ELEMENTARY FACTORIZATION IN π-REGULAR RINGS

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1. Introduction. This paper extends the results of A. L. Foster **(1)** on elementary factorization in Boolean-like rings to commutative π-regular rings. After proving some preliminary lemmas we proceed to the partition of the set of non-units of a π-regular ring into irreducible and composite elements. Finally, we prove a number of theorems concerning factorization rings, weakly unique factorization rings, principal ideal rings, etc. The principal result is that a π-regular ring is a weakly unique factorization ring if and only if it is a principal ideal ring.

In this paper, ring will always mean a commutative ring with identity. Following van der Waerden **(5)**, we say that two elements b, c of a ring R are associates provided there exists a unit $u \in R$ such that $b = uc$. In this event, we write $b \sim c$.

2. Preliminary lemmas. In each of the lemmas of this section, R denotes a ring.

LEMMA 2.1. *Let $b, e \in R$ with $e^2 = e$. If $bR = eR$, then $b \sim e$.*

Proof. Since e is idempotent, $b = be$ and $b = e(1 - e + b)$. Let $e = bx$. It is easily verified that $(1 - e + b)(1 - e + ex) = 1$. Hence, $b \sim e$.

LEMMA 2.2. *Let $b \in R$ and let n be a natural number. Then the following are equivalent:*

I. *$b^n = b^n x b^n$ for some $x \in R$.*

II. *$b \sim e + \beta$ for some $e, \beta \in R$ with $e^2 = e$ and $\beta^n = 0$.*

III. *$b^n \sim f$ for some idempotent $f \in R$.*

Proof. I $\Rightarrow$ II. Let $e = xb^n$. Then $e = e^2 = bexb^{n-1}$ and $e \in beR$. Thus, $eR = beR$ and, by Lemma 2.1, $e \sim be$. Now let $be = ue$ where u is a unit in R, and let $\gamma = b(1 - e)$. Then $\gamma^n = 0$ and $b = be + \gamma$. Hence,

$$b = ue + \gamma = u(e + u^{-1}\gamma).$$

Therefore, $b \sim e + \beta$ where $\beta = u^{-1}\gamma$ and $\beta^n = 0$.

II $\Rightarrow$ III. $(e + \beta)^n = e(1 + \delta)$ where $\delta = n\beta + \ldots + n\beta^{n-1}$. Hence, δ is nilpotent and $1 + \delta$ is a unit. Therefore, $b^n \sim (e + \beta)^n \sim e$.

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III $\Rightarrow$ I. Let $f = vb^n$ with v a unit in R. Therefore, $(vb^n)^2 = vb^n$ and $b^n = b^n v b^n$.

LEMMA 2.3. *Let $b, e \in R$ with $e^2 = e$. If $b^n \sim e$, then, for every $m \geqslant n$, $b^m \sim e$.*

Proof. Let k be a natural number such that $kn \geqslant m$. Hence,

$$eR = b^{kn}R \subseteq b^m R \subseteq b^n R = eR.$$

Therefore, $b^m R = eR$ and $b^m \sim e$.

It follows, from the proof of Lemma 2.2 and the fact that associated idempotents are equal, that if R is a ring and $b \in R$, then there is at most one idempotent $e \in R$ such that, for some nilpotent $\beta \in R$, $b \sim e + \beta$. Further, by Lemma 2.3, e is the unique idempotent associated with all sufficiently high powers of b.

3. Irreducible and composite elements. N. H. McCoy **(3)** defined a ring S to be π-regular provided that for every $b \in S$ there is an $x \in S$ and a natural number n such that $b^n = b^n x b^n$. Lemma 2.2 provides two alternative characterizations of π-regular rings. Included among π-regular rings are regular rings with Boolean rings and p-rings as special cases; rings with descending chain condition on ideals (in particular, finite rings); and Boolean-like rings. The following facts are well known and easily verified. (1) The homomorphic image (in particular, a direct summand) of a π-regular ring is π-regular. (2) A finite direct sum of π-regular rings is π-regular. This need not be the case for infinite direct sums. (3) A ring is both an integral domain and π-regular if and only if it is a field. (4) In a π-regular ring any prime ideal (other than the entire ring) is a maximal ideal.

