subsetneq \Pi$ *is normal. The other non-empty normal subsets of $\Pi$ are $\mathrm{F}_{4,12}$, $\mathrm{F}_{4,14}$, and $\mathrm{F}_{4,34}$.*
(3) *Assume that $\mathfrak{g}$ is of type $\mathrm{E}_6$. The non-empty normal subsets of $\Pi$ are $\mathrm{E}_{6,3}$, $\mathrm{E}_{6,6}$, $\mathrm{E}_{6,15}$.*
(4) *Assume that $\mathfrak{g}$ is of type $\mathrm{E}_7$ If* $\operatorname{card}\Theta = 6$, $\Theta \subsetneq \Pi$ *is normal. The other non-empty normal subsets of $\Pi$ are $\mathrm{E}_{7,27}$, $\mathrm{E}_{7,67}$, $\mathrm{E}_{7,127}$, and $\mathrm{E}_{7,2467}$.*
(5) *Assume that $\mathfrak{g}$ is of type $\mathrm{E}_8$. If* $\operatorname{card}\Theta = 7$, $\Theta \subsetneq \Pi$ *is normal. The other non-empty normal subsets of $\Pi$ are $\mathrm{E}_{8,12}$, $\mathrm{E}_{8,18}$, $\mathrm{E}_{8,38}$, and $\mathrm{E}_{8,1238}$.*

We put

$$K(\Theta) = \{w \in W \mid w\Theta \subseteq \Pi\}.$$

We give some characterizations of normality.

Proposition 3.3.4. *For $\Theta \subsetneq \Pi$. the following conditions are equivalent:*

(1) $\Theta \subsetneq \Pi$ *is normal;*
(2) $K(\Theta) = W(\Theta)'$;
(3) $K(\Theta) = W(\Theta)$;
(4) ${}^{u}\Sigma_\Theta = \Sigma_\Theta$.

*Proof.* First, we assume condition (1). Then, using Propositions 3.3.2 and 3.3.3, we obtain conditions (2) and (4) via case-by-case analysis. Condition (2) obviously implies condition (3). Next, we assume condition (3). For $\alpha \in \Pi - \Theta$, we easily see $\sigma_\alpha^2(\Pi) \subseteq \Delta^+$. Hence $\sigma_\alpha$ is an involution. This means that $\alpha \in {}^{u}\Delta_\Theta^+$. So, we have condition (1). Condition (4) is clearly stronger than condition (1). □

Corollary 3.3.5. *If $\Theta \subsetneq \Pi$ is normal, then $W(\Theta)' = W(\Theta)$.*

Since $\Delta^+ \cap (-w\Delta^+) = \{\alpha \in \Delta^+ \mid \alpha|_{\mathfrak{a}_\Theta} \in \Sigma_\Theta^+ \cap (-w\Sigma_\Theta^+)\}$ for each $w \in W(\Theta)$, we easily see the following lemma.

Lemma 3.3.6. *We assume that $\Theta \subsetneq \Pi$ is normal. Then for each $w \in W(\Theta)$, we have*

$$\Delta^+ \cap (-w\Delta^+) = \bigcup_{\gamma \in {}^{ru}\Sigma_\Theta^+ \cap (-w\,{}^{ru}\Sigma_\Theta^+)} \{\alpha \in \Delta^+ \mid \exists c > 0\, \alpha|_{\mathfrak{a}_\Theta} = c\gamma\}.$$

Hence, we have the following result.

Proposition 3.3.7. *If $\Theta \subsetneq \Pi$ is normal, then $(W(\Theta)', S(\Theta))$ is a quasi-subsystem of $(W, S)$.*

As a corollary of Proposition 3.3.4, we easily see the following corollary.

Corollary 3.3.8. *We have that $\Theta \subsetneq \Pi$ is normal if and only if any two parabolic subalgebras with the Levi part $\mathfrak{l}_\Theta$ are conjugate under an inner automorphism of $\mathfrak{g}$.*

### 3.4 Comparison of Bruhat orderings

In this subsection, we use the notation in § 1.

