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DISTRIBUTION OF WEIERSTRASS POINTS ON RATIONAL CUSPIDAL CURVES

BY

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ABSTRACT. We study the set $W(\mathcal{L})$ of Weierstrass points of all positive tensor powers of an invertible sheaf $\mathcal{L}$ on an irreducible rational curve X with $g \geqq 2$ ordinary cusps. Using an idea from B. Olsen's study of the analogous question on smooth curves, and an explicit formula for the "theta function" of a cuspidal rational curve, we show that $W(\mathcal{L})$ is never dense on X (in contrast to the case of smooth curves of genus $g \geqq 2$).

1. **Introduction.** There has recently been an increase of interest in the problem of extending well-known notions from the theory of smooth algebraic curves, such as the notion of Weierstrass points for invertible sheaves, to the setting of singular curves. In the papers [2], [3], and [4], R. F. Lax and C. Widland have generalized the classical definition of the Weierstrass points of the canonical sheaf of a smooth curve via the order of vanishing of a certain Wronskian. Using this method, they have defined Weierstrass points for any invertible sheaf $\mathcal{L}$ with dim $H^0(\mathcal{L}) > 0$ on an integral, complex projective Gorenstein curve. Recall that a curve X is said to be Gorenstein if its dualizing sheaf is locally free. For instance, all curves with only planar singularities (e.g. nodes and ordinary cusps) are Gorenstein.

The basic results of this study of Weierstrass points on a singular curve X are as follows. For smooth points P, if $s = \dim H^0(X, \mathcal{L}) > 0$, then P is a Weierstrass point of $\mathcal{L}$ if and only if dim $H^0(X, \mathcal{L}(-sP)) > 0$. On the other hand, if $s > 1$, then the singular points of X are automatically Weierstrass points of high weight.

Our major concern in this note will be the distribution of the smooth Weierstrass points of such a sheaf $\mathcal{L}$. Given an invertible sheaf $\mathcal{L}$ of positive degree on a smooth curve of genus $g \geqq 2$, it is known by a result of B. Olsen ([6]) that the set

$$W(\mathcal{L}) = \{P \in X | P \text{ is a Weierstrass point of } \mathcal{L}^{\otimes n} \text{ for some } n \geqq 1\}$$

is dense on X in its complex topology. For some singular curves, the picture can be quite different. Indeed, in [3], Lax exhibited an example of a 2-nodal rational curve X and a particular $\mathcal{L}$ on X such that the analogous set $W(\mathcal{L})$ of smooth Weierstrass points of all the tensor powers $\mathcal{L}^{\otimes n}$ is not dense on X. The present author and K. A. Furio showed in [5] that this example is not an isolated phenomenon. In fact, if X is

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an irreducible g-nodal rational curve with $g \geqq 2$, and $\mathcal{L}$ is any invertible sheaf on X with $\dim H^0(X, \mathcal{L}) > 0$ and $\dim H^1(X, \mathcal{L}) = 0$, then $W(\mathcal{L})$ is not dense. The limit points of $W(\mathcal{L})$ lie on a real one-dimensional subset of X determined by $\mathcal{L}$ and the locations of the nodes.

Lax and Widland have also studied the Weierstrass points of the canonical sheaf on cuspidal rational curves in [4], and have noted the following general pattern. Weierstrass points on cuspidal rational curves tend to be even scarcer than on rational nodal curves, because even more of the total Weierstrass weight is accounted for by the singular points (see 3 below). In line with this phenomenon, in this note we will show that on an irreducible cuspidal rational curve of arithmetic genus $g \geqq 2$, for all invertible sheaves $\mathcal{L}$ with $\dim H^0(X, \mathcal{L}) > 0$, and $\dim H^1(X, \mathcal{L}) = 0$, $W(\mathcal{L})$ is never dense. We will show that $W(\mathcal{L})$ may be partitioned (though of course not in a unique way) into at most g sequences, each converging to a point in X.

Our technique, as in [5], is to use a criterion for Weierstrass points borrowed from Olsen's work for smooth X. For this, we will need to begin with some preliminaries about the generalized Jacobian of a cuspidal curve, and the theta function associated to X. Following this, in 3 we will formulate and prove our main result.

