Glasgow Math. J. **51** (2009) 101–108. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004576. Printed in the United Kingdom

THE DUALITY PROBLEM FOR THE CLASS OF ORDER WEAKLY COMPACT OPERATORS

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(Received 5 March, 2008; accepted 13 September, 2008)

Abstract. We study the duality problem for order weakly compact operators by giving sufficient and necessary conditions under which the order weak compactness of an operator implies the order weak compactness of its adjoint and conversely.

2000 *AMS Classification.* 46A40; 46B40; 46B42.

1. Introduction and notation. The class of order weakly compact operators was introduced by Dodds [**6**] on Banach lattices. It contains the subspace of weakly compact operators as well as AM-compact operators. Next, this class of operators was studied by Duhoux [**7**] on locally convex solid lattices and later Ercan [**8**] gave some new results relatively to this class. Recently, Nowak [**12**] considered order weakly compact operators from a vector-valued function space into a Banach space and gave a characterization of an order weakly compact operator T in terms of the continuity of its adjoint relatively to some weak topologies.

Let us recall that an operator T from a Banach lattice E into a Banach space F is said to be order weakly compact if for each $x \in E^+$, the subset $T([0, x])$ is relatively weakly compact in F, where $E^+ = \{x \in E : 0 \leq x\}$.

Contrarily to weakly compact operators [**2**, **13**], the class of order weakly compact operators satisfies the domination problem. Indeed, if S and T are two operators from a Banach lattice E into another F such that $0 \leq S \leq T$ and T is order weakly compact, then S is order weakly compact [**3**].

Also, the class of order weakly compact operators does not satisfy the duality property, that is, there exist order weakly compact operators whose adjoints are not order weakly compact. In fact, the identity operator of the Banach lattice l^1 is order weakly compact, but its adjoint, which is the identity operator of the Banach lattice l^∞, is not order weakly compact. And conversely, there exist operators that are not order weakly compact but their adjoints are order weakly compact. In fact, the identity operator of the Banach lattice l^∞ is not order weakly compact but its adjoint, which is the identity operator of the topological dual $(l^\infty)'$, is order weakly compact.

In [**16**], Zaanen investigated the duality problem for semi-compact operators. Also, in [**5**] and [**16**], the duality problem of AM-compact operators on Banach lattices was studied. They gave sufficient and necessary conditions for which the AM-compactness

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of an operator implies the AM-compactness of its dual and conversely. These results are natural analogues of Gantmacher's theorem for weakly compact operators.

In the same direction, the aim of this paper is to resolve the duality problem for the class of order weakly compact operators. For this, we prove that if E and F are two Banach lattices and T is any order-bounded operator from E into F, then its adjoint T' from F' into E' is order weakly compact whenever T is order weakly compact if and only if the norm of E' or F' is order continuous. And conversely, whenever E and F are order σ-complete, we show that an order-bounded operator T from E into F is order weakly compact whenever its adjoint T' from F' into E' is order weakly compact if and only if the norm of E or F is order continuous.

To state our results, we need to fix some notation and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace F of a vector lattice E is said to be a sublattice if for every pair of elements a, b of F the supremum of a and b taken in E belongs to F. A subset B of a vector lattice E is said to be solid if it follows from $|y| \leq |x|$ with $x \in B$ and $y \in E$ that $y \in B$. An order ideal of E is a solid subspace. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E \ : \ x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A vector lattice is said to be order σ-complete if every non-empty countable subset that is bounded from above has a supremum. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space. We refer to Zaanen [**16**] for unexplained terminology on the Banach lattice theory.

2. Main Results. We use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$, where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous. For more information about positive operators, we refer the reader to the book of Aliprantis and Burkinshaw [**3**].

Recall that a norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. For example, the norm of the Banach lattice l^1 is order continuous but the norm of the Banach lattice l^∞ is not.