Definition 3.4.1. We call $\Theta \subsetneq \Pi$ *seminormal* if there exists some $\Psi$ such that $\Theta \subseteq \Psi \subseteq \Pi$ and ${}^u\Pi_\Theta = \{\alpha|_{\mathfrak{a}_\Theta} \mid \alpha \in \Psi - \Theta\}$.

So, $S(\Theta) = \{\sigma_\alpha \mid \alpha \in \Theta - \Psi\}$.

We have that $\Theta \subsetneq \Pi$ is seminormal if and only if there is a $\alpha \in \Pi \cap {}^{ru}\Delta^+$ such that $\alpha|_{\mathfrak{a}_\Theta} = \gamma$ for each $\gamma \in {}^u\Pi_\Theta$.

We immediately see the following result from Proposition 3.3.7.

Corollary 3.4.2. *If $\Theta \subsetneq \Pi$ is seminormal, then $(W(\Theta)', S(\Theta))$ is a quasi-subsystem of $(W, S)$.*

We fix a connected complex reductive Lie group $G$ whose Lie algebra is $\mathfrak{g}$. For $\Theta \subsetneq \Pi$, we denote by $P_\Theta$ (respectively $H$) the parabolic subgroup (respectively the Cartan subgroup) of $G$ corresponding to $\mathfrak{p}_\Theta$ (respectively $\mathfrak{h}$). We denote by $N_G(H)$ the normalizer of $H$ in $G$. Since the Weyl group $W$ is identified with the quotient group $N_G(H)/H$, for each $w \in W$ we can fix a representative in $N_G(H)$. We denote the representative by the same letter '$w$'.

For $x \in W$, we put $U_x = P_\Theta x / P_\Theta$. Namely, $U_x$ is a $P_\Theta$-orbit in $G/P_\Theta$ through $x/P_\Theta \in G/P_\Theta$. We denote by $\overline{U}_x$ the closure of $U_x$ in $G/P_\Theta$. If $w \in W(\Theta)$, then $\ell(ws_\alpha) > \ell(w)$ for all $\alpha \in \Theta$. Hence, we have the following lemma.

Lemma 3.4.3.

(1) *For $w \in W(\Theta)$, we have $\dim U_w = \ell(w)$.*
(2) *For $x, y \in W(\Theta)$, $x \leqslant y$ if and only if $\overline{U}_x \subseteq \overline{U}_y$.*

Next we show the following lemma.

Lemma 3.4.4. *Assume that $\Theta \subsetneq \Pi$ is seminormal. We choose $\Theta \subseteq \Psi \subseteq \Pi$ as in Definition 3.4.1. Fix $x \in W(\Theta)'$. Let $\alpha \in \Psi - \Theta$ be such that $\ell(\sigma_\alpha x) < \ell(x)$. Then we have $\overline{U}_x = P_{\Theta \cup \{\alpha\}} \overline{U}_x = P_{\Theta \cup \{\alpha\}} \overline{U}_{\sigma_\alpha x}$.*

*Proof.* We may choose a reduced expression $x = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$ such that $\alpha_1 = \alpha$. We consider a contraction map as follows:

$$F : P_{\Theta \cup \{\alpha_1\}} \times_{P_\Theta} P_{\Theta \cup \{\alpha_2\}} \times_{P_\Theta} \cdots \times_{P_\Theta} P_{\Theta \cup \{\alpha_k\}} / P_\Theta \to G/P_\Theta.$$

We easily see that:

(a) $\mathrm{Image}(F)$ is an irreducible Zariski closed set in $G/P_\Theta$;
(b) $\dim \overline{U}_x = \ell(x) = \dim P_{\Theta\cup\{\alpha_1\}} \times_{P_\Theta} \cdots \times_{P_\Theta} P_{\Theta\cup\{\alpha_k\}}/P_\Theta$;
(c) $\overline{U}_x \subseteq \mathrm{Image}(F)$.

Hence, we have $\overline{U}_x = \mathrm{Image}(F)$. So, we have the lemma immediately. □

The following result is the main result of this section.

