Canad. Math. Bull. Vol. **33** (2), 1990

THE HEXAGONAL PACKING LEMMA AND DISCRETE POTENTIAL THEORY

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ABSTRACT. One of the questions concerning the Hexagonal Packing Lemma ([1], [3], [4]) is the rate of convergence of S_n. It was suggested in [3] and [4] that $S_n = O(1/n)$. In the following we prove this conjecture under the additional condition of some "nice" behaviour of the "circle function".

1. **Introduction.** In [5] Burt Rodin and Dennis Sullivan proved Thurston's conjecture that his scheme converges to the Riemann mapping. The Hexagonal Packing Lemma ([3], [4]) is a key result in this proof. A weaker result, which is closely related was earlier proved in [1]. In [5], Rodin and Sullivan, suggested, following ideas of Thurston, to investigate the rate of convergence of S_n. It was suggested ([3], [4], [5]) that $S_n = O(1/n)$. Rodin used discrete potential theory, as developed by him for the Hexagonal case, to find some interesting results concerning this question, and related problems. In any case, it seems that for the time being nothing is known about the rate of convergence of S_n (except, of course, the deep result, $Sn \longrightarrow 0$, which is equivalent to the Hexagonal Packing Lemma [5]).

In what follows we use freely the notations of Rodin in [3] and [4]. Our aim in the following note, is to show that under certain restrictions, the conjecture $S_n = O(1/n)$ is indeed, true. The main idea is the observation that if both u and $1/u$ are subharmonic for a positive u, then u behaves "similarly" to an harmonic function in some sense that will be clear later.

2. **Preliminary results.** We first quote some results that will be needed for our purposes.

THEOREM A. *Let HCP'_1 packing consists of an inner circle of radius r, surrounded by six tangent circles of radii $\{r_1, \ldots, r_6\}$, Then*

(2.1) $$r \leqq \frac{1}{6}\sum_{j=1}^{6} rj$$

(2.2) $$\frac{1}{r} \leqq \frac{1}{6}\sum_{j=1}^{6} \frac{1}{rj}$$

Received by the editors January 11, 1989 and, in revised form, April 20, 1989.
AMS (1980) Subject Classification Number: 30C35, 31C20.

Both statements appeared in [1]. The proof of (2.2) appeared in [1] and the proof of (2.1) appeared in [3].

THEOREM B. (Rodin, [3], Prop. 3.4).

Let u, v be functions defined on H_N, the first N generations of the Hexagonal lattice of mesh h. Then

$$\sum_{\alpha\in H_{N-1}} v(\alpha)D_h u(\alpha) - u(\alpha)D_h v(\alpha) = \frac{2}{3h^2}\sum_{(\alpha,\beta)} v(\alpha)u(\beta) - u(\alpha)v(\beta) \tag{2.3}$$

where the second sum is over all pairs (α, β) such that $\alpha \in \partial H_{N-1}$, $\beta \in \partial H_N$ and α is a neighbor of β.

3. **Statement of the result.** Following Rodin, we denote by HCP_N the first N generations of the regular hexagonal circle packing. Also, HCP'_N will be a circle packing that is combinatorially isomorphic to HCP_N. We now claim

THEOREM 3.1. *There exists an absolute constant $\rho > 0$, independent of n, s.t. if the ratio $R_n/r_n < 1+\rho$, then, indeed, $S_n = O(1/n)$*

In the above $R_n(r_n)$ stands for the maximum (minimum) of the radii of the circle packing HCP'_N

We first prove the following

LEMMA 3.2. *Let $u > 0$ and $D_1 u \geqq 0$, $D_1(1/u) \geqq 0$.*
Then

$$|D_1(\log u)| \leqq 2/3 \sum_{k=0}^{5} \frac{(u(\alpha+\omega^k) - u(\alpha))^2}{u(\alpha+\omega^k)u(\alpha)} \tag{3.1}$$

PROOF OF LEMMA 3.2. Denote $v = \log u$. We have:

$$\begin{aligned} D_1 v &= 2/3 \sum_{k=0}^{5} \log\left[\frac{u(\alpha+\omega^k)}{u(\alpha)}\right] \\ &= 2/3 \sum_{k=0}^{5} \log\left[1 + \frac{(u(\alpha+\omega^k) - u(\alpha))}{u(\alpha)}\right]. \end{aligned}$$