For the remainder of this paper, S will always denote a π-regular ring. In addition, the following conventions will be observed. Letters followed by an asterisk, $b^*, c^*, d^*, \ldots$, will denote idempotents; Greek letters, $\beta, \gamma, \delta, \ldots$, will denote nilpotents; and, lastly, if $b \in S$, the unique idempotent associated with all sufficiently high powers of b will be denoted by b^*.

LEMMA 3.1. *Let $b, c \in S$. Then $(bc)^* = b^*c^*$.*

Proof. By Lemma 2.3, for some n, $b^n \sim b^*$ and $c^n \sim c^*$. Hence $(bc)^n \sim b^*c^*$.

LEMMA 3.2. *Let $u \in S$. Then u is a unit if and only if $u^* = 1$.*

Proof. If $u^* = 1$, then $u^n \sim 1$ for some n and u is a unit. Conversely, if u is a unit, then $u \sim 1$, $u^* \sim 1$, and $u^* = 1$.

LEMMA 3.3. *Let $b, c \in S$. If $b \sim c$, then $b^* = c^*$.*

Proof. Let $b = uc$ with u a unit. Then $b^* = u^*c^* = c^*$.

LEMMA 3.4. *Let $b^*, \beta \in S$. Then $b^* + \beta \sim b^*$ if and only if $b^*\beta = \beta$.*

Proof. If $b^*\beta = \beta$, then $b^* + \beta = b^* + b^*\beta = b^*(1 + \beta)$. Conversely, if $b^* + \beta \sim b^*$, let $b^* + \beta = ub^*$. Multiplying by $1 - b^*$, we obtain

$$(1 - b^*)\beta = 0.$$

Thus, $b^*\beta = \beta$.

DEFINITION 3.1. *Let R be a ring, $b \in R$, and b not be a unit. Then, b is irreducible provided its only divisors are associates and units; b is composite provided there are elements $s, t \in R$ such that $s \nsim b$, $t \nsim b$, and $b = st$.*

It is clear that a non-unit cannot be both irreducible and composite. It may, however, be neither. For example, in $Z \dotplus Z$, Z = integers, the non-unit $(1, 0)$ is neither irreducible nor composite. The definitions above are equivalent to the classical ones for integral domains where the cancellation law provides a trivial proof that a non-zero non-unit is either irreducible or composite. The next theorem will characterize the irreducible and composite elements of a π-regular ring. It is then an immediate consequence that in a π-regular ring every non-unit (including zero) is either irreducible or composite. In the partition of the non-units of a π-regular ring into irreducible and composite elements we may, by Lemmas 2.2 and 3.2, restrict our attention to elements of the form $b^* + \beta$ with $b^* \neq 1$. Recall, finally, that if R is a ring and J is the set of idempotents in R, then $\langle J, \cap, ' \rangle$ is a Boolean algebra where $a \cap b = ab$ and $a' = 1 - a$ **(2)**. An element of J is said to be a co-atom (= prime) if its only divisors in J are itself and 1.

THEOREM 3.1. *Let $b^* + \beta \in S$ with $b^* \neq 1$. Then* (1) *$b^* + \beta$ is irreducible if b^* is a co-atom, and either $b^*\eta = \eta$ for every nilpotent $\eta \in S$ or $(1 - b^*)\beta$ cannot be expressed as the product of two nilpotents; and* (2) *$b^* + \beta$ is composite if either b^* is not a co-atom, or $b^*\eta \neq \eta$ for some nilpotent $\eta \in S$ and $(1 - b^*)\beta$ can be expressed as the product of two nilpotents.*

Proof. (1) Suppose $b^*\eta = \eta$ for every nilpotent $\eta \in S$. Let $c^* + \gamma$ be a non-unit divisor of $b^* + \beta$. Since b^* is a co-atom, by Lemmas 3.1 and 3.2, $c^* = b^*$. By Lemma 3.4, $c^* + \gamma \sim b^* \sim b^* + \beta$. On the other hand, assume that $(1 - b^*)\beta$ cannot be expressed as the product of two nilpotents, and let

$$b^* + \beta = (c^* + \gamma)(d^* + \delta)$$

where $c^* + \gamma$ and $d^* + \delta$ are non-units. Multiplying by $1 - b^*$, we see that

$$(1 - b^*)\beta = [(1 - b^*)\gamma][(1 - b^*)\delta],$$

a contradiction. Hence, $b^* + \beta$ is irreducible.

