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NUMERICAL INVARIANTS IN HOMOTOPICAL ALGEBRA, I

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Introduction. Classically CW-complexes were found to be the best suited objects for studying problems in homotopy theory. Certain numerical invariants associated to a CW-complex X such as the Lusternik-Schnirelmann Category of X, the index of nilpotency of $\Omega(X)$, the cocategory of X, the index of co-nilpotency of $\Sigma(X)$ have been studied by Eckmann, Hilton, Berstein and Ganea, etc. Recently D. G. Quillen [**6**] has developed homotopy theory for categories satisfying certain axioms. In the axiomatic set up of Quillen the duality observed in classical homotopy theory becomes a self-evident phenomenon, the axioms being so formulated. In addition to the category of based topological spaces there are at least two other familiar categories which satisfy the axioms of Quillen. For any ring A the category $C_+(A)$ of chain complexes over A which are bounded below is known to satisfy the axioms of Quillen [**6**]. The category of semi-simplicial sets also satisfies the axioms of Quillen. This paper is devoted to the development of the theory of Lusternik-Schnirelmann Category and Cocategory etc. in the axiomatic set up of Quillen.

Part of §1 deals with some known facts about categories which we need later. A reference for this is [**5**]. For the sake of completeness we include them here without proofs.

1. Preliminaries about categories. Let $\mathscr{C}$ be an arbitrary category. Recall that a map $j \in \text{Hom}(X, Y)$ is called monic if $\varphi, \theta \in \text{Hom}(A, X)$, $j \circ \varphi = j \circ \theta \Rightarrow \varphi = \theta$. By a subobject of X in $\mathscr{C}$ we mean a pair (E, i) where E is an object in $\mathscr{C}$ and $i : E \to X$ is monic.

1.1 LEMMA. *Suppose*

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "i_1"'] & & E_2 \arrow[dl, "i_2"] \\ & X & \end{tikzcd}$$

Diagram 1

is a commutative diagram in $\mathscr{C}$ with i_1, i_2 monic. Then φ is unique and monic.

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Let $f : X \to Y$ be a given map in $\mathscr{C}$. Let $\mathscr{F}$ be the family of subobjects (E, i) of Y satisfying the condition that there exists a map $g : X \to E$ with $i \circ g = f$. Since i is monic it follows that whenever such a g exists it is unique. For any (E_1, i_1) and (E_2, i_2) in $\mathscr{F}$ a map of (E_1, i_1) into (E_2, i_2) is defined to be a map $\varphi : E_1 \to E_2$ in $\mathscr{C}$ satisfying $i_2 \circ \varphi = i_1$ From 1.1 it follows that φ, whenever it exists, is unique and monic.

1.2 LEMMA. *Let (E_1, i_1) and (E_2, i_2) be objects in $\mathscr{F}$ and $\varphi : E_1 \to E_2$ be a map in $\mathscr{F}$. Suppose $g_1 : X \to E_1$ and $g_2 : X \to E_2$ are the unique maps satisfying $i_1 \circ g_1 = f = i_2 \circ g_2$. Then $\varphi \circ g_1 = g_2$.*

Proof. We have $i_2 \circ (\varphi \circ g_1) = (i_2 \circ \varphi) \circ g_1) = i_1 \circ g_1 = f = i_2 \circ g_2$. The fact that i_2 is monic now yields $\varphi \circ g_1 = g_2$.

1.3 *Definition.* The image of f is defined to be a universal object in $\mathscr{F}$, that is to say it is an object (D, j) in $\mathscr{F}$ with the property that if (E, i) is any other object in $\mathscr{F}$ there exists a map θ of (D, j) into (E, i).

Such a θ is necessarily unique by (1.1). Also the image of f when it exists is unique up to an isomorphism in the category $\mathscr{F}$ since it is defined by a universal property. We denote the image of f by $\operatorname{Im} f$.

1.4. *Definition.* Let (E_1, i_1) and (E_1, i_2) be subobjects of X in $\mathscr{C}$. By their union (in X) we mean a subobject (E, i) of X having the following properties:

(a) There exist maps $j_1 : E_1 \to E$, $j_2 : E_2 \to E$ satisfying $i \circ j_1 = i_1$ and $i \circ j_2 = i_2$.

(b) If (F, k) is any subobject of X with maps $k_1 : E_1 \to F$, $k_2 : E_2 \to F$ satisfying $k \circ k_1 = i_1$, $k \circ k_2 = i_2$ then there exists a unique map $l : E \to F$ satisfying $k \circ l = i$.

1.5. *Definition.* Two subobjects (E, i) and (E', i') of X are said to be isomorphic as subobjects of X if there exists an isomorphism $\theta : E \to E'$ in $\mathscr{C}$ satisfying $i' \circ \theta = i$.

The union of subobjects (E_1, i_1), (E_2, i_2) of X being defined by a universal property, it follows that whenever there is a union it is unique up to an isomorphism as a subobject of X.

Let A_1, A_2 denote the following axioms.

(A_1) Any map $f : X \to Y$ in $\mathscr{C}$ admits an $\operatorname{Im} f$.

(A_2) Any two subobjects (E_1, i_1), (E_2, i_2) of any object X of $\mathscr{C}$ have a union.

1.6. *Remark.* It is clear that in Definition 1.4 the roles of (E_1, i_1) and (E_2, i_2) can be interchanged. Thus if (E, i) is the union of (E_1, i_1) and (E_2, i_2) it follows that (E, i) is also the union of (E_2, i_2) and (E_1, i_1).

We denote the union of the subobjects (E_1, i_1), (E_2, i_2) of X by $(E_1, i_1) \cup (E_2, i_2)$.

1.7. PROPOSITION. *Let (E_q, i_q) $q = 1, 2, 3$ be any three subobjects of X. Let*

$(E, i) = (E_1, i_1) \cup (E_2, i_2)$; $(F, j) = (E, i) \cup (E_3, i_3)$ and $(E', i') = (E_2, i_2) \cup (E_3, i_3)$; $(F', j') = (E_1, i_1) \cup (E', i')$. Then (F, j) and (F', j') are isomorphic as subobjects of X.