Theorem 3.4.5. *Let $\Theta \subsetneq \Pi$ be seminormal. For $x, y \in W(\Theta)'$, $x \leqslant y$ if and only if $x \leqslant_\Theta y$.*

*Proof.* We choose $\Theta \subseteq \Psi \subseteq \Pi$ as in Definition 3.4.1. From Lemmas 3.1.2, 3.1.3, and Corollary 3.4.2, we have only to show the condition (C) in the statement of Lemma 3.1.3 holds for $(W(\Theta)', S(\Theta))$. So we choose $x, y \in W(\Theta)'$ and $\alpha \in \Psi - \Theta$ such that $x \leqslant y$, $\ell(\sigma_\alpha y) < \ell(y)$, and $\ell(\sigma_\alpha x) > \ell(x)$. From $x \leqslant y$, we have $\overline{U}_x \subseteq \overline{U}_y$ by Lemma 3.4.3(2). Hence $P_{\Theta\cup\{\alpha\}}\overline{U}_x \subseteq P_{\Theta\cup\{\alpha\}}\overline{U}_y$. From Lemma 3.4.4, we have $\overline{U}_y = P_{\Theta\cup\{\alpha\}}\overline{U}_y$ and $\overline{U}_{\sigma_\alpha x} = P_{\Theta\cup\{\alpha\}}\overline{U}_x$. So, we have $\overline{U}_{\sigma_\alpha x} \subseteq \overline{U}_y$. This means that $\sigma_\alpha x \leqslant y$. Hence, the condition (C) holds for $\Theta$. □

## 4. Elementary homomorphisms

### 4.1 Elementary homomorphisms

We fix a subset $\Theta$ of $\Pi$ and $\alpha \in {}^{ru}\Delta^+_\Theta$. We define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta\in\Delta(\alpha)} \mathfrak{g}_\beta, \quad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of $\mathfrak{g}$ whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

We denote by $\omega_\alpha \in \mathfrak{a}^*_\Theta \subseteq \mathfrak{h}^*$ the fundamental weight for $\alpha$ with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely $\omega_\alpha$ satisfies that $\langle \omega_\alpha, \beta\rangle = 0$ for $\beta \in \Theta$, $\langle \omega_\alpha, \alpha^\vee\rangle = 1$, and $\omega_\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number $a$ such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta}$, since $\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{\lambda \in \mathfrak{a}^*_\Theta \mid \langle\lambda, \omega_\alpha\rangle = 0\}$.

Put $\rho(\alpha) = \frac{1}{2}\sum_{\beta\in\Delta^+(\alpha)} \beta$. For $\nu \in \mathfrak{a}^*_\Theta$, we denote by $\mathbb{C}_\nu$ the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$-module corresponding to $\nu$. For $\nu \in \mathfrak{a}^*_\Theta$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows:

$$M^{\mathfrak{g}(\alpha)}_\Theta(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu-\rho(\alpha)}.$$

Then, we have the following theorem.

Theorem 4.1.1 [Mat06]. *Let $\nu$ be an arbitrary element in $V_\alpha$, and let $c$ be either $1$ or $\frac{1}{2}$. Assume that $M^{\mathfrak{g}(\alpha)}_\Theta(\rho_\Theta - c\omega_\alpha) \subseteq M^{\mathfrak{g}(\alpha)}_\Theta(\rho_\Theta + c\omega_\alpha)$. Then, we have $M_\Theta(\rho_\Theta + \nu - (c+n)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + (c+n)\omega_\alpha)$ for all $n \in \mathbb{N}$.*

We call the above homomorphism of $M_\Theta(\rho_\Theta + \nu - (c+n)\omega_\alpha)$ into $M_\Theta(\rho_\Theta + \nu + (c+n)\omega_\alpha)$ *an elementary homomorphism.* In [Mat06], homomorphisms between scalar generalized Verma modules associated with a maximal parabolic subalgebra are classified. So, elementary homomorphisms are understood.

The following conjecture is proposed in [Mat06] as a working hypothesis.