But $1/6\ \log[\pi_{j=0}^{5}(1+\beta_j)] = \log[\pi_{j=0}^{5}(1+\beta_j)]^{1/6} \leqq \log[1+\sum_{j=0}^{5}\beta_{j/6}] \leqq \sum_{j=0}^{5}\beta_{j/6}$, provided $1+\beta_j \geqq 0$, $\sum_{j=0}^{5}\beta_j \geqq 0$. Using this for $\beta_j = [u(\alpha+\omega^j) - u(\alpha)]/u(\alpha)$ (Noting that $\sum_{j=0}^{5}\beta_j \geqq 0$ follows from $D_1 u \geqq 0$) we get:

$$D_1(v) \leqq (2/3)\sum_{k=0}^{5} \frac{u(\alpha+\omega^k) - u(\alpha)}{u(\alpha)} = D_1(u)/u. \tag{3.2}$$

We also have:

$$D_1(u)/u + D_1(u^{-1})/u^{-1} = (2/3)\sum_{k=0}^{5} \frac{u(\alpha+\omega^k) - u(\alpha))^2}{u(\alpha)u(\alpha+\omega^k)}$$

as follows by a simple calculation. Hence, using $D_1(u^{-1})/u^{-1} \geqq 0$,

$$0 \leqq D_1(u)/u \leqq (2/3)\sum_{k=0}^{5} \frac{u(\alpha+\omega^k) - u(\alpha))^2}{u(\alpha)u(\alpha+\omega^k)} \tag{3.3}$$

Applying (3.3) to (3.2) we get

$$D_1(v) \leqq (2/3)\sum_{k=0}^{5} \frac{u(\alpha+\omega^k) - u(\alpha))^2}{u(\alpha)u(\alpha+\omega^k)}$$

But

$$D_1 \ \log(1/u) = -D_1 \ \log u \leqq (2/3)\sum_{k=0}^{5}\left(\frac{1}{u(\alpha+\omega^k)} - \frac{1}{u(\alpha)}\right)^2 \Big/ \left(\frac{1}{u(\alpha)}\cdot\frac{1}{u(\alpha+\omega^k)}\right)$$

(changing the roles of u and $1/u$), and thus the lemma is proved.

REMARK. It is worthwhile to note that a similar statement to the above lemma is easily established also in the continuous case. Indeed, if $T = \log\ u$, $\Delta u \geqq 0$, $\Delta(1/u) \geqq 0$ for a positive u, it follows by an easy calculation that $|\Delta T| \leqq T_x^2 + T_y^2$, where $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the Laplacian operator. (In fact, the same property holds in R^n, as follows at once).

PROOF OF THEOREM 3.1. The proof will be by induction on the number of generations. Assume that the theorem has been proved up to $N-1$, i.e. for some $A > 0$

$$|S_k| < \frac{A}{k}, \quad 1 \leqq k \leqq N-1 \tag{3.4}$$

Let $v = \log u$, where u is the "circle function". Let $1 \leqq N_1 < N-1$ be fixed later. By Lemma 3.2 and the induction assumption we have

$$|D_1(v)| \leqq \frac{4A^2}{(N-N_1-1)^2} \tag{3.5}$$

On the diagonal lattice $HL(1,\ N_1)$ (where we used, as above, Rodin's notation, [3] p. 274).

We now consider a discrete harmonic function h which is defined on $HL(1,\ N_1)$ and agrees with v on the boundary $\partial HL(1,\ N_1)$.

Apply *Th. B* for the two functions $\phi = v - h$ and the discrete Green's function's $g_{N_1}(\alpha, 0)$ for the domain $HL(1, N_1)$. We get

$$(3.6) \qquad \sum_{\alpha \in H_{N_1 - 1}} [D_1 v(\alpha) g_{N_1}(\alpha, 0) - D_1 g_{N_1}(\alpha, 0)\phi(\alpha)] = 0$$

Here we have used the fact that both ϕ and g_{N_1} vanish on the boundary. Also $D_1 v = D_1(v - h) = D_1 \phi$, since h is harmonic by our construction. Similarly,

$$(3.7) \qquad \sum_{\alpha \in H_{N_1 - 1}} [D_1 v(\alpha) g_{N_1}(\alpha, 1) - D_1 g_{N_1}(\alpha, 1)\phi(\alpha)] = 0$$

It is not difficult to deduce from Rodin's results (cf. [3] Th. 3.2 and Prop. 3.3) that there exists an absolute constant B such that

$$(3.8) \qquad |g_{N_1}(\alpha, 1) - g_{N_1}(\alpha, 0)| < B/(|\alpha| + 1)$$

If we subtract (3.6) from (3.7) we get, using (3.5) and (3.8), that:

$$(3.9) \qquad |(v - h)(0) - (v - h)(1)| \frac{2\sqrt{3}}{3} < \frac{4A^2 B}{(N - N_1 - 1)^2} \cdot (6N_1)$$

(Recall that $|D_1 g_{N_1}(\alpha, 0)| = 2\sqrt{3}/3$ for $\alpha = 0$ and equals zero other wise. [3], p. 275).