1.8. THEOREM. *Let $\mathscr{C}$ be any category satisfying axioms A_1 and A_2. Let*

$$\begin{tikzcd} B \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \arrow[d, "\mu_1"] \\ X_2 \arrow[r, "\mu_2"] & Y \end{tikzcd}$$

Diagram 2

be a push-out diagram in $\mathscr{C}$. Let $(E_q, i_q) = \mathrm{Im}\, \mu_q (q = 1, 2)$ and $(E, i) = (E_1, i_1) \cup (E_2, i_2)$. Then $i : E \to Y$ is an isomorphism.

Proof. Let $\alpha_q : X_q \to E_q$ be the map satisfying $i_q \circ \alpha_q = \mu_q$ $(q = 1, 2)$. Since $(E, i) = (E_1, i_1) \cup (E_2, E_2)$ there exist maps $j_q : E_q \to E$ such that $i \circ j_q = i_q$. Write θ_q for $j_q \circ \alpha_q$. Then $i \circ \theta_q \circ f_q = i \circ j_q \circ \alpha_q \circ f_q = i_q \circ \alpha_q \circ f_q = \mu_q \circ f_q$. The commutativity of Diagram 2 gives $i \circ \theta_1 \circ f_1 = i \circ \theta_2 \circ f_2$. Since i is monic, $\theta_1 \circ f_1 = \theta_2 \circ f_2$. Hence

$$\begin{tikzcd} B \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \arrow[d, "\theta_1"] \\ X_2 \arrow[r, "\theta_2"] & E \end{tikzcd}$$

Diagram 3

is a commutative diagram. Since by assumption Diagram 2 is a push-out diagram it follows that there exists a unique map $\lambda : Y \to E$ such that

$$\begin{tikzcd} & X_1 \arrow[dl, "\mu_1"'] \arrow[dr, "\theta_1"] & \\ Y \arrow[rr, "\lambda"] & & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y \arrow[dd, "\lambda"] \\ X_2 \arrow[ur, "\mu_2"] \arrow[dr, "\theta_2"'] & \\ & E \end{tikzcd}$$

Diagram 4 Diagram 5

are commutative. From $i \circ \lambda \circ \mu_1 = i \circ \theta_1 = i \circ j_1\alpha_1 = i_1 \circ \alpha_1 = \mu_1$ and $i \circ \lambda \circ \mu_2 = i \circ \theta_2 = i \circ j_2 \circ \alpha_2 = i_2 \circ \alpha_2 = \mu_2$ we see that

$$\begin{tikzcd} & X_1 \arrow[dl, "\mu_1"'] \arrow[dr, "\theta_1"] & \\ Y \arrow[rr, "i \circ \lambda"] & & Y \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dd, "i \circ \lambda"] \\ X_2 \arrow[ur, "\mu_2"] \arrow[dr, "\mu_2"'] & \\ & Y \end{tikzcd}$$

Diagram 6 Diagram 7

are commutative. From the fact that Diagram 2 is a push-out diagram it follows immediately that $i \circ \lambda = \mathrm{Id}_Y$.

Also $i \circ (\lambda \circ i) = (i \circ \lambda) \circ i = \mathrm{Id}_Y \circ i = i = i \circ \mathrm{Id}_E$. Since i is monic we get $\lambda \circ i = \mathrm{Id}_E$.

Thus $i : E \to Y$ is an isomorphism with $\lambda : Y \to E$ as its inverse.

1.9. *Definition.* Suppose $f : X \to Y$ is a map in $\mathscr{C}$ and (E, i) a subobject of Y. The inverse image of (E, i) by f is defined to be a subobject (F, j) of X satisfying the following conditions:

(i) $f \circ j$ factors through i, i.e. there exists a map $\varphi : F \to E$ such that $i \circ \varphi = f \circ j$.

(ii) If (F', j') is any other subobject of X with the property that there exists a map $\varphi' : F' \to E$ with $i \circ \varphi' = f \circ j'$ then there exists a unique map $\mu : F' \to F$ satisfying $j \circ \mu = j'$.

Remarks. (a) Since $i : E \to Y$ is monic, whenever a map $\varphi: F \to E$ exists satisfying $i \circ \varphi = f \circ j$ then it has to be unique.

(b) The inverse image being defined by a universal property is unique up to an isomorphism as a subobject of X, whenever it exists.

(c) The map $\mu : F' \to F$ postulated to exist in (ii) above is monic by (1.1).

Axiom A_3. For every map $f : X \to Y$ in $\mathscr{C}$ and every subobject (E, i) of Y there exists an inverse image by f.

1.10. Lemma. *Let $\mathscr{C}$ be a category satisfying axioms A_1 and A_3. Let $f : X \to Y$ be any map in $\mathscr{C}$ and $(E, i) = \mathrm{Im} f$. Let (F, j) be the inverse image of (E, i) by f. Then $j : F \to X$ is an isomorphism.*

Axiom A_4. Let $f : X \to Y$ be any map in $\mathscr{C}$ and $\mathrm{Im}\, f = (E, j)$. For any subobject (F, i) of E there exists a subobject (D, μ) of X satisfying

(00) $\quad \mathrm{Im}\, f \circ \mu = (F, j \circ i)$.

1.11. Proposition. *Let $\mathscr{C}$ be a category satisfying axioms A_1, A_3 and A_4 and $f : X \to Y$ any map in $\mathscr{C}$. Let $\mathrm{Im}\, f = (E, i)$ and (F, j) any subobject of E. Let (C, ν) be the inverse image of $(F, j \circ i)$ by f. Then $\mathrm{Im}\, f \circ \nu = (F, j \circ i)$.*

1.12. Proposition. *Let $\mathscr{C}$ be a category for which axioms A_1, A_2 and A_3 are valid. Let $f : X \to Y$ be any map in $\mathscr{C}$. Let (E_k, i_k) be subobjects of X and (B_k, j_k) subobjects of Y (for $k = 1, 2$). Let $(E, i) = (E_1, i_1) \cup (E_2, i_2)$ and $(B, j) = (B_1, j_1) \cup (B_2, j_2)$ with $\lambda_k : E_k \to E$, $\mu_k : B_k \to B$ the unique maps satisfying $i \circ \lambda_k = i_k$, $j \circ \mu_k = j_k (k = 1, 2)$. Suppose there exists maps $\theta_k : E_k \to B_k$ satisfying $f \circ i_k = j_k \circ \theta_k (k = 1, 2)$. Then there exists a map $\theta : E \to B$*