There exist operators that are not order weakly compact. In fact, the identity operator of the Banach lattice l^∞ is not order weakly compact. The following results give some sufficient conditions under which each operator is order weakly compact:

THEOREM 2.1. *Let E and F be two Banach lattices. Then we have the following assertions:*

(1) *If the norm of F is order continuous, then each order bounded operator T from E into F is order weakly compact.*
(2) *If the norm of E is order continuous, then each operator T from E into F is order weakly compact.*
(3) *If $E = F$, then the following conditions are equivalent:*

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(i) *Each operator* $T : E \longrightarrow E$ *is order weakly compact.*
(ii) *The identity operator of E is order weakly compact.*
(iii) *The norm of E is order continuous.*

Proof. (1) For each $x \in E^+$, the subset $T[0, x]$ is order bounded. Since the norm of F is order continuous, it follows from Theorem 22.1 of [**1**] that $T[0, x]$ is relatively weakly compact, and then T is order weakly compact.

(2) If the norm of E is order continuous, it follows from Theorem 22.1 of [**1**] that for each $x \in E^+$, the order interval $[0, x]$ is weakly compact. Hence, $T[0, x]$ is weakly compact.

(3) (i) $\Longrightarrow$ (ii) is evident.

(ii) $\Longrightarrow$ (iii) Let $x \in E^+$, since the identity operator $Id_E : E \longrightarrow E$ is order weakly compact, then $Id_E([0, x]) = [0, x]$ is relatively compact for the topology $\sigma(E, E')$. But the order interval $[0, x]$ is weakly closed and thus $[0, x]$ is compact for the topology $\sigma(E, E')$. Finally, Theorem 22.1 of [**1**] implies that the norm of E is order continuous.

(iii) $\Longrightarrow$ (i) It is exactly Theorem 2.1 (2). □

To give some examples, let us recall that a non-zero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements.

REMARKS 2.2. There exist Banach lattices E and F and a regular operator T from E into F that is not order weakly compact; however,

(i) the topological dual E' is discrete. In fact, if we take $E = F = c$ the Banach lattice of all convergent sequences, then $E' = c'$ is discrete, but since the norm of c is not order continuous, the identity operator Id_c is not order weakly compact.
(ii) F is discrete. In fact, if we take $E = F = l^\infty$, then F is discrete but the identity operator Id_{l^∞} is not order weakly compact.
(iii) the norm of E' is order continuous. In fact, if we take $E = F = l^\infty$, the norm of $(l^\infty)'$ is order continuous but the identity operator Id_{l^∞} is not order weakly compact.

REMARKS 2.3. If E and F are two Banach lattices such that each regular operator T from E into F is order weakly compact, then

(i) the topological dual E' is not necessary discrete. In fact, for $E = l^1$, each operator $T : E \longrightarrow F$ is order weakly compact, but the topological dual $E' = l^\infty$ is not discrete.
(ii) the Banach lattices E and F are not necessary discrete. In fact, for $E = F = l^1$, each operator $T : E \longrightarrow F$ is order weakly compact, but the topological dual $E' = F' = l^\infty$ is not discrete.
(iii) the norms of E' and F' are not necessary order continuous. In fact, for $E = F = l^1$, each operator $T : E \longrightarrow F$ is order weakly compact, but the norm of the topological dual $E' = F' = l^\infty$ is not order continuous.
(iv) the norm of E is not necessary order continuous. In fact, for $E = l^\infty$ and F of finite-dimensional, then each operator $T : E \longrightarrow F$ is compact and hence order weakly compact, but the norm of l^∞ is not order continuous.
(v) the Banach lattices E and F are not necessary reflexive. In fact for $E = l^\infty$ and $F = c_0$, each operator from E into F is weakly compact and hence order weakly compact, but E and F are not reflexive.

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Note that the subspace of order weakly compact operators is a two-sided ideal in the space of all order-bounded operators on a Banach lattice. But it is not necessary a right ideal in the space of all operators on a Banach lattice. To give some examples, we need a lemma.

For this, recall that for every infinite-dimensional Banach space E, there exits a sequence (x'_n) in E' such that $\|x'_n\| = 1$ for all n, but $x'_n \longrightarrow 0$ for the weak topology $\sigma(E', E)$ (see Josefson-Nissenzweig theorem [**9**, **11**]). This is equivalent to say that a Banach space E is either finite-dimensional or there exist a normalized sequence (x'_n) in E' such that $x'_n(x) \longrightarrow 0$ for every $x \in E$.