We now claim that there exists an absolute constant C such that

$$(3.10) \qquad |h(0) - h(1)| < \frac{C \log(M_N/m_N)}{N_1}$$

(M_N, m_N where defined in the statement of the theorem). Indeed, consider the function $h_1 = h - \log m_N$. Obviously, $|h_1(0) - h_1(1)| = |h(0) - h(1)|$. Also $0 \leqq h_1 \leqq \log M_N/m_N$ which follows at once from the way that h was defined. It is well known that

$$(3.11) \qquad \left| \frac{h_1(0) - h_1(1)}{h_1(0)} \right| < \frac{D}{N_1}$$

where D is an absolute constant (cf. [2] Th. 5 for a similar statement). In [3] (Th. 3.5) Rodin proved a "mean value property" which we apply for h_1. Thus

$$(3.12) \qquad |h_1(0)| < E \log(M_N/m_N)$$

for some absolute constant E. Hence (3.10) follows at once from (3.11) and (3.12).

In order to finish the induction proof we have to find an estimate for $|v(0) - v(1)|$ using (3.9) and (3.10). Indeed.

$$\left| \log \left(\frac{u(1)}{u(0)} \right) \right| = \left| v(0) - v(1) \right| = \left| \log \left[1 + \left(\frac{u(1) - u(0)}{u(0)} \right) \right] \right| > F \left| \frac{u(1) - (u(0)}{u(0)} \right|$$

https://doi.org/10.4153/CMB-1990-039-5 Published online by Cambridge University Press

for some absolute constant $F > 0$ (assuming an obvious limitation on the growth of $|(u(1) - u(0))/u(0)|$). Hence we get from (3.9) and (3.10):

$$\left|\frac{u(1) - u(0)}{u(0)}\right| < \frac{C_1 \log (M_N/m_N)}{N_1} + \frac{B_1 A^2 N_1}{(N - N_1 - 1)^2} \tag{3.13}$$

where B_1 and C_1 are absolute constants depending on the previous constants. It now remains to show that if N_1 is chosen properly, and if $\log (M_N/m_N)$ is small enough then

$$\frac{C_1 \log (N_N/m_N)}{N_1} + \frac{B_1 A^2 N_1}{(N - N_1 - 1)^2} \leqq \frac{A}{N}\,. \tag{3.14}$$

But $M_N/m_N < 1 + \rho$ implies $\log (M_N/m_N) < \rho$ and thus (3.14) will be established if

$$\frac{C_1 \rho}{N_1} + \frac{B_1 A^2 N_1}{(N - N_1 - 1)^2} \leqq \frac{A}{N} \tag{3.15}$$

for a small enough ρ. It will be convenient to write $N_1 = \lambda N/A$ and to choose the optimal λ instead of the optimal N_1. We thus have from (3.15).

$$\frac{C_1 \rho A}{N\lambda} + \frac{B_1 A^2 \lambda N}{A(N - \lambda N/A - 1)^2} \leqq \frac{A}{N}\,.$$

Hence

$$\frac{C_1 \rho}{\lambda} + \frac{B_1 \lambda}{(1 - \lambda/A - 1/N)^2} \leqq 1.$$

If we make the obvious restriction $\lambda/A + N^{-1} \leqq 1/2$ we get $C_1\rho\lambda^{-1} + 4B_1\lambda \leqq 1$. We now take $\lambda^2 = C_1\rho/4B_1$ to get the desired restriction $\rho = (16B_1C_1)^{-1}$. Since B_1 and C_1 are absolute constants independent of N our proof is complete.

REMARK. Obviously N_1 has to satisfy $N_1 \geqq 1$. Also N_1 has to be an integer. While the limitation "N_1 is an integer" is a minor technical detail, the limitation $N_1 \geqq 1$ needs some care. We have:

$$\lambda^2 = \frac{C_1 \rho}{4B_1} = \frac{C_1}{4B_1}(16B_1C_1)^{-1} = \frac{1}{64B_1^2}\,.$$

Hence $\lambda = (8B_1)^{-1}$ and $N_1 = N/8B_1A$. It follows that $N_1 \geqq 1$ is equivalent to $A/N \leqq 1/8B_1$. But since $S_N \longrightarrow 0$ is known and B_1 is an absolute constant the condition $S_N \leqq (8B_1)^{-1}$ is obviously satisfied for N large enough. (In fact we don't need the full strength of the result $S_N \longrightarrow 0$, but only the fact that S_n is "small enough", i.e. $S_N \leqq (8B_1)^{-1}$ for N sufficiently large).

ACKNOWLEDGEMENT. The research was supported by L. Edelstein Research Fund.

ADDED IN PROOF. Recently He, Zheng-Xu proved similar results using an entirely different method.

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