(*necessarily unique*) *such that the following diagram is commutative.*

$$\begin{tikzcd}
 & X \arrow[r, "f"] & Y & \\
E_1 \arrow[rrr, "\theta_1"] \arrow[dr, "\lambda_1"'] & & & B_1 \arrow[dl, "\mu_1"] \\
 & E \arrow[uu, "i"] \arrow[r, "\theta"] & B \arrow[uu, "j"'] & \\
E \arrow[ur, "\lambda_2"] \arrow[rrr, "\theta_2"] & & & B \arrow[ul, "\mu_2"']
\end{tikzcd}$$

Diagram 8

1.13. LEMMA. *Let $\mathscr{C}$ be a category satisfying axiom A_3. Let $f : X \to Y$ and $g : X \to Z$ be any two maps in $\mathscr{C}$. Let (E, i) be any subobject of Z. Let (F, j) be the inverse image of (E, i) by g and (C, ν) the inverse image of (F, j) by f. Then (C, ν) is the inverse image of (E, i) by $g \circ f$.*

2. Some propositions on model categories. In this section $\mathscr{C}$ denotes a model category in the sense of D. G. Quillen [**6**]. The notations and the terminology we follow are those of [**6**]. In particular by a trivial fibration we mean a fibration which is also a weak equivalence. We use the abbreviation *w.e.* to denote a weak equivalence. We briefly recall the definition of a model category.

Definition. By a model category we mean a category $\mathscr{C}$ together with three classes of maps in $\mathscr{C}$, called fibrations, cofibrations and weak equivalences, satisfying the following axioms.

M_0: $\mathscr{C}$ is closed under finite limits and colimits.

M_1: Given a solid arrow diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Diagram 9

where i is a cofibration, p a fibration and where either i or p is a *w.e.*, then the dotted arrow exists.

M_2: Any map f may be factored $f = pi$ where i is a cofibration and *w.e.* and p is a fibration. Also $f = qj$ where j is a cofibration and p a fibration and *w.e.*

M_3: Fibrations are stable under composition, base change, and any isomorphism is a fibration. Cofibrations are stable under composition, cobase change and any isomorphism is a cofibration.

M_4: The base extension of a map which is both a fibration and a *w.e.* is a *w.e.* The cobase extension of a map which is both a cofibration and a *w.e.* is a *w.e.*

M_5: Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be maps in $\mathscr{C}$. Then if two of the maps f, g and $g \circ f$ are weak equivalences then the third is. Any isomorphism is a *w.e.*

2.1. Proposition. *Let $f, g \in$ Hom (X, Y) and $f \sim^l g$. Let $p : E \to X$ be any trivial fibration. Then $f \circ p \sim^l g \circ p$.*

Proof. Let $X \times I$ be a cylinder object for X such that there exists a left homotopy $h : X \times I \to I$ between f and g. Let

$$X \vee X \xrightarrow{\partial_0 + \partial_1} X \times I \longrightarrow X$$

be such that $\partial_0 + \partial_1$ is a cofibration, σ a *w.e.*, $\sigma \circ (\partial_0 + \partial_1) = \nabla_X$ and Diagram 9(a) below commutative.

$$\begin{tikzcd}
X \vee X \arrow[rr, "f + g"] \arrow[d, "\nabla_X"'] \arrow[drr, "\partial_0 + \partial_1"] & & Y \\
X & & X \times I \arrow[ll, "\sigma"] \arrow[u, "\eta"']
\end{tikzcd}$$

Diagram 9(a)

Let

$$\begin{tikzcd}
P \arrow[r, "\sigma'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "\sigma"] & X
\end{tikzcd}$$

Diagram 10

denote the pull-back of p by σ. Since p is a *w.e.* and a fibration by axioms M_3 and M_4 for model categories [**6**] it follows that p' is a trivial fibration. By axiom M_5, $\sigma \circ p'$ is a *w.e.* But $\sigma \circ p' = p \circ \sigma'$. Hence $p \circ \sigma'$ is a *w.e.* Since p is a *w.e.* again by axiom M_5 σ' is a *w.e.*

In Diagram 11 below we have $\sigma \circ (\partial_0 \circ p + \partial_1 \circ p) = p \circ \nabla_E$.

$$\begin{tikzcd}
E \vee E \arrow[dr, dashed, "\partial_0' + \partial_1'", "(a)"'] \arrow[drr, "\nabla_E"] \arrow[ddr, "\partial_0 \circ p + \partial_1 \circ p"'] & & \\
& P \arrow[r, "\sigma'"] \arrow[d, "p"] & E \arrow[d, "p"] \\
& X \times I \arrow[r, "\sigma"] & X
\end{tikzcd}$$

Diagram 11

Since the inner square in Diagram 11 is a pull-back diagram there exists a unique map $\partial_0' + \partial_1' : E \vee E \to P$ along the dotted arrow making Diagram 11 commutative. Let $h' = h \circ p' : P \to Y$. Then

$$(*) \qquad \begin{cases} h' \circ \partial_0' = h \circ p' \circ \partial_0' = h \circ \partial_0 \circ p = f \circ p \\ h' \circ \partial_1' = h \circ p' \circ \partial_1' = h \circ \partial_1 \circ p = g \circ p. \end{cases}$$

From the commutativity of the triangle marked (a) in Diagram 11 and the equations (*) we see immediately that

$$\begin{tikzcd}
E \vee E \arrow[rr, "f \circ p + g \circ p"] \arrow[d, "\nabla_E"'] \arrow[drr, "\partial_0' + \partial_1'"] & & Y \\
E & & P \arrow[ll, "\sigma'"] \arrow[u, "h'"']
\end{tikzcd}$$

Diagram 12

is a commutative diagram. In here σ' is a *w.e.* Hence $f \circ p \sim^l g \circ p$.

2.2. *Remark. Proposition* 2.1 *can be contrasted with the following facts already proved in* [**6**].