Conjecture 4.1.2. An arbitrary non-trivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.

The conjecture in the case of the Verma modules is nothing but the result of Bernstein *et al.* [BGG71]. We do not know a counterexample for the above working hypothesis, and we obtain partial affirmative results in this article. A weaker version is given in the following conjecture.

Conjecture 4.1.3. Let $\Theta \subseteq \Pi$ be normal and let $\mu, \nu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ and $\rho_\Theta + \nu$ are regular integral. If $M_\Theta(\rho_\Theta + \nu) \subseteq M_\Theta(\rho_\Theta + \mu)$, then it is a composition of elementary homomorphisms.

Later, we confirm the conjecture for strictly normal case (see § 5) and exceptional Lie algebras (see §§ 5 and 6).

For example, I do not know whether a homomorphism of the form $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$ $(\mu \in \mathfrak{a}_\Theta^*)$ is always elementary. We have a weak result.

Proposition 4.1.4. *We fix $\mu \in \mathfrak{a}_\Theta^*$ such that $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$ and $\rho_\Theta + \mu$ is regular and integral. If $\{\beta \in \Sigma_\Theta - \mathbb{R}\alpha|_{\mathfrak{a}_\Theta} \mid \langle \mu, \beta\rangle > 0\} = \{\beta \in \Sigma_\Theta - \mathbb{R}\alpha|_{\mathfrak{a}_\Theta} \mid \langle \sigma_\alpha\mu, \beta\rangle > 0\}$, then $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$ is an elementary homomorphism.*

*Proof.* Put $\nu_0 = \mu - \langle \mu, \alpha^\vee\rangle\omega_\alpha$. Then $\nu_0 \in V_\alpha$. Since $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$, we have $\mu - \sigma_\alpha\mu = 2\langle \mu, \alpha^\vee\rangle\omega_\alpha \in \mathsf{Q}^+$. Hence, $2\langle \mu, \alpha^\vee\rangle\omega_\alpha$ is integral. So, we can write $\langle \mu, \alpha^\vee\rangle = c + n_0$. Here, $c$ is either 1 or $\frac{1}{2}$ and $n_0$ is a positive integer. Put $\kappa = 2(\mu + \sigma_\alpha\mu)$. Since $2\rho_\Theta$ and $\rho_\Theta + \mu$ are integral, so is $\kappa$. Moreover, we have $\kappa \in V_\alpha$ and $\langle \kappa, \beta\rangle > 0$ for all $\beta \in \Sigma_\Theta - \mathbb{R}\alpha|_{\mathfrak{a}_\Theta}$ such that $\langle \mu, \beta\rangle > 0$. From the translation principle, we have $M_\Theta(\rho_\Theta + (\nu_0 + m\kappa) - (c + n_0)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + (\nu_0 + m\kappa) + (c + n_0)\omega_\alpha)$ for all $m \in \mathbb{N}$. Hence $\{a \in \mathbb{C} \mid M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) - (c + n_0)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) + (c + n_0)\omega_\alpha)\}$ is Zariski dense in $\mathbb{C}$. So, we can prove $M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) - (c + n_0)\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + (\nu_0 + a\kappa) + (c + n_0)\omega_\alpha)$ for all $a \in \mathbb{C}$ in the same way as [Lep75b, Lemma 5.4]. If $a \in \mathbb{C}$ is generic, then the integral root system for $\rho_\Theta + (\nu_0 + a\kappa) - (c + n_0)\omega_\alpha$ is $\Delta(\alpha)$. Hence, Lemma 2.2.1 implies that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - (c + n_0)\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + (c + n_0)\omega_\alpha)$. Applying [Mat06, Lemma 2.2.6], we have $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - c\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + c\omega_\alpha)$. □

### 4.2 Θ-excellent roots

We retain the notations in § 4.1.

Definition 4.2.1.