LEMMA 2.4. *There exists an operator from c into c_0 that is not order weakly compact.*

Proof. In fact, it follows from Josefson-Nissenzweig theorem [**9**, **11**] that the existence of a sequence (x'_n) of the topological dual of c such that $\|x'_n\| = 1$ for all n and $x'_n \longrightarrow 0$ for the weak topology $\sigma(c', c)$. We consider the operator T, which we can find in Wnuk ([**15**], p. 170), defined by

$$T : c \longrightarrow c_0, x \longmapsto T(x) = (x'_n(x))_{n=1}^{\infty}.$$

The operator T is not compact. In fact, if T is compact, then its adjoint T' defined by

$$T' : l^1 \longrightarrow c', (\lambda_n)_{n=1}^{\infty} \longmapsto T'\big((\lambda_n)_{n=1}^{\infty}\big) = \sum_{n=1}^{\infty} \lambda_n x'_n$$

would be compact. Hence, the sequence $(T'(e_n)) = (x'_n)$ has a subsequence that converges to 0 for the norm where (e_n) is the canonical basis of l^1. But this is in contradiction with the condition that $\|x'_n\| = 1$ for all n. Therefore, the operator T is not compact.

On the other hand, according to Wnuk ([**15**], p. 171), who stated that an operator from c into c_0 is regular if and only if it is compact if and only if it is Dunford-Pettis (i.e. carries weakly compact subsets of c onto compact subsets of c_0), and since c has the Dunford-Pettis property (i.e. each weakly compact operator defined on c, and taking its values in another Banach space, is Dunford-Pettis), it follows that T is not weakly compact. Finally, as c is an AM-space with unit, we deduce that T is not order weakly compact. □

Now, we are in position to give our examples.

EXAMPLES 2.5. (1) Let $T : c \longrightarrow c_0$ be an operator from c into c_0 that is not order weakly compact (this operator exists by Lemma 2.4). Since c_0 has an order-continuous norm, Theorem 2.1 (1) implies that the identity operator $Id_{c_0} : c_0 \longrightarrow c_0$ is order weakly compact. But the product $Id_{c_0} \circ T = T$ is not order weakly compact.

(2) Let E be the Banach lattice $c \oplus c_0$ and let $T : c \longrightarrow c_0$ be an operator from c into c_0 that is not order weakly compact. We consider the operators S_1 and S_2 from E into E defined by the following:

$$S_2 = \begin{pmatrix} 0 & 0 \\ 0 & Id_{c_0} \end{pmatrix} \quad \text{and} \quad S_1 = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix},$$

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where $Id_{c_0} : c_0 \longrightarrow c_0$ is the identity operator of c_0. It is clear that S_2 is order weakly compact but S_1 is not.

On the other hand, the composed operator

$$S_2 \circ S_1 = \begin{pmatrix} 0 & 0 \\ 0 & Id_{c_0} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ Id_{c_0} \circ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} = S_1$$

is not order weakly compact.

Let us recall that if an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

As we have discussed in the introduction, the class of order weakly compact operators does not satisfy the duality property. The following result gives sufficient and necessary conditions under which an order weakly compact operator has an adjoint that is order weakly compact:

THEOREM 2.6. *Let E and F be two Banach lattices and let T be an order-bounded operator from E into F. The following conditions are equivalent:*

1. *The adjoint T' from F' into E' is order weakly compact whenever T is order weakly compact.*
2. *One of the following assertions is valid:*
   - (i) *The norm of E' is order continuous.*
   - (ii) *The norm of F' is order continuous.*

Proof. $2 \Longrightarrow 1$. It is just a consequence of Theorem 2.4 (1) and (2).

$1 \Longrightarrow 2$. Assume that the norms of E' and F' are not order continuous. Then Theorem 2.4.14 and Proposition 2.3.11 of Meyer-Nieberg [**10**] imply that E (resp. F) contains a sublattice isomorphic to l^1 and there exists a positive projection $P_1 : E \longrightarrow l^1$ (resp. $P_2 : F \longrightarrow l^1$).

Since F' is order σ-complete, it follows from Corollary 2.4.3 of Meyer-Neiberg [**10**] that F' contains a sublattice isomorphic to l^∞.

