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AN UPPER BOUND FOR THE NUMBER OF ODD MULTIPERFECT NUMBERS

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Abstract

A natural number n is called k-perfect if $\sigma(n) = kn$. In this paper, we show that for any integers $r \geq 2$ and $k \geq 2$, the number of odd k-perfect numbers n with $\omega(n) \leq r$ is bounded by $\binom{\lfloor 4^r \log_3 2 \rfloor + r}{r} \sum_{i=1}^{r} \binom{\lfloor kr/2 \rfloor}{i}$, which is less than 4^{r^2} when r is large enough.

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1. Introduction

Let $k \geq 2$ be a positive integer. A natural number N is said to be k-perfect (or multiperfect of abundancy k) if $\sigma(N) = kN$, where $\sigma(N)$ denotes the sum of all the divisors of N. We say N is perfect when $k = 2$. The even perfect numbers were completely classified by Euler. Namely, N is an even perfect number if and only if $N = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime. However we know less about odd perfect numbers. We do not have a single example, and we do not have a proof that they do not exist.

Let $\omega(N)$ denote the number of distinct prime factors of a natural number N. In 1913, Dickson [4] proved that there are only finitely many odd perfect numbers with k distinct prime factors. In 1977, Pomerance [8] gave an explicit upper bound in terms of k. Heath-Brown [5] improved the bound to $N < 4^{4^k}$, and Cook [2] reduced this bound to $N < D^{4^k}$ with $D = (195)^{1/7}$. Nielsen [6] slightly improved and generalised Cook's method; he proved that if N is an odd multiperfect number with k distinct prime factors, then

$$N < 2^{4^k}. \tag{1.1}$$

In addition to an upper bound on the *size* of such N, Pollack [7] proved that for each positive integer k the number of odd perfect numbers N with $\omega(N) \leq k$ is bounded

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by 4^{k^2}. This result was generalised by Chen and Luo [1], who showed that, for any integer $r \geq 1$, the number of odd k-perfect numbers n with $\omega(n) \leq r$ is bounded by $(k-1) \cdot 4^{r^3}$. More recently, Dai *et al.* [3] improved the bound of Chen and Luo to $4^{r^2}(k-1)^{2r^2+3}$. The purpose of this paper is to improve the above result. We prove the following estimate.

Theorem 1.1. *For any integers $r \geq 2$ and $k \geq 2$, the number of odd k-perfect numbers n with $\omega(n) \leq r$ is bounded by $\binom{4^r\lfloor \log_3 2\rfloor + r}{r} \sum_{i=1}^{r} \binom{\lfloor kr/2\rfloor}{i}$, which is less than 4^{r^2} when r is large enough .*

2. The proof

The proof is essentially in the spirit of Pollack's work [7], and is a modification of Wirsing's method [9], but with a different counting argument. Let x be a positive real number. Suppose that $N < x$ is an odd k-perfect number and $\omega(N) \leq r$. Write $N = AB$, where $A := \prod_{p^e \| N, p > kr} p^e$ and $B := \prod_{p^e \| N, p \leq kr} p^e$. We have

$$\frac{\sigma(A)}{A} = \prod_{p^e \| A}\Big(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\Big) < \prod_{p \mid A}\Big(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\Big),$$

and so

$$\frac{A}{\sigma(A)} > \prod_{p \mid A}\Big(1 - \frac{1}{p}\Big) \geq 1 - \sum_{p \mid A} \frac{1}{p} \geq 1 - \frac{r}{kr+1} > \frac{k-1}{k}, \tag{2.1}$$

which implies that $B > 1$. Since N is k-perfect, $\sigma(AB) = kAB$, and hence

$$(k-1)B = \frac{k-1}{k}kB < \frac{A}{\sigma(A)}kB = \sigma(B) \leq kB, \tag{2.2}$$

with equality on the right precisely when $A = 1$. Suppose $A \neq 1$. By the previous inequality,