(i) *If* $f, g \in \operatorname{Hom}(B, C)$ *satisfy* $f \sim^l g$ *then* $u \circ f \sim^l u \circ g$ *for any map* $u : C \to D$ *in* $\mathscr{C}$.

ii() *If further* C *is fibrant then* $f \circ v \sim^l g \circ v$ *for any* $v : A \to B$ *in* $\mathscr{C}$.

2.3. Proposition. *Let* $f, g \in \operatorname{Hom}(X, Y)$ *be such that* $f \sim^r g$. *Let* $i : Y \to Y'$ *denote any trivial cofibration. Then* $i \circ f \sim^r i \circ g$.

This is precisely the dual of 2.1.

2.4. Proposition. *Let* Y, Z *be fibrant and* $i : Y \to Z$ *a w.e. If* $f, g \in \operatorname{Hom}(X, Y)$ *are such that* $i \circ f \sim^r i \circ g$ *then* $f \sim^r g$.

Proof. Let Z^I be a path object for Z with a right homotopy $k : X \to Z^I$ between $i \circ f$ and $i \circ g$. Let

$$Z \xrightarrow{\tau} Z^I \xrightarrow{(d_0,\, d_1)} Z \times Z$$

be such that τ is a *w.e.*, (d_0, d_1) is a fibration, $(d_0, d_1) \circ \tau = \nabla_Z$ and Diagram 13 below commutative.

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "{(i \circ f,\, i \circ g)}"'] & Z^I \arrow[dl, "{(d_0,\, d_1)}"'] \\
Z \times Z & Z \arrow[l, "\Delta_Z"'] \arrow[u, "\tau"']
\end{tikzcd}$$

Diagram 13

Let

$$\begin{tikzcd}
N \arrow[r, "\beta"] \arrow[d, "{(\varphi_0,\, \varphi_1)}"] & Z^I \arrow[d, "{(d_0,\, d_1)}"] \\
Y \times Z \arrow[r, "i \times 1_Z"] & Z \times Z
\end{tikzcd}$$

Diagram 14

and

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "{(\nu_0,\, \nu_1)}"] & N \arrow[d, "{(\varphi_0,\, \varphi_1)}"] \\
Y \times Y \arrow[r, "1_Y \times i"] & Y \times Z
\end{tikzcd}$$

Diagram 15

be pull-back diagrams. If $\theta = \beta \circ \alpha$ then it follows that Diagram 16 below is a pull-back diagram.

$$\begin{tikzcd}
M \arrow[r, "\theta"] \arrow[d, "{(\nu_0,\, \nu_1)}"] & Z^I \arrow[d, "{(d_0,\, d_1)}"] \\
Y \times Y \arrow[r, "i \times i"] & Z \times Z
\end{tikzcd}$$

Diagram 16

We have $(d_0, d_1) \circ \tau \circ i = (i \times i) \circ \Delta_Y$ in Diagram 17 below.

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\mu"] \arrow[drr, "\tau \circ i"] \arrow[ddr, "\Delta_Y"'] \arrow[dr, phantom, "(a)"', shift right=6] & & \\
& M \arrow[r, "\theta"] \arrow[d, "{(\nu_0,\, \nu_1)}"] & Z^I \arrow[d, "{(d_0,\, d_1)}"] \\
& Y \times Y \arrow[r, "i \times i"] & Z \times Z
\end{tikzcd}$$

Diagram 17

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Since the inner square is a pull-back diagram it follows that there exists a unique map $\mu : Y \to M$ along the dotted arrow making Diagram 17 commutative. Clearly

$$\begin{tikzcd} Y \times Z \arrow[r, "i \times 1_Z"] \arrow[d, "p_Y"] & Z \times Z \arrow[d, "p_1"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

Diagram 18

where $p_1 : Z \times Z \to Z$, $p_Y : Y \times Z \to Y$ are projections to the first factors, is a pull-back diagram. This together with the fact that Diagram 14 is a pull-back immediately yields that Diagram 19 below is a pull-back.

$$\begin{tikzcd} N \arrow[r, "\beta"] \arrow[d, "\varphi_0"] & Z^I \arrow[d, "d_0"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

Diagram 19

Now, $d_0 : Z^I \to Z$ is a trivial fibration since Z is fibrant. Hence by axioms M_3 and M_4 it follows that $\varphi_0 : N \to Y$ is a trivial fibration. In particular φ_0 is a *w.e.* Moreover $i \circ \varphi_0 = d_0 \circ \beta$ and i, d_0, φ_0 are weak equivalences. By axiom M_5 we see immediately that β is a *w.e.* From $d_1 \circ \beta = \varphi_1$ (Diagram 14) and the fact that d_1 is a *w.e.* it follows that φ_1 is a *w.e.*

Similarly it is possible to show that

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\nu_1"] & N \arrow[d, "\varphi_1"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

Diagram 20

is a pull-back. Since Y is fibrant the projection $p_2 : Y \times Z \to Z$ to the second factor is a fibration. Moreover $\varphi_1 = p_2 \circ (\varphi_0, \varphi_1)$. Hence by axiom M_3 the map φ_1 is a fibration. We have earlier shown that φ_1 is a *w.e.* Thus φ_1 is a trivial fibration. By axiom M_4, $\nu_1 : M \to Y$ is a *w.e.* From $\nu_1 \circ \mu = 1_Y$ (Diagram 17) and axiom M_5 we immediately see that μ is a *w.e.*

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In Diagram 21 we have $(d_0, d_1) \circ k = (i \times i) \circ (f, g)$.

$$
\begin{tikzcd}
X \arrow[dr, dashed, "k'"] \arrow[drr, bend left, "k"] \arrow[ddr, "{(f, g)}"'] \arrow[dr, phantom, "(b)", shift right=3] & & \\
& M \arrow[r, "\theta"] \arrow[d, "{(\nu_0, \nu_1)}"] & Z^I \arrow[d, "{(d_0, d_1)}"] \\
& Y \times Y \arrow[r, "i \times i"] & Z \times Z
\end{tikzcd}
$$

Diagram 21

Since the inner square is a pull-back, there exists a unique map $k' : X \to M$ along the dotted arrow making Diagram 21 commutative.