We denote by $i_1 : l^1 \longrightarrow E$ (resp. $i_2 : l^1 \longrightarrow F$) the canonical injection of l^1 into E (resp. l^1 into F). We consider the composed operator

$$i_2 \circ P_1 : E \longrightarrow l^1 \longrightarrow F.$$

It is an order weakly compact operator because $i_2 \circ P_1 = i_2 \circ Id_{l^1} \circ P_1$ and the identity operator Id_{l^1} is order weakly compact. But the operator $P_1' \circ i_2'$ is not order weakly compact. If not, that is, if

$$P_1' \circ i_2' : F' \longrightarrow l^\infty \longrightarrow E'$$

is order weakly compact, then the composed operator

$$i_1' \circ P_1' \circ i_2' : F \longrightarrow l^\infty$$

would be order weakly compact and hence its restriction to l^∞, which is just the identity operator Id_{l^∞}, would be order weakly compact. But this is impossible. □

REMARKS 2.7. Let E and F be two Banach lattices and let T be an operator from E into F. Then the adjoint T' is not necessary order weakly compact whenever T is order weakly compact in the following situations:

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(1) If the topological bidual F'' has an order-continuous norm (in particular if F has an order-continuous norm). In fact, for $F = l^1$, the topological bidual $F'' = (l^\infty)'$ has an order-continuous norm. However, the identity operator $Id_{l^1} : l^1 \longrightarrow l^1$ is order weakly compact but its adjoint $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ is not.
(2) If E has an order-continuous norm. In fact, for $E = l^1$, the identity operator $Id_{l^1} : l^1 \longrightarrow l^1$ is order weakly compact but its adjoint $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ is not.
(3) If E' is discrete. In fact, for $E = l^1$, the topological dual $E' = l^\infty$ is discrete. However, the identity operator $Id_{l^1} : l^1 \longrightarrow l^1$ is order weakly compact but its adjoint $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ is not.

Now, we study the converse whenever E and F are two order σ -complete Banach lattices. In fact, the following result gives a sufficient and necessary conditions under which an operator is order weakly compact if its adjoint is order weakly compact:

THEOREM 2.8. *Let E and F be order σ-complete Banach lattices and let T be an operator from E into F. The following conditions are equivalent:*
(1) *The operator T is order weakly compact whenever its adjoint T' from F' into E' is order weakly compact.*
(2) *One of the following assertions is valid:*
  (i) *The norm of E is order continuous.*
  (ii) *The norm of F is order continuous.*

Proof. $2 \Longrightarrow 1$. It is just a consequence of Theorem 2.1 (1) and (2).

$1 \Longrightarrow 2$. Assume that the norms of E and F are not order continuous. Then, it follows from the proof of Theorem 1 of Wickstead [**14**] that E (resp. F) contains a sublattice isomorphic to l^∞ and there exists a positive projection $P_1 : E \longrightarrow l^\infty$ (resp. $P_2 : F \longrightarrow l^\infty$) . We denote by $i : l^\infty \longrightarrow F$ the canonical injection of l^∞ into F, and we consider the operator T defined by

$$T = i \circ P_1 : E \longrightarrow l^\infty \longrightarrow F.$$

Since $(l^\infty)'$ is an AL-space, its norm is order continuous and hence Theorem 2.4 (2) implies that the identity operator $Id_{(l^\infty)'} : (l^\infty)' \longrightarrow (l^\infty)'$ is order weakly compact. Now, as the subspace of order weakly compact operators is a two-sided ideal, the adjoint

$$T' = P_1' \circ i' = P_1' \circ Id_{(l^\infty)'} \circ i' : F' \longrightarrow (l^\infty)' \longrightarrow (l^\infty)' \longrightarrow E'$$

is order weakly compact. However, the operator

$$T = i \circ P : E \longrightarrow l^\infty \longrightarrow F$$

is not order weakly compact. If not, that is, if T is order weakly compact, then the composed operator

$$P_2 \circ i \circ P_1 : E \longrightarrow l^\infty \longrightarrow F \longrightarrow l^\infty$$

would be order weakly compact. Hence, its restriction to l^∞, which is just the identity operator of l^∞, would be order weakly compact. But this is false. Then, the norm of E is order continuous or the norm of F is order continuous. □

REMARKS 2.9. Let E and F be two Banach lattices and let T be an operator from E into F. Then, T is not necessary order weakly compact whenever its adjoint T' from F' into E' is order weakly compact in the following situations:

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(1) if E' has an order-continuous norm. In fact, if we take $E = l^\infty$, its topological dual $E' = (l^\infty)'$ has an order-continuous norm. But the identity operator $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ is not order weakly compact, however, its adjoint $Id_{(l^\infty)'} : (l^\infty)' \longrightarrow (l^\infty)'$ is order weakly compact.

