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APPLICATION OF SZEBEHELY'S INVERSE PROBLEM TO NON-STATIONARY DYNAMICAL SYSTEMS

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ABSTRACT

A first-order linear partial differential equation is presented, giving the non-stationary potential functions $U=U(x,y,t)$ which give rise to a given family of evoling planar orbits $f(x,y,t) = c$ in two-dimensional dynamical system. It is shown, that this equation is applied in celestial mechanics of variable mass.

INTRODUCTION

A new approach to the classical inverse problem of finding the potential from the orbits was made by V.Szebehely (1974).

Let

$$f(x,y) = c = \text{const.} \tag{1}$$

is a given monoparametric family of planar orbits in dynamical system determined by the equations of motion in rectangular coordinates x and y

$$\ddot{x} = U_x \;, \quad \ddot{y} = U_y \tag{2}$$

then the potential $U = U(x,y)$ may be determined from Szebehely's equation

$$f_x U_x + f_y U_y + 2\,\frac{f_x^2 f_{yy} - 2 f_{xy} f_x f_y + f_y^2 f_{xx}}{f_x^2 + f_y^2}\,(U+h) = 0 \qquad (3)$$

where dots denote derivatives with respect to the time, subscripts correspond to partial derivatives and h is the total energy per unit mass of the body.

In the present paper we shall consider the inverse problem for two-dimensional dissipative system.

THE DERIVATION OF THE EQUATION

Let us consider the two-dimensional dissipative system determined by the equations of motion

$$\ddot{x} = U_x + \alpha\dot{x}\ , \quad \ddot{y} = U_y + \alpha\dot{y}\ , \qquad (4)$$

where $U = U(x,y,t)$ and $\alpha = \alpha(t)$ is an arbitrary function of time. These equations admit the non-stationary analogy of the angular momentum integral

$$x\dot{y} - y\dot{x} = k, \quad k = \text{const.}\ \exp\left(\int_{t_o}^{t} \alpha(t)dt\right) \qquad (5)$$

For the family of orbits given by a twice differentiable function $f(x,y,t) = c = \text{const.}$, we have along each orbit

$$\dot{x}f_x + \dot{y}f_y + f_t = 0 \qquad (6)$$

From equations (5) and (6) the components of the velocity may be expressed as

$$\dot{x} = \frac{-kf_y - xf_t}{xf_x + yf_y}\ , \qquad (7)$$

and

$$\dot{y} = \frac{kf_x - yf_t}{xf_x + yf_y}$$

The time-derivative of equation (6) is

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$$\ddot{x}f_x + \ddot{y}f_y + \dot{x}^2 f_{yy} + 2\dot{x}\dot{y}f_{xy} + 2\dot{x}f_{xt} + 2\dot{y}f_{yt} + f_{tt} = 0 \qquad (8)$$

Substituting equations (4) and (7) into the equation (8) one obtains

$$f_x U_x + f_y U_y + k\,\frac{f_x^2 f_{yy} - 2f_{xy}f_x f_y + f_y^2 f_{xx}}{(xf_x + yf_y)^2}$$

$$+ \frac{2k\,f_t\,(f_y f_{xx} - f_x f_{xy})x + (f_y f_{xy} - f_x f_{yy})y}{(xf_x + yf_y)^2} \qquad (9)$$

$$+ 2k\,\frac{f_x f_{yt} - f_y f_{xt}}{xf_x + yf_y} + f_t^2\,\frac{x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}}{(xf_x + yf_y)^2}$$

$$- 2f_t\,\frac{xf_{xt} + yf_{yt}}{xf_x + yf_y} - \alpha f_t + f_{tt} = 0$$

Equation (9) in polar coordinates r and θ is

$$f_r U_r + \frac{f_\theta}{r^2} U_\theta + \frac{k^2}{r^5 f_r^2}\,(rf_{rr}f_\theta^2 + rf_r^2 f_{\theta\theta}$$

$$- 2rf_r f_{r\theta} f_\theta + r^2 f_r^3 + 2f_r f_\theta^2) + 2\,\frac{kf_t}{r^3 f_r^2}\,(f_r f_\theta$$

$$- rf_r f_{r\theta} + rf_{rr} f_\theta) - \frac{2f_{rt} f_t}{r} - \alpha f_t + f_{tt} = 0 \qquad (10)$$

The equation (9) (or (10)) is a first-order linear partial differential equation, giving the non-stationary potentials. The solution of this equation is not unique.