From the commutativity of the triangle marked (a) in Diagram 17 and of the triangle marked (b) in Diagram 21 we immediately see that Diagram 22 below is commutative.

$$
\begin{tikzcd}
X \arrow[r, "k'"] \arrow[d, "{(f, g)}"'] & M \arrow[dl, "{(\nu_0, \nu_1)}"'] \\
Y \times Y & Y \arrow[l, "\Delta_Y"] \arrow[u, "\mu"']
\end{tikzcd}
$$

Diagram 22

Moreover μ is a *w.e.* This proves that $f \sim^r g$.

2.5. COROLLARY. *Let Y, Z be fibrant and $i : Y \to Z$ a w.e. Then $i_* : \Pi^r(X, Y) \to \Pi^r(X, Z)$ is a set theoretic injection for any X in $\mathscr{C}$.*

2.6. PROPOSITION. *Let A, B be cofibrant and $h : A \to B$ any w.e. If $f, g \in$ Hom (B, C) are such that $f \circ h \sim^l g \circ h$ then $f \sim^l g$.*

This is the dual of Proposition 2.4.

2.7. COROLLARY. *Let A, B be cofibrant and $h : A \to B$ any w.e. Then $h^* : \pi^l(B, C) \to \pi^l(A, C)$ is a set theoretic injection for any C in $\mathscr{C}$.*

As in [**6**] for any object A of $\mathscr{C}$, $p_A : Q(A) \to A$ (respectively $i_A : A \to R(A)$) will denote a trivial fibration (respectively a trivial cofibration) with $Q(A)$ cofibrant (respectively $R(A)$ fibrant). Then as explained in [**6**] given any map $f : A \to B$ in $\mathscr{C}$ it is possible to find a map $Q(f) : Q(A) \to Q(B)$ (respectively

$R(f) : R(A) \to R(B))$ such that $f \circ p_A = p_B \circ Q(f)$ (respectively $R(f) \circ i_A = i_B \circ f$). Moreover such a $Q(f)$ (respectively $R(f)$) is unique up to left-homotopy (respectively right homotopy).

2.8. LEMMA. *Let* $f, g \in \operatorname{Hom}(A, B)$. *Then* $RQ(f) \sim RQ(g)$ *if and only if* $i_B \circ f \circ p_A \sim i_B \circ g \circ p_A$.

Proof. Diagram 23 below is clearly a commutative diagram for any $\varphi \in \operatorname{Hom}(A, B)$.

$$\begin{tikzcd}
RQ(A) \arrow[r, "RQ(\varphi)"] & RQ(B) & \\
Q(A) \arrow[u, "i_{Q(A)}"] \arrow[r, "Q(\varphi)"] \arrow[d, "p_A"] & Q(B) \arrow[u, "i_{Q(B)}"] \arrow[r, "i_{Q(B)}"] \arrow[d, "P_B"] & RQ(B) \arrow[d, "R(p_B)"] \\
A \arrow[r, "\varphi"] & B \arrow[r, "i_B"] & R(B)
\end{tikzcd}$$

Diagram 23

Since $RQ(B)$ is fibrant and $i_{Q(A)} : Q(A) \to RQ(A)$ is a trivial cofibration, by the dual of Lemma 7, § 1, Chapter I of [**6**] it follows that

$$i_{Q(A)}{}^* : \pi^r(RQ(A), RQ(B)) \to \pi^r(Q(A), RQ(B))$$

is a set theoretic bijection. Since both $Q(A)$ and $RQ(A)$ are cofibrant and $RQ(B)$ is fibrant we have

$$\pi^r(RQ(A), RQ(B)) = \pi^l(RQ(A), RQ(B)) = \pi(RQ(A), RQ(B)) \quad \text{and}$$
$$\pi^r(Q(A), RQ(B)) = \pi^r(Q(A), RQ(B)) = \pi(Q(A), RQ(B)).$$

Thus $i_{Q(A)}{}^* : \pi(RQ(A), RQ(B)) \to \pi(Q(A), RQ(B))$ is a set theoretic bijection. From the commutativity of the upper square in Diagram 23 for f and g separately in place of φ we now get

(5) $\quad RQ(f) \sim RQ(g) \Leftrightarrow i_{Q(B)} \circ Q(f) \sim i_{Q(B)} \circ Q(g).$

Let us assume that $RQ(f) \sim RQ(g)$. Then $i_{Q(B)} \circ Q(f) \sim i_{Q(B)} \circ Q(g)$. From (i) of Remark 2.2 we immediately get $R(p_B) \circ i_{Q(B)} \circ Q(f) \sim^l R(p_B) \circ i_Q(_B) \circ Q(g)$. However, since $Q(A)$ is cofibrant and $R(B)$ is fibrant $\pi^l(Q(A), R(B)) = \pi^r(Q(A), R(B)) = \pi(Q(A), R(B))$. Hence $R(p_B) \circ i_{Q(B)} \circ Q(f) \sim R(p_B) \circ i_{Q(B)} \circ Q(g)$. From Diagram 23 we see that $R(p_B) \circ i_{Q(B)} \circ Q(f) = i_B \circ f \circ p_A$ and $R(p_B) \circ i_{Q(B)} \circ Q(g) = i_B \circ g \circ p_A$. Hence $RQ(f) \sim RQ(g) \Rightarrow i_B \circ f \circ p_A \sim i_B \circ g \circ p_A$.

Conversely, assume $i_B \circ f \circ p_A \sim i_B \circ g \circ p_A$. Then $R(p_B) \circ i_{Q(B)} \circ Q(f) \sim R(p_B) \circ i_{Q(B)} \circ Q(g)$. Since $RQ(B)$ and $R(B)$ are fibrant and $R(p_B) : RQ(B) \to$

$R(B)$ is a *w.e.* from Proposition 2.4 we get $i_{Q(B)} \circ Q(f) \sim^r i_{Q(B)} \circ Q(g)$. Again since $\pi^r(Q(B), RQ(B)) = \pi(Q(B), RQ(B))$ it follows that $i_{Q(B)} \circ Q(f) \sim i_{Q(B)} \circ Q(g)$. Now (5) gives $RQ(f) \sim RQ(g)$. This completes the proof of Lemma 2.8.

From now on we will assume that the model category $\mathscr{C}$ in addition to satisfying the axioms M_1, M_2, M_3, M_4, M_5 of Quillen also satisfies the axiom W mentioned below.