2. **Rational cuspidal curves.** Let X be the irreducible cuspidal rational curve formed by creating simple cusps (singularities analytically isomorphic to the origin on $y^2 = x^3$) at g distinct points $a_i \in \mathbf{P}^1$. We will view the normalization $(\mathbf{P}^1)$ of X as the Riemann sphere, or extended complex plane in what follows, and we assume that no $a_i = \infty$ for simplicity.

Let ω denote the dualizing sheaf of X. Then $H^0(X, \omega)$ is spanned by the differentials $-dz/(z-a_i)^2$, $i = 1, \dots, g$. Since X is homeomorphic to $\mathbf{P}^1$, there is no period lattice, and the generalized Jacobian of X is $J(X) = \mathbf{C}^g$. Let X_0 denote the set of smooth points of X. Using $x_0 = \infty \in X_0$ as our base point, we can define the Abel mapping $\varphi : X_0 \longrightarrow J(X)$ by $\varphi(x) = (\int_{x_0}^{x} -dz/(z-a_1)^2, \dots, \int_{x_0}^{x} -dz/(z-a_g)^2)$, which has as its image the parametric rational normal curve $W_1 = (1/(x-a_1), \dots, 1/(x-a_g))$ in $\mathbf{C}^g$. By means of the group operation in $\mathbf{C}^g$, we extend φ to a mapping on effective divisors supported in X_0 in the usual way:

$$\varphi : X_0^{(m)} \longrightarrow J(X)$$
$$\sum n_k x_k \longrightarrow \left(\sum n_k/(x_k - a_1), \dots, \sum n_k/(x_k - a_g)\right).$$

Finally, if $D = D_1 - D_2$ with D_i effective, we define $\varphi(D) = \varphi(D_1) - \varphi(D_2)$.

The image of $X_0^{(g-1)}$ under φ, which we will denote by W_{g-1}, is a Zariski-open subset of an irreducible (algebraic!) hypersurface $\Theta \subset \mathbf{C}^g$, whose closure in the compactification of $J(X)$ plays the role of the theta divisor on a smooth curve. We will now derive a defining equation of Θ that plays the role of the theta function in the smooth curve case.

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PROPOSITION 1. *Let X, a_i, g be as above, and let $X_1, \ldots, X_g$ be coordinates on $\mathbf{C}^g$. Consider the function*

$$(1)\quad \theta_X(X_1,\ldots,X_g) = \det\begin{bmatrix} a_1^{g-1}X_1 + (g-1)a_1^{g-2} & a_1^{g-2}X_1 + (g-2)a_1^{g-3} & \cdots & X_1 \\ \vdots & \vdots & & \vdots \\ a_g^{g-1}X_g + (g-1)a_g^{g-2} & a_g^{g-2}X_g + (g-2)a_g^{g-3} & \cdots & X_g \end{bmatrix}.$$

Then the hypersurface $\theta_X(X_1,\ldots,X_g) = 0$ is reduced and irreducible, and contains W_{g-1} as a Zariski-open subset.

PROOF. By definition, W_{g-1} is the translation hypersurface with parametrization:

$$(2)\qquad \begin{aligned} X_1 &= 1/(x_1 - a_1) + \cdots + 1/(x_{g-1} - a_1) \\ &\vdots \\ X_g &= 1/(x_1 - a_g) + \cdots + 1/(x_{g-1} - a_g) \end{aligned}$$

obtained by summing $g-1$ general points $\varphi(x_i) \in W_1$. We can obtain an equation for W_{g-1} by eliminating the x_j in the equations (2).

For each k, $0 \leqq k \leqq g-1$, let σ_k denote the k^{th} elementary symmetric polynomial in the x_j. By putting all the terms over a common denominator, the i^{th} equation in (2) can be rewritten as

$$X_i = \left[\sum_{k=1}^{g-1}(-1)^{k-1}\sigma_{g-1-k}ka_i^{k-1}\right] \Big/ \left[\sum_{k=0}^{g-1}(-1)^k\sigma_{g-1-k}a_i^k\right],$$

or

$$0 = \sum_{k=0}^{g-1}[a_i^kX_i + ka_i^{k-1}]\sigma_{g-1-k}.$$

Since the σ_{g-1-k} are not identically zero, this implies that the determinant in equation (1), which is the determinant formed from the coefficients of this system of linear equations for the σ_{g-1-k}, must be zero. It is easy to check that $\theta_X(X_1,\ldots,X_g) = 0$ defines an irreducible, reduced hypersurface. □