AN EXAMPLE

As an example of solution of the equation (10) one may consider the motion along evolving spiral orbits

$$f(r,\theta,t) = r\gamma(1+e\cos\theta) = P = \text{const.}, \qquad (11)$$

where $\gamma = \gamma(t)$ is a given function of time, e = const.

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Substituting (11) in (10) we obtain

$$(1+e\cos\theta)U_r - \frac{\gamma e}{r}\sin\theta U_\theta + \frac{k^2}{r^3} - WP = 0, \tag{12}$$

where

$$W = W(t) = \frac{\alpha\dot{\gamma}}{\gamma} + 2\frac{\dot{\gamma}^2}{\gamma^2} - \frac{\ddot{\gamma}}{\gamma} \tag{13}$$

Equation (12) can be solved directly by the method of characteristics.

Then

$$\frac{dr}{\gamma(1+e\cos\theta)} = -\frac{r\,d\theta}{\gamma e\sin\theta} = \frac{dU}{-\frac{k^2}{r^3}\gamma + W.P} \tag{14}$$

The general solution of equation (14) is

$$U = \frac{k^2\gamma}{P.r} + \frac{W.r^2}{2} + \phi\left(\frac{r^e(1-\cos\theta)}{(\sin\theta)^{1-e}}\right), \tag{15}$$

where ϕ is an arbitrary function of its argument.

THE EQUATIONS OF MODEL PROBLEMS IN CELESTIAL MECHANICS OF VARIABLE MASS

Let us put in (15) $\phi = 0$, then equation (15) determines the force of the form

$$\vec{F} = -\frac{k^2\gamma}{Pr^3}\vec{r} + W\vec{r} \tag{16}$$

From equation (16) under various values of the functions $k(t)$, $\alpha(t)$, $\gamma(t)$ we can obtain the following equations of motion:

1. When $k(t) = \sqrt{GM(t)}$, $\gamma(t) = P = \text{const.}$, (17)

then we have from (16) the equation of aperiodic motion along a conic section

$$\frac{d^2\vec{r}}{dt^2} = -GM(t)\frac{\vec{r}}{r^3} + \frac{1}{2M}\frac{dM}{dt}\frac{d\vec{r}}{dt}; \tag{18}$$

2. When $\alpha(t) = 0$, $k(t) = \sqrt{P} = const.$, $\gamma(t) = \mu(t)$, (19)

then in this case we obtain the equation of the model problem for a spiral motion

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu\vec{r}}{r^3} + \mu\vec{r}\,\frac{d^2}{dt^2}\left(\frac{1}{\mu}\right); \tag{20}$$

3. When

$$k(t) = \sqrt{\frac{\mu(t)}{\gamma(t)}}, \quad \gamma(t) = \frac{1}{\gamma(t)}, \tag{21}$$

then we receive the equation of motion of the model problem of the evolution of binary system inside gravitating resistant medium

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3} + \frac{1}{2}\left(\frac{\dot{\mu}}{\mu} + \frac{\dot{\gamma}}{\gamma}\right)\dot{\vec{r}} + \left[\frac{\ddot{\gamma}}{\gamma} - \frac{1}{2}\left(\frac{\dot{\mu}}{\mu} + \frac{\dot{\gamma}}{\gamma}\right)\frac{\dot{\gamma}}{\gamma}\right]\vec{r} \tag{22}$$

The equations (18), (20), (22) play important role in celestial mechanics of variable mass.

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