(W) If $f : A \to B$, $g : C \to D$ are weak equivalences then $f \times g : A \times B \to C \times D$ and $f \vee g : A \vee B \to C \vee D$ are also weak equivalences.

2.9. Lemma. *Let $f_i, g_i \in$ Hom (A_i, B_i) and $f_i \sim^l g_i (i = 1, 2)$. Then $f_1 \times f_2 \sim^l g_1 \times g_2$ and $f_1 \vee f_2 \sim^l g_1 \vee g_2$.*

There exist commutative diagrams

$$\begin{tikzcd}
A_1' \arrow[r, "h_1"] \arrow[d, "\sigma_1"'] & B_1 \\
A_1 & A_1 \vee A_1 \arrow[l, "\nabla_{A_1}"] \arrow[u, "f_1 + g_1"'] \arrow[ul, "\partial_0 + \partial_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2' \arrow[r, "h_2"] \arrow[d, "\sigma_2"'] & B_2 \\
A_2 & A_2 \vee A_2 \arrow[l, "\nabla_{A_2}"] \arrow[u, "f_2 + g_2"'] \arrow[ul, "\partial_0' + \partial_1'"']
\end{tikzcd}$$

Diagram 24 Diagram 25

with σ_1 and σ_2 weak equivalences. It follows that Diagrams 26 and 27 below are commutative.

$$\begin{tikzcd}
A_1' \times A_2' \arrow[r, "h_1 \times h_2"] \arrow[d, "\sigma_1 \times \sigma_2"'] & B_1 \times B_2 \\
A_1 \times A_2 & (A_1 \times A_2) \vee (A_1 \times A_2) \arrow[l, "\nabla_{A_1 \times A_2}"] \arrow[u, "f_1 \times f_2 + g_1 \times g_2"'] \arrow[ul, "\partial_0 \times \partial_0' + \partial_1 \times \partial_1'"']
\end{tikzcd}$$

Diagram 26

$$\begin{tikzcd}
A_1' \vee A_2' \arrow[r, "h_1 \vee h_2"] \arrow[d, "\sigma_1 \vee \sigma_2"'] & B_1 \vee B_2 \\
A_1 \vee A_2 & (A_1 \vee A_2) \vee (A_1 \vee A_2) \arrow[l, "\nabla_{A_1 \vee A_2}"] \arrow[u, "f_1 \vee f_2 + g_1 \vee g_2"'] \arrow[ul, "\partial_0 \vee \partial_0' + \partial_1 \vee \partial_1'"']
\end{tikzcd}$$

Diagram 27

From axiom (W) we see that $\sigma_1 \times \sigma_2$ and $\sigma_1 \vee \sigma_2$ are weak equivalences. Hence $f_1 \times f_2 \sim^l g_1 \times g_2$ and $f_1 \vee f_2 \sim^l g_1 \vee g_2$.

2.10. LEMMA. *Let $f_i, g_i \in \operatorname{Hom}(A_i, B_i)$ and $f_i \sim^r g_i (i = 1, 2)$. Then $f_1 \times f_2 \sim^r g_1 \times g_2$ and $f_1 \vee f_2 \sim^r g_1 \vee g_2$.*

Proof. There exist commutative diagrams

$$\begin{tikzcd} A_1 \arrow[r, "k_1"] \arrow[d, "(f_1, g_1)"'] & B_1' \arrow[dl, "(d_0, d_1)"'] \\ B_1 \times B_1 & B_1 \arrow[l, "\Delta_{B_1}"] \arrow[u, "\tau_1"'] \end{tikzcd}$$

Diagram 28

$$\begin{tikzcd} A_2 \arrow[r, "k_2"] \arrow[d, "(f_2, g_2)"'] & B_2' \arrow[dl, "(d_0', d_1')"'] \\ B_2 \times B_2 & B_2 \arrow[l, "\Delta_{B_2}"] \arrow[u, "\tau_2"'] \end{tikzcd}$$

Diagram 29

with τ_1 and τ_2 weak equivalences. It follows that Diagrams 30 and 31 below are commutative.

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, "k_1 \times k_2"] \arrow[d, "(f_1 \times f_2, g_1 \times g_2)"'] & B_1' \times B_2' \arrow[dl, "(d_0 \times d_0', d_1 \times d_1')"'] \\ (B_1 \times B_2) \times (B_1 \times B_2) & B_1 \times B_2 \arrow[l, "\Delta_{B_1 \times B_2}"] \arrow[u, "\tau_1 \times \tau_2"'] \end{tikzcd}$$

Diagram 30

$$\begin{tikzcd} A_1 \vee A_2 \arrow[r, "k_1 \vee k_2"] \arrow[d, "(f_1 \vee f_2, g_1 \vee g_2)"'] & B_1' \vee B_2' \arrow[dl, "(d_0 \vee d_0', d_1 \vee d_1')"'] \\ (B_1 \vee B_2) \times (B_1 \vee B_2) & B_1 \vee B_2 \arrow[l, "\Delta_{B_1 \vee B_2}"] \arrow[u, "\tau_1 \vee \tau_2"'] \end{tikzcd}$$

Diagram 31

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By axiom (W) $\tau_1 \times \tau_2$ and $\tau_1 \vee \tau_2$ are weak equivalences. Hence $f_1 \times f_2 \sim^r g_1 \times g_2$ and $f_1 \vee f_2 \sim^r g_1 \vee g_2$.

2.11. *Remark.* The three categories which we mentioned in the introduction do satisfy axiom (W) also. Hence our results apply to all these three categories.

2.12 PROPOSITION. *Let $f_k, g_k \in$ Hom (A_k, B_k) $(k = 1, 2)$ be such that $RQ(f_k) \sim RQ(g_k)$. Then*
(a) $RQ(f_1 \times f_2) \sim RQ(g_1 \times g_2)$, *and*
(b) $RQ(f_1 \vee f_2) \sim RQ(g_1 \vee g_2)$.