We observe that $\theta_X(X_1,\ldots,X_g)$ is a polynomial of degree g in $X_1,\ldots,X_g$ of the following form:

$$(3)\qquad \theta_X(X_1,\ldots,X_g) = \sum_{I\subseteq\{1,\ldots,g\}} c_I \cdot \prod_{i\in I} X_i,$$

where $c_{\{1,\ldots,g\}}$ is the Vandermonde determinant of the a_i. Hence $c_{\{1,\ldots,g\}} \neq 0$. The constant term of θ_X is always zero.

Our major use of θ_X will be in the following criterion for Weierstrass points. (See [6], p. 362, and Lemma 2 of [5] for related results.)

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PROPOSITION 2. *Let* $\mathcal{L}$ *be an invertible sheaf on* X *such that* $s = \dim H^0(X, \mathcal{L}) > 0$ *and* $\dim H^1(X, \mathcal{L}) = 0$. *Let* P *be a smooth Weierstrass point of* X. *Then if* P *is a Weierstrass point of* $\mathcal{L}$, *then* $\varphi(\mathcal{L}) - s\varphi(P) \in \Theta$.

PROOF. We begin with a comment regarding the notation $\varphi(\mathcal{L})$ in the statement of the Proposition. The generalized Jacobian of X is isomorphic to $\mathrm{Pic}^0(X)$, the group of isomorphism classes of invertible sheaves of degree zero on X. Given any invertible sheaf $\mathcal{L}$ on X, $\mathcal{L} \cong O_X(D)$ for some (not necessarily effective) divisor D supported in X_0. Furthermore, if D and D' are linearly equivalent divisors supported in X_0, then Abel's Theorem on X implies that $\varphi(D) = \varphi(D')$. Thus, we may define $\varphi(\mathcal{L})$ to be the image $\varphi(D)$ for any divisor D supported in X_0 with $\mathcal{L} \cong O_X(D)$. Similarly, if D is any Cartier divisor on X, we may take the associated invertible sheaf $\mathcal{L} = O_X(D)$ and define $\varphi(D) = \varphi(\mathcal{L})$.

Now we proceed to the proof of the Proposition. Since P is a smooth Weierstrass point of $\mathcal{L}$, there is a nonzero section $\sigma \in H^0(X, \mathcal{L}(-sP))$. Then $D = \mathrm{div}(\sigma) - sP$ is an effective Cartier divisor on X. Its degree is $\deg(\mathcal{L}) - s = g - 1$ by the Riemann Roch Theorem for Gorenstein curves (using the assumption that $\mathcal{L}$ is nonspecial).

Even though D may not be supported entirely in X_0, since it is a Cartier divisor, it can be viewed as a "limit" of effective divisors of degree $g-1$ supported in X_0, as in [1]. The image $\varphi(D)$ can then be computed by a limiting process as follows. By the linearity of the abelian sums, we can reduce to the case in which Q is a cusp of X, D is supported at Q, and $f \in O_{X,Q}$ is a local equation for D. For every $\epsilon \in \mathbf{C} \setminus \{0\}$ sufficiently small in absolute value, the divisor $D_\epsilon = \mathrm{div}(f - \epsilon)$ will consist of $\deg(D)$ smooth points. As in Theorem 2 of [1], we will have $\varphi(D) = \lim_{\epsilon \to 0} \varphi(D_\epsilon)$.

As a result, $\varphi(D) = \varphi(\mathcal{L}) - s\varphi(P)$ lies in the closure of $\varphi(X_0^{(g-1)})$, hence in Θ. □

Say $\varphi(\mathcal{L}) = (b_1, \ldots, b_g) \in \mathbf{C}^g = J(X)$. Then from the explicit form of θ_X in equation (3), we have the following corollary.