(2) Suppose b^* is not a co-atom. Then there is c^* such that $c^* \neq 1$, $c^* \neq b^*$, and $b^* = b^*c^*$. One verifies that

$$b^* + \beta = [c^* + (1 - c^* + b^*)\beta][(1 - c^* + b^*) + (c^* - b^*)\beta]$$

and that $1 - c^* + b^*$ is idempotent. Further, $1 - c^* + b^* \neq b^*$ since $c^* \neq 1$. By Lemma 3.3, $b^* + \beta$ is reducible.

Finally, assume that $b^*\eta \neq \eta$ for some $\eta \in S$ and that $(1 - b^*)\beta$ can be expressed as the product of two nilpotents. There are two cases: $b^*\beta = \beta$ and $b^*\beta \neq \beta$. If $b^*\beta = \beta$, let $\gamma = (1 - b^*)\eta$. Then $\gamma \neq 0$ and $b^*\gamma = 0$. Let $n =$ the least natural number such that $\gamma^n = 0$. Since $\gamma \neq 0$, $n > 1$. Clearly,

$$b^* = (b^* + \gamma)(b^* + \gamma^{n-1}).$$

Assume that $b^* \sim b^* + \gamma$ and let $ub^* = b^* + \gamma$. Multiplying by $1 - b^*$, we see that $\gamma = 0$, a contradiction. Similarly, $b^* \nsim b^* + \gamma^{n-1}$. Thus, b^* is reducible. The assumption that $b^*\beta = \beta$ implies that $b^* + \beta \sim b^*$. Thus, $b^* + \beta$ is reducible.

If $b^*\beta \neq \beta$, let $(1 - b^*)\beta = \rho\sigma$. One verifies that

$$b^* + \beta = [b^* + (1 - b^*)\rho][b^* + b^*\beta + (1 - b^*)\sigma].$$

Assume that $b^* + \beta \sim b^* + (1 - b^*)\rho$ and let $b^* + (1 - b)^*\rho = u(b^* + \beta)$. Multiplying by $1 - b^*$, we see that

$$(1 - b^*)\rho = u(1 - b^*)\beta = u(1 - b^*)\rho\sigma.$$

Hence,

$$[(1 - b^*)\rho][1 - u\sigma] = 0.$$

Since $1 - u\sigma$ is a unit, $(1 - b^*)\rho = 0$. Therefore $(1 - b^*)\beta = 0$ and $b^*\beta = \beta$, a contradiction. Similarly,

$$b^* + \beta \nsim b^* + b^*\beta + (1 - b^*)\sigma.$$

Thus, $b^* + \beta$ is reducible.

4. Factorization in π-regular rings.

LEMMA 4.1. *Let $b, c \in S$ with $bS = cS$. Then $b \sim c$.*

Proof. Let $b = rc$ and $c = sb$. Then, for every natural number n, $b = r^n s^n b$ and $c = r^n s^{n+1} b$. Therefore, $b \sim r^*s^*b$, $c \sim r^*s^*b$, and $b \sim c$.

Recall that a non-unit p of a ring R is said to be a prime provided pR is a prime ideal in R. For a π-regular ring, it follows from Lemma 4.1 that a composite cannot be a prime; and since in a π-regular ring a non-unit is either irreducible or composite it further follows that a prime is necessarily irreducible. The important special case in which, conversely, each irreducible element is necessarily a prime will be discussed later in this section.

LEMMA 4.2. *Let p be a prime in S and let $p^m \mid p^n f$ where $m > n$ and $p \nmid f$. Then $p^m \sim p^n$.*

Proof. Since each prime ideal in S is maximal, for some $x, y \in S$, $px + fy = 1$. Therefore, $p^{n+1}x + p^n f = p^n$. Since $n + 1 \leqslant m$, $p^{n+1} \mid p^n f$. Hence $p^{n+1} \mid p^n$ and $p^{n+1} \sim p^n$. The result follows by induction.

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DEFINITION 4.1. *A ring R is said to be a factorization ring if each non-unit of R can be expressed as the product of irreducible elements.*

THEOREM 4.1. *If S satisfies the ascending chain condition, then S is a factorization ring.*

We omit the proof, which can be patterned after the proof of the corresponding theorem for integral domains. The only properties of a π-regular ring that are needed are: (1) a non-unit is either irreducible or composite and (2) two elements generating the same principal ideal are associates.