Proof of (a). By hypothesis, $RQ(f_k) \sim RQ(g_k)$. Lemma 2.8 gives $i_{B_k} \circ f_k \circ p_{A_k} \sim i_{B_k} \circ g_k \circ p_{A_k}$. Lemma 2.10 now yields

$$(i_{B_1} \circ f_1 \circ p_{A_1}) \times (i_{B_2} \circ f_2 \circ p_{A_2}) \sim^r (i_{B_1} \circ g_1 \circ p_{A_1}) \times (i_{B_2} \circ g_2 \circ p_{A_2}).$$

In other words,

$$(6) \qquad (i_{B_1} \times i_{B_2}) \circ (f_1 \times f_2) \circ (p_{A_1} \times p_{A_2}) \sim^r (i_{B_1} \times i_{B_2}) \circ (g_1 \times g_2) \circ (p_{A_1} \times p_{A_2}).$$

Since p_{A_1} and p_{A_2} are fibrations it follows that $p_{A_1} \times p_{A_2} : Q(A_1) \times Q(A_2) \to A_1 \times A_2$ is a fibration. By axiom (W), $p_{A_1} \times p_{A_2}$ is also a *w.e.* Since $Q(A_1 \times A_2)$ is cofibrant it follows from axiom M_1 that there exists a map $\lambda : Q(A_1 \times A_2) \to Q(A_1) \times Q(A_2)$ along the dotted arrow in Diagram 32 making it commutative.

$$\begin{tikzcd}
\phi \arrow[r] \arrow[d] & Q(A_1) \times Q(A_2) \arrow[d, "p_{A_1} \times p_{A_2}"] \\
Q(A_1 \times A_2) \arrow[r, "p_{A_1 \times A_2}"'] \arrow[ur, dashed, "\lambda"] & A_1 \times A_2
\end{tikzcd}$$

Diagram 32

From (6) and the dual of (i), Remark 2.2 we get

$$(i_{B_1} \times i_{B_2}) \circ (f_1 \times f_2) \circ (p_{A_1} \times p_{A_2}) \circ \lambda \sim^r (i_{B_1} \times i_{B_2}) \circ (g_1 \times g_2) \circ (p_{A_1} \times p_{A_2}) \circ \lambda.$$

But $(p_{A_1} \times p_{A_2}) \circ \lambda = p_{A_1 \times A_2}$. Hence

$$(7) \qquad (i_{B_1} \times i_{B_2}) \circ (f_1 \times f_2) \circ p_{A_1 \times A_2} \sim^r (i_{B_1} \times i_{B_2}) \circ (g_1 \times g_2) \circ p_{A_1 \times A_2}.$$

Since $i_{B_1 \times B_2}$ is a trivial cofibration and $R(B_1) \times R(B_2)$ is fibrant by axiom M_1 there exists a map $\mu : R(B_1 \times B_2) \to R(B_1) \times R(B_2)$ along the dotted arrow in Diagram 33, making it commutative.

$$\begin{tikzcd}
B_1 \times B_2 \arrow[r, "i_{B_1} \times i_{B_2}"] \arrow[d, "i_{B_1 \times B_2}"'] & R(B_1) \times R(B_2) \arrow[d] \\
R(B_1 \times B_2) \arrow[r] \arrow[ur, dashed, "\mu"] & *
\end{tikzcd}$$

Diagram 33

By axiom (W) $i_{B_1} \times i_{B_2}$ is a *w.e.* It follows from axiom M_5 that μ is a *w.e.* Now, relation (7) is the same as

(8) $\quad \mu \circ i_{B_1 \times B_2} \circ (f_1 \times f_2) \circ p_{A_1 \times A_2} \sim^r \mu \circ i_{B_1 \times B_2} \circ (g_1 \times g_2) \circ p_{A_1 \times A_2}.$

Also, both $R(B_1 \times B_2)$ and $R(B_1) \times R(B_2)$ are fibrant. Hence Proposition 2.4 yields

(8) $\quad i_{B_1 \times B_2} \circ (f_1 \times f_2) \circ p_{A_1 \times A_2} \sim^r i_{B_1 \times B_2} \circ (g_1 \times g_2) \circ p_{A_1 \times A_2}.$

Since $Q(A_1 \times A_2)$ is cofibrant and $R(B_1 \times B_2)$ is fibrant (8) gives

$$i_{B_1 \times B_2} \circ (f_1 \times f_2) \circ p_{A_1 \times A_2} \sim i_{B_1 \times B_2} \circ (g_1 \times g_2) \circ p_{A_1 \times A_2}.$$

Lemma 2.8 now gives $RQ(f_1 \times f_2) \sim RQ(g_1 \times g_2)$.

Proof of (b). As in (a),

$$RQ(f_k) \sim RQ(g_k) \Rightarrow i_{B_k} \circ f_k \circ p_{A_k} \sim i_{B_k} \circ g_k \circ p_{A_k}.$$

The second part of Lemma 2.10 yields

$$(i_{B_1} \circ f_1 \circ p_{A_1}) \vee (i_{B_2} \circ f_2 \circ p_{A_2}) \sim^l (i_{B_1} \circ g_1 \circ p_{A_1}) \vee (i_{B_2} \circ g_2 \circ p_{A_2}).$$

In other words,

(9) $\quad (i_{B_1} \vee i_{B_2}) \circ (f_1 \vee f_2) \circ (p_{A_1} \vee p_{A_2}) \sim^l (i_{B_1} \vee i_{B_2}) \circ (g_1 \vee g_2) \circ (p_{A_1} \vee p_{A_2}).$

Since

$$Q(A_1 \vee A_2) \xrightarrow{p_{A_1 \vee A_2}} A_1 \vee A_2$$

is a trivial fibration and $Q(A_1) \vee Q(A_2)$ is cofibrant, by axiom M_1 there exists a map $\alpha : Q(A_1) \vee Q(A_2) \to Q(A_1 \vee A_2)$ along the dotted arrow in Diagram

34 below making it commutative.

$$\begin{tikzcd}
\phi \arrow[r] \arrow[d] & Q(A_1 \vee A_2) \arrow[d, "p_{A_1 \vee A_2}"] \\
Q(A_1) \vee Q(A_2) \arrow[r, "p_{A_1} \vee p_{A_2}"'] \arrow[ur, dashed, "\alpha"] & A_1 \vee A_2
\end{tikzcd}$$