COROLLARY *Let* $\mathcal{L}$ *be as in Proposition 2, and write* $d = \deg(\mathcal{L})$. *If* x *is a smooth Weierstrass point of* $\mathcal{L}^{\otimes n}$, *then* $\dim H^0(X, \mathcal{L}^{\otimes n}) = nd + 1 - g$, *and*

$$\sum_{I \subseteq \{1,\ldots,g\}} c_I \cdot \prod_{i \in I} [nb_i - (nd + 1 - g)/(x - a_i)] = 0. \tag{4}$$

3. **The location and distribution of the Weierstrass points.** Using the equation (4), we will now make two observations about the location and distribution of the smooth Weierstrass ponts of $\mathcal{L}^{\otimes n}$ on X as $n \longrightarrow \infty$.

Our first observation is that for each n, the equation (4) has at most g roots, and that for generic X and $\mathcal{L}$, there will be exactly g roots. This agrees with the results of [4], since by Proposition 4 of that paper, if Q is a unibranch singular point, then for all $\mathcal{L}$, the $\mathcal{L}$-Weierstrass weight of Q is at least $\delta_Q s(s-1) + (s-1)$, where s, as always, denotes $\dim H^0(X, \mathcal{L})$. If X has only ordinary cusps ($\delta_Q = 1$), $\mathcal{L}$ has degree d, and

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all the cusps have the minimum possible $\mathcal{L}$-Weierstrass weight, then the number of smooth Weierstrass points (counted with their weights) will be the total Weierstrass weight of $\mathcal{L}$, minus the contribution from the cusps, or

$$ds + (g - 1)s(s - 1) - g[s(s - 1) + (s - 1)],$$

which equals g, using the Riemann Roch Theorem. It is in this sense that Weierstrass points are scarcer on cuspidal curves than on nodal curves of the same arithmetic genus.

Our main result is the following second observation.

THEOREM. *Let X be an irreducible rational curve of arithmetic genus $g \geqq 2$ with g ordinary cusps as singularities. Let $\mathcal{L}$ be an invertible sheaf of degree d on X with $s = \dim H^0(X, \mathcal{L}) > 0$, and $\dim H^1(X, \mathcal{L}) = 0$. Let $\varphi(\mathcal{L}) = (b_1, \ldots, b_g) \in J(X)$. Then if $W(\mathcal{L}) = \{P \in X_0 | P$ is a Weierstrass point of $\mathcal{L}^{\otimes n}$ for some $n \geqq 1\}$ is nonempty, its limit points are contained in the set $S = \{d/b_1 + a_1, \ldots, d/b_g + a_g\}$.*

PROOF. Rewrite equation (4) as

$$0 = \sum_{I \subseteq \{1,\ldots,g\}} c_I n^{|I|} \cdot \prod_{i \in I} [b_i - (d + (1 - g)/n)/(x - a_i)].$$

since $c_{\{1,\ldots,g\}} \neq 0$, the leading (that is, the fastest growing) term as $n \longrightarrow \infty$ is the term with $i = \{1, \ldots, g\}$. Dividing by n^g, and letting n go to infinity, we see that the equation is approaching

$$0 = \prod_{i=1}^{g} [b_i - d/(x - a_i)].$$

The claim follows. $\square$

In particular, the set $W(\mathcal{L})$ is not dense on X, and is, in a sense, even less evenly distributed than in the case of nodal curves.

In conclusion, we note that by Olsen's result, it may be seen that if the geometric genus of X is 2 or greater, then the set $W(\mathcal{L})$ will be dense in X. In this paper, and in [5], we have considered only rational curves. One natural question is: What happens when the geometric genus of X is 1? It seems reasonable to conjecture that that case will be more like the rational examples here than the case of geometric genus $\geqq 2$.

REFERENCES

1. V. Ancona, *On the Abel-Jacobi Theorem for Singular Curves (Italian)*, Seminari di Geometria 1984, Univ. Stud. Bologna, Bologna, 1985, 3–6.

2. R. F. Lax, *Weierstrass Points on Rational Nodal Curves*, Glasgow Math. J., **29** (1987), 131–140.

3. ———, *On the Distribution of Weierstrass Points on Singular Curves*, Israel J. Math. **57** (1987), 107–115.

4. R. F. Lax, and C. Widland, *Weierstrass Points on Rational Cuspidal Curves*, to appear Bolletino U.M. Italiano.

5. J. Little, and K. Furio, *On the Distribution of Weierstrass Points on Irreducible Rational Nodal Curves*, preprint.

6. B. Olsen, *On Higher Order Weierstrass Points*, Ann. of Math. **95** (1972), 351–364.

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