$$\sigma(B) > (k-1)B \quad \text{and} \quad \sigma(B) \mid kAB. \tag{2.3}$$

If $\gcd(A, \sigma(B)) = 1$, then by the second formula of (2.3), $\sigma(B) \mid kB$, and so $\sigma(B) \leq kB/2 \leq (k-1)B$, which contradicts (2.3). Therefore, there is a prime p dividing $\gcd(A, \sigma(B))$, which means that $\sigma(B)$ has a prime factor p with $p > kr$ and $\gcd(p, B) = 1$ by the definition of A. Let p_1 be the least such prime factor of $\sigma(B)$. Suppose $p_1^{e_1} \| A$, where $e_1 \geq 1$. Then, if we put

$$A' := A/p_1^{e_1} \quad \text{and} \quad B' := Bp_1^{e_1},$$

it is clear that (2.1)–(2.3) hold with A' and B' replacing A and B. By the same argument as in [7], continuing the above procedure, we eventually obtain a factorisation

$$A = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t},$$

where $t = \omega(A) = \omega(N) - \omega(B) \leq r - 1$.

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We note that the prime p_1 depends only on B, while for $i > 1$, the prime p_i depends only on B and the exponents $e_1, \ldots, e_{i-1}$. It follows that for a given B, the cofactor A (if $A > 1$) is entirely determined by $e_1, \ldots, e_t$, and we have $e_i \le \log_5 x$, $i = 1, \ldots, t$.

Let $B = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$. Then $f_j \le \log_3 x$, $j = 1, \ldots, s$, $s + t = r$. Let m be the number of odd primes not exceeding kr, so $m < kr/2$. To estimate the number of possibilities for B and $e_1, \ldots, e_t$, we first choose s, $1 \le s \le r$, odd primes from the first m odd primes, then choose positive integers $f_j \le \log_3 x$, $j = 1, \ldots, s$, and nonnegative integers $e_i \le \log_5 x$, $i = 1, \ldots, t$, with $s + t = r$ and obviously $e_1 + \cdots + e_t + f_1 + \cdots + f_s \le \log_3 x$. The number of possibilities for $e_1 + \cdots + e_t + f_1 + \cdots + f_s \le \log_3 x$ is not larger than the number of nonnegative integer solutions of the equation

$$e_1 + \cdots + e_t + f_1 + \cdots + f_s + y = \lfloor \log_3 x \rfloor,$$

which is $\binom{\lfloor \log_3 x \rfloor + r}{r}$. It follows that the number of possibilities for B and $e_1, \ldots, e_t$ is bounded by

$$\binom{\lfloor \log_3 x \rfloor + r}{r} \sum_{i=1}^{r} \binom{m}{i} \le \binom{\lfloor \log_3 x \rfloor + r}{r} \sum_{i=1}^{r} \binom{\lfloor kr/2 \rfloor}{i}.$$

Recall Mertens' formula: for $x \ge 2$

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma} \log x + O(1),$$

where $\gamma = 0.577\ldots$ is Euler's constant. Recall also the prime number theorem: if p_n denotes the nth prime number, then $p_n \sim n \log n$. We have

$$\begin{aligned} k = \frac{\sigma(N)}{N} &< \prod_{p \mid N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \mid N} \left(1 - \frac{1}{p}\right)^{-1} \\ &\le \frac{1}{2} \prod_{p \le p_r} \left(1 - \frac{1}{p}\right)^{-1} \sim \frac{e^{\gamma}}{2} \log r. \end{aligned} \tag{2.4}$$

By (1.1), we take $x = 2^{4^r}$ so that the number of odd k-perfect numbers n with $\omega(n) \le r$ is bounded by

$$\binom{\lfloor 4^r \log_3 2 \rfloor + r}{r} \sum_{i=1}^{r} \binom{\lfloor kr/2 \rfloor}{i} \le \frac{2^{kr/2}}{r!} \lfloor 4^r \log_3 2 + r \rfloor^r \le \frac{2^{kr/2}}{r!} 4^{r^2}.$$

By (2.4) and the fact that the Taylor series for $\exp(2^{k/2}) = \sum_{i=0}^{\infty} 2^{ki/2}/i!$ converges, $2^{kr/2}/r!$ must go to 0 as $r \to \infty$. This proves the theorem.

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