(1) We call $\alpha \in {}^{ru}\Delta = \Theta^+$ Θ-excellent if $\sigma_\alpha$ is a Duflo involution ([Dix77], cf. [Jos83]) in $W(\alpha)$.
(2) We put ${}^e\Delta_\Theta^+ = \{\alpha \in {}^{ru}\Delta_\Theta^+ \mid \alpha \text{ is } \Theta\text{-excellent}\}$.
(3) We put ${}^e\Sigma_\Theta^+ = \{\alpha|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^* \mid \alpha \in {}^e\Delta_\Theta^+\}$ and ${}^e\Sigma_\Theta = {}^e\Sigma_\Theta^+ \cup (-{}^e\Sigma_\Theta^+)$.
(4) We denote by ${}^eW(\Theta)$ the subgroup of $W(\Theta)'$ generated by $\{\sigma_\alpha \mid \alpha \in {}^e\Delta_\Theta^+\}$.
(5) For $\alpha \in {}^{ru}\Delta_\Theta^+$, we put $c_\alpha = 1$ (respectively $c_\alpha = \frac{1}{2}$) if $\rho_\Theta$ is integral (respectively not integral) with respect to $\Delta(\alpha)$. Then, $\rho_\Theta + (c_\alpha + n)\omega_\alpha$ is integral with respect to $\Delta(\alpha)$ for all $n \in \mathbb{Z}$.

We have the following proposition.

Proposition 4.2.2. *Let $\alpha \in {}^e\Delta_\Theta^+$ and let $\mu \in \mathfrak{a}_\Theta^*$ be such that $\rho_\Theta + \mu$ is integral and $\langle \mu, \alpha\rangle > 0$. Then, we have an elementary homomorphism $M_\Theta(\rho_\Theta + \sigma_\alpha\mu) \subseteq M_\Theta(\rho_\Theta + \mu)$.*

*Proof.* Put $\nu_0 = \mu - \langle \mu, \alpha^\vee\rangle\omega_\alpha$. Then $\nu_0 \in V_\alpha$. Since $\rho_\Theta + \mu$ is integral, we have $\langle \rho_\Theta + \mu, \alpha^\vee\rangle \in \mathbb{Z}$. From the definition of $c_\alpha$, we have $\langle \rho_\Theta, \alpha^\vee\rangle \in c_\alpha + \mathbb{Z}$. Hence, we can write $\mu = \nu_0 + (c_\alpha + n)\omega_\alpha$ for some $n \in \mathbb{N}$. So, from $\alpha \in {}^e\Delta_\Theta^+$, Theorem 4.1.1 and [Mat93, Proposition 2.1.2], we have the proposition. □

For a simple Lie algebra of the type A, every involution is a Duflo involution [Duf77]. Hence, we have the following corollary.

COROLLARY 4.2.3. *If $\mathfrak{g}$ is a simple Lie algebra of the type A, we have ${}^{ru}\Delta^+_\Theta = {}^{e}\Delta^+_\Theta$ for all $\Theta \subsetneq \Pi$.*

## 5. Strictly normal case

### 5.1 Strictly normal subset of $\Pi$

DEFINITION 5.1.1. We call $\Theta \subsetneq \Pi$ *strictly normal* if $\Theta$ is normal and ${}^{e}\Delta^+_\Theta = {}^{ru}\Delta^+_\Theta$. A standard parabolic subalgebra $\mathfrak{p}_\Theta$ is called strictly normal when $\Theta$ is strictly normal.

Before stating the main result, we prove the following lemma.

LEMMA 5.1.2. *Let $\Theta \subsetneq \Pi$ be normal and let $\mu \in \mathfrak{a}^*_\Theta$ be such that $\rho_\Theta + \mu$ is integral. Then, $\mu$ is integral with respect to ${}^{ru}\Sigma_\Theta$.*

*Proof.* Since $\rho_\Theta + \mu$ is integral, we have $w(\rho_\Theta + \mu) - \rho_\Theta - mu = w\mu - \mu \in \mathsf{Q}$ for all $w \in W(\Theta)'$. Since $\mathsf{Q} \cap \mathfrak{a}^*_\Theta$ is contained in the root lattice for ${}^{ru}\Sigma_\Theta$, we have the result. □

The following result is the main result.