THEOREM 4.2. *If S is a factorization ring, then S has only a finite number of idempotents.*

Proof. Let $0 = pq \ldots r$ where $p, q, \ldots, r$ are irreducible. Then

$$0 = p^*q^* \ldots r^*$$

where $p^*, q^*, \ldots, r^*$ are co-atoms. The conclusion follows from a well-known theorem of Boolean algebra.

The following example shows that the converse of Theorem 4.2 is false. Let $R = K[x_1, x_2, \ldots, x_n, \ldots]$ where K is a field. Let

$$A = (x_1^2, x_1 - x_2^2, \ldots, x_n - x_{n+1}^2, \ldots).$$

Then R/A is a π-regular ring with 0 and 1 as its only idempotents but R/A is not a factorization ring.

In all that follows, exponents are assumed to be non-negative integers.

DEFINITION 4.2. *Let R be a ring, $b \in R$, and suppose that*

$$b \sim p_1^{i_1}p_2^{i_2} \ldots p_n^{i_n}$$

with each p_i irreducible and $p_i \nsim p_j$ if $i \neq j$. Then $p_1^{i_1}p_2^{i_2} \ldots p_n^{i_n}$ is said to be an irredundant factorization of b if

$$j_1 \leqslant i_1, j_2 \leqslant i_2, \ldots, j_n \leqslant i_n, \text{ and } \sum_1^n j_k < \sum_1^n i_k$$

implies $b \nsim p_1^{j_1}p_2^{j_2} \ldots p_n^{j_n}$.

DEFINITION 4.3. *R is a weakly unique factorization ring provided that R is a factorization ring and that if*

$$p_1^{i_1}p_2^{i_2} \ldots p_n^{i_n} \quad \text{and} \quad p_1^{j_1}p_2^{j_2} \ldots p_n^{j_n}$$

are irredundant factorizations of some $b \in R$, then $i_1 = j_1, i_2 = j_2, \ldots, i_n = j_n$.

We note, as is customary, that an irreducible factor with zero exponent may be added or discarded as is convenient.

THEOREM 4.3. *Let S be a factorization ring. Then the following are equivalent:*

I. *Each irreducible element of S is a prime.*

II. *S has weakly unique factorization.*

III. *For every $b^* \in S$, if $b^* + \beta$ and $b^* + \gamma$ are irreducible, then*

$$b^* + \beta \sim b^* + \gamma.$$

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Proof. I $\Rightarrow$ II. Let $b \in S$ and let

$$p_1^{i_1}p_2^{i_2}\ldots p_n^{i_n} \quad \text{and} \quad p_1^{j_1}p_2^{j_2}\ldots p_n^{j_n}$$

be two irredundant factorizations of b. Suppose $i_1 > j_1$. By Lemma 4.2, $p_1^{i_1} \sim p_1^{j_1}$. Hence, $b \sim p_1^{j_1}p_2^{i_2}\ldots p_n^{i_n}$, a contradiction. Therefore, $i_1 \leqslant j_1$ and similarly, $j_1 \leqslant i_1$. In the same way $i_s = j_s$ for $s = 1, 2, \ldots, n$.

II $\Rightarrow$ III. Let $b^* + \beta$ and $b^* + \gamma$ be irreducible. For some natural number n, $b^* \sim (b^* + \beta)^n \sim (b^* + \gamma)^n$. By assumption II, $b^* + \beta \sim b^* + \gamma$.

III $\Rightarrow$ I. Let p be irreducible and let $p \mid bc$. Also, let $b = p_1 p_2 \ldots p_m$ and $c = q_1 q_2 \ldots q_n$ be factorizations of b and c into the product of irreducible elements. Then, $p^* \mid p_1^*p_2^* \ldots p_m^*q_1^*q_2^* \ldots q_n^*$ with p^*, p_i^*, q_j^* necessarily co-atoms. Therefore $p^* = p_s^*$ for some s or $p^* = q_t^*$ for some t. Say $p^* = p_1^*$. By hypothesis, $p \sim p_1$. Hence $p \mid b$ and p is a prime.

THEOREM 4.4. *If S is a weakly unique factorization ring, then each pair of elements in S has a greatest common divisor (necessarily unique to within associates by Lemma* 4.1).