(2) if E' is discrete. In fact, if we take $E = c$ the Banach lattice of all convergent sequences, the topological dual $E' = c'$ is discrete and the identity operator $Id_{c'}$ is order weakly compact (because c' has an order-continuous norm), but the identity operator Id_c, of the Banach lattice c, is not order weakly compact since the norm of c is not order continuous.

REMARKS 2.10. Let us recall that if G is an order σ-complete Banach lattice such that its topological dual G' is discrete, then the norm of G is order continuous ([**4**], Proposition 3.7). Hence, it is natural to ask the following question: if the Banach lattices E and F are not necessary order σ-complete in Theorem 2.8, is an operator T from E into F order weakly compact whenever its adjoint T' from F' into E' order weakly compact if E' is discrete or F' is discrete and conversely. The answer is no in general. In fact,

(i) there exist Banach lattices E and F such that E' and F' are discrete and there exists an operator T from E into F that is not order weakly compact but its operator adjoint T' from F' into E' is order weakly compact. In fact, for $E = F = c$, we have $E' = F' = c'$ is discrete, but the identity operator of c is not order weakly compact; however, its adjoint that is the identity operator of c' is order weakly compact.

(ii) conversely, there exist Banach lattices E and F such that E' and F' are not discrete, but each operator from E into F is order weakly compact whenever its operator adjoint T' from F' into E' is order weakly compact. In fact, for $E = F = L^2[0, 1]$, we have $E' = F' = L^2[0, 1]$ is discrete, but each operator $T : E \longrightarrow F$ is order weakly compact whenever its adjoint $T' : F' \longrightarrow E'$ is order weakly compact.

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces* (Academic Press, New York, 1978).

2. C. D. Aliprantis and O. Burkinshaw, On weakly compact operators on Banach lattices, *Proc. Amer. Math. Soc.* **83**(3), (1981) 573–578.

3. C. D. Aliprantis and O. Burkinshaw, *Positive operators, in Pure and applied mathematics*, Vol. 119 (Academic Press, Orlando, FL, 1985).

4. B. Aqzzouz and R. Nouira, Sur les opérateurs précompacts positifs, C. R. Acad. Sc. Paris **337**(8) (2003) 527–530.

5. B. Aqzzouz, R. Nouira and L. Zraoula, The duality problem for the class of AM-compact operators on Banach lattices, *Can. Math. Bull.* **51**(1) (2008) 15–20.

6. P. G. Dodds, o-weakly compact mappings of Riesz spaces, *Trans. Amer. Math. Soc.* **214** (1975), 389–402.

7. M. Duhoux, o-weakly compact mappings from a Riesz space to a locally convex space, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.)* **22**(4)(70) (1978), 371–378.

8. Z. Ercan, Interval-bounded operators and order weakly compact operators on Riesz spaces, *Demonstr. Math.* **31**(4) (1998), 805–812.

9. B. Josefson, Weak sequential convergence in the dual of a Banach space does not imply norm convergence, *Ark. Mat.* **13** (1975), 79–89.

https://doi.org/10.1017/S0017089508004576 Published online by Cambridge University Press

10. P. Meyer-Nieberg, *Banach lattices* (Universitext, Springer-Verlag, Berlin, 1991).

11. A. Nissenzweig, w*-sequential convergence, *Israel J. Math.* **22**(3–4) (1975), 266–272.

12. M. Nowak, Order-weakly compact operators from vector-valued function spaces to Banach spaces, *Proc. Amer. Math. Soc.* **135**(9) (2007), 2803–2809.

13. A. W. Wickstead, Extremal structure of cones of operators, *Quart. J. Math. Oxford* **32**(2) (1981), 239–253.

14. A. W. Wickstead, Converses for the Dodds-Fremlin and Kalton-Saab theorems, *Math. Proc. Camb. Phil. Soc.* **120** (1996), 175–179.

15. W. Wnuk, A note on the positive Schur property, *Glasgow Math. J.* **31** (1989), 169–172.

16. A. C. Zaanen, *Riesz spaces II* (North Holland Publishing Company, 1983).

https://doi.org/10.1017/S0017089508004576 Published online by Cambridge University Press