THEOREM 5.1.3. *We assume that $\Theta \subsetneq \Pi$ is strictly normal. Let $\mu \in \mathfrak{a}^*_\Theta$ be such that $\rho_\Theta + \mu$ is dominant integral and regular. Let $x, y \in W(\Theta)'$. Then, we have:*

(1) *$M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ if and only if $y \leqslant_\Theta x$;*
(2) *if $y \leqslant_\Theta x$, then $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ is a composition of elementary homomorphisms.*

*Proof.* First, we assume that $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$. Hence, $L(\rho_\Theta + x\mu)$ is an irreducible constituent of $M(\rho_\Theta + y\mu)$. From [BGG71], we have $M(\rho_\Theta + x\mu) \subseteq M(\rho_\Theta + y\mu)$, namely $y \leqslant x$. Hence from Theorem 3.4.5, we have $y \leqslant_\Theta x$.

Next, we assume that $y \leqslant_\Theta x$. Since $\mu$ is regular dominant integral with respect to ${}^{ru}\Sigma_\Theta$, there exist $\alpha_1, \dots, \alpha_k \in {}^{ru}\Delta^+_\Theta$ such that $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k} y = x$, $\langle y\mu, \alpha_k \rangle > 0$, and $\langle \sigma_{\alpha_{r+1}} \cdots \sigma_{\alpha_k} y\mu, \alpha_r \rangle > 0$ for $1 \leqslant r \leqslant k-1$. So, from Proposition 4.2.2, we can construct embedding $M_\Theta(\rho_\Theta + x\mu) \subseteq M_\Theta(\rho_\Theta + y\mu)$ as a composition of elementary homomorphisms. □

*Remark.* The argument in § 2.2 does not imply the conclusion of the theorem for a non-integral infinitesimal character, since the corresponding parabolic subalgebras of $\mathfrak{g}'$ are normal but not necessary strictly normal.

### 5.2 Classification of the strictly normal parabolic subalgebras

From [Mat06], we can determine $\Theta$-excellent roots, and we can obtain the following result.

PROPOSITION 5.2.1. *The following is the list of the strictly normal standard parabolic subalgebras of a classical Lie algebra:*

(a) $\mathfrak{p}(A_{n-1,k})$ $(k|n)$;
(b) $\mathfrak{p}(B_{n,2k,m})$ $(k \leqslant m)$;
(c) $\mathfrak{p}(B_{n,2k+1,m})$ $(k \geqslant m)$;
(d) $\mathfrak{p}(C_{n,2k,m})$ $(k \leqslant m)$;
(e) $\mathfrak{p}(C_{n,2k+1,m})$ $(k \geqslant m)$;
(f) $\mathfrak{p}(D_{n,2k-1,m})$ $(k \leqslant m, 2 \leqslant m)$;

(g) $\mathfrak{p}(D_{n,2k,m})$ $(k \geqslant m, 2 \leqslant m)$;

(h) $\mathfrak{p}(D_{n,1,0})$.

Next, we state the classification of strictly normal parabolic subalgebras for exceptional Lie algebras. It is obtained by more or less straightforward calculation from [Mat06].

Proposition 5.2.2. *Let $\mathfrak{g}$ be an exceptional Lie algebra. We assume $\Theta \subsetneq \Pi$ is normal and* $\mathrm{card}(\Pi - \Theta) \geqslant 2$. *Moreover, we assume that $\Theta$ is not strictly normal. Then $\Theta$ is $F_{4,14}$, $E_{7,27}$, or $E_{8,18}$.*

*Remark.* If $\mathrm{card}(\Pi - \Theta) = 1$, then $\mathfrak{p}_\Theta$ is a maximal parabolic subalgebra. In this case, the homomorphisms between scalar generalized Verma modules are classified in [Mat06]. So, we neglect them.

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Hisayosi Matumoto hisayosi@ms.u-tokyo.ac.jp

Graduate School of Mathematical Sciences, University of Tokyo,
3-8-1 Komaba, Tokyo, 153-8914, Japan