Proof. Let $b, c \in S$. If either b or c is a unit, then it is clear that b and c have a greatest common divisor. Therefore, assume that b and c are non-units and let $p_1^{i_1}p_2^{i_2}\ldots p_n^{i_n}$ and $p_1^{j_1}p_2^{j_2}\ldots p_n^{j_n}$ be irredundant factorizations of b and c respectively. Let $m_s = \min\{i_s, j_s\}$ for $s = 1, 2, \ldots, n$. Clearly, if $d = p_1^{m_1}p_2^{m_2}\ldots p_n^{m_n}$, then d is a common divisor of b and c. Now let f be a common divisor of b and c and let q be an irreducible divisor of f. By Theorem 4.3, $q \sim p_t$ for some t, $1 \leqslant t \leqslant n$. Therefore, we may let $p_1^{k_1}p_2^{k_2}\ldots p_n^{k_n}$ be an irredundant factorization of f. By Lemma 4.2, $k_1 > i_1$ implies $p_1^{k_1} \sim p_1^{i_1}$ and thus, $f \sim p_1^{i_1}p_2^{k_2}\ldots p_n^{k_n}$, a contradiction. Hence, $k_1 \leqslant i_1$ and, similarly, $k_1 \leqslant j_1$. Thus, $k_1 \leqslant m_1$ and, in the same way, $k_s \leqslant m_s$ for $s = 1, 2, \ldots, n$. Thus $f \mid d$ and d is a greatest common divisor for b and c.

Unlike the situation with integral domains, the converse of Theorem 4.4 is false as the following example shows. Let $R = K[x, y]$ where K is a field and let $A = (x^2, xy, y^2)$. Then R/A is a π-regular factorization ring in which each pair of elements has a greatest common divisor but R/A does not have weakly unique factorization.

LEMMA 4.3. *Let S be a weakly unique factorization ring and let $S = S_1 \dot{+} S_2$. Then S_1 is a π-regular weakly unique factorization ring.*

Proof. That S_1 is π-regular is clear. Let $b \in S_1$ with b a non-unit. Since S is a factorization ring, let $p_1 p_2 \ldots p_n$ be a factorization of b into irreducible elements in S. If h is the natural homomorphism of S onto S_1, then

$$b = bh = (p_1 h)(p_2 h)\ldots(p_n h)$$

where $p_i h$ is either a unit in S_1 or an irreducible element in S_1 for

$$i = 1, 2, \ldots, n.$$

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Hence, S_1 is a factorization ring. Now let $q_1 \in S_1$ with q_1 irreducible in S_1 and let e_2 be the identity in S_2. Then $q_1 + e_2$ is irreducible in S and, by Theorem 4.3, $(q_1 + e_2)S$ is a prime ideal in S. Since $(q_1 + e_2)S \supseteq S_2$ (= kernel h) and $(q_1 + e_2)Sh = q_1 S_1$, it follows that q_1 is a prime in S_1. Hence, S_1 has weakly unique factorization.

LEMMA 4.4. *Let S be a weakly unique factorization ring. If J is the set of idempotents in $S = \{0, 1\}$, then S is a principal ideal ring.*

Proof. Since 0 is the only co-atom in J, by Theorem 4.3, there is only one irreducible element p (to within associates) in S. Hence, if $b \in S$, then $b \sim p^k$ for some k. Let A be an ideal in S and let n be the smallest non-negative integer such that $p^n \in A$. Clearly, $A = p^n S$.

THEOREM 4.5. *S is a weakly unique factorization ring if and only if S is a principal ideal ring.*

Proof. If S is a principal ideal ring, then S satisfies the ascending chain condition for ideals. Also, each irreducible element in S is a prime. By Theorems 4.1 and 4.3, S is a weakly unique factorization ring.

Conversely, let S have weakly unique factorization. By Theorem 4.2, there are only a finite number of idempotents in S, and we may let

$$S = S_1 \dotplus S_2 \dotplus \ldots \dotplus S_n$$

where S_i has only two idempotents for $i = 1, 2, \ldots, n$. By Lemmas 4.3 and 4.4, S_i is a principal ideal ring for $i = 1, 2, \ldots, n$. Hence, S is a principal ideal ring.

We conclude with a structure theorem that is an easy consequence of a structure theorem of G. Pollak **(4)**. The proof is omitted.

THEOREM 4.6. *R is a π-regular principal ideal ring if and only if R is the direct sum of a finite number of completely primary rings in which the unique prime ideal is principal.*

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