Diagram 34

By axiom (W), $p_{A_1} \vee p_{A_2}$ is a *w.e.* Since $p_{A_1 \vee A_2}$ is a *w.e.* it follows from M_5 that α is a *w.e.* Also $p_{A_1} \vee p_{A_2} = p_{A_1 \vee A_2} \circ \alpha$ together with (9) yields

$$(i_{B_1} \vee i_{B_2}) \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \circ \alpha \sim^l (i_{B_1} \vee i_{B_2}) \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2} \circ \alpha.$$

Both $Q(A_1) \vee Q(A_2)$ and $Q(A_1 \vee A_2)$ are cofibrant and α is a *w.e.* Hence Proposition 2.6 yields

$$(i_{B_1} \vee i_{B_2}) \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim^l (i_{B_1} \vee i_{B_2}) \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$$

From (i), Remark 2.2 we now get

$$i_{R(B_1) \vee R(B_2)} \circ (i_{B_1} \vee i_{B_2}) \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim^l i_{R(B_1) \vee R(B_2)} \circ (i_{B_1} \vee i_{B_2}) \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$$

Writing θ for the composite $i_{R(B_1) \vee R(B_2)}$ $(i_{B_1} \vee i_{B_2})$ the above relation can be written as

(10) $\quad \theta \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim^l \theta \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$

By axiom (W) $i_{B_1} \vee i_{B_2}$ is a *w.e.* Hence θ is also a *w.e.* by M_5.

Since $R(R(B_1) \vee R(B_2))$ is fibrant and

$$B_1 \vee B_2 \xrightarrow{B_1 \vee B_2} R(B_1 \vee B_2)$$

is a trivial cofibration it follows by axiom M_1 that there exists a map $\gamma : R(B_1 \vee B_2) \rightarrow R(R(B_1) \vee R(B_2))$ along the dotted arrow in Diagram 35 making it commutative.

$$\begin{tikzcd}
B_1 \vee B_2 \arrow[r, "\theta"] \arrow[d] & R(R(B_1) \vee R(B_2)) \arrow[d] \\
R(B_1 \vee B_2) \arrow[r] \arrow[ur, dashed, "\gamma"] & *
\end{tikzcd}$$

Diagram 35

Since θ and $i_{B_1 \vee B_2}$ are weak equivalences by M_5 we see that γ is a *w.e.* Since $\theta = \gamma\, i_{B_1 \vee B_2}$ relation (10) is the same as

$$\gamma \circ i_{B_1 \vee B_2} \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim^r \gamma \circ i_{B_1 \vee B_2} \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$$

Since $Q(A_1 \vee A_2)$ is cofibrant, Lemma 5.1, § 1, Chapter I of [**6**] gives

$$\gamma \circ i_{B_1 \vee B_2} \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim^r \gamma \circ i_{B_1 \vee B_2} \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$$

Both $R(B_1 \vee B_2)$ and $R(R(B_1) \vee R(B_2))$ are fibrant and γ a *w.e.* From Proposition 2.4 we now get

$$i_{B_1 \vee B_2} \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim^r i_{B_1 \vee B_2} \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$$

The fact that $Q(A_1 \vee A_2)$ is cofibrant and $R(B_1 \vee B_2)$ is fibrant now gives

$$i_{B_1 \vee B_2} \circ (f_1 \vee f_2) \circ p_{A_1 \vee A_2} \sim i_{B_1 \vee B_2} \circ (g_1 \vee g_2) \circ p_{A_1 \vee A_2}.$$

An application of Lemma 2.8 now yields

$$RQ(f_1 \vee f_2) \sim RQ(g_1 \vee g_2).$$

2.13. *Definition.* Let A, X be objects of $\mathscr{C}$. We say that X dominates A (or X dominates A in homotopy) if there exist maps $f : A \to X$, $g : X \to A$ such that $RQ(g \circ f) \sim 1_{RQ(A)}$.

Since $RQ(g \circ f) \sim RQ(g) \circ RQ(f)$ the above condition is equivalent to $RQ(g) \circ RQ(f) \sim 1_{RQ(A)}$. By Lemma 2.8, the condition $RQ(g \circ f) \sim 1_{RQ(A)}$ is equivalent to $i_A \circ (g \circ f) \circ p_A \sim i_A \circ p_A$. We write $X > A$ (or $A < X$) to denote that X dominates A. If $\mathscr{T}$ is the model category of topological spaces, every object in $\mathscr{T}$ is fibrant. CW-complexes are also cofibrant. When X and A are CW-complexes the concept of homotopy domination introduced here agrees with the classical concept introduced by J. H. C. Whitehead [**8**]. When X and A are not CW-complexes, in general the concept of homotopy domination introduced by us differs from the concept introduced by J. H. C. Whitehead. But it appears that Definition 2.13 is the best suited for our purposes.

2.14. PROPOSITION. *Let $X > A$ and $Y > B$. Then*
(i) $X \times Y > A \times B$
(ii) $X \vee Y > A \vee B$.

Proof. Let

$$A \xrightarrow{f} X \xrightarrow{g} A \quad \text{and} \quad B \xrightarrow{\theta} Y \xrightarrow{\varphi} B$$

be such that $RQ(g \circ f) \sim 1_{RQ(A)}$ and $RQ(\varphi \circ \theta) \sim 1_{RQ(B)}$.

Consider the diagrams

$$A \times B \xrightarrow{f \times \theta} X \times Y \xrightarrow{g \times \varphi} A \times B \text{ and}$$

$$A \times B \xrightarrow{f \vee \theta} X \times Y \xrightarrow{g \vee \varphi} A \times B.$$

Now, $1_{RQ(A)} \sim RQ(1_A)$ and $1_{RQ(B)} \sim RQ(1_B)$ and hence

$$RQ(g \circ f) \sim RQ(1_A), \quad RQ(\varphi \circ \theta) \sim RQ(1_B).$$

By Proposition 2.12 we get

$$RQ((g \circ f) \times (\varphi \circ \theta)) \sim RQ(1_A \times 1_B) = RQ(1_{A\times B}) \sim 1_{RQ(A\times B)} \quad \text{and}$$
$$RQ((g \circ f) \vee (\varphi \circ \theta)) \sim RQ(1_A \vee 1_B) = RQ(1_{A\vee B}) \sim 1_{RQ(A\vee B)}.$$

In other words,

$$RQ((g \times \varphi) \circ (f \times \theta)) \sim 1_{RQ(A\times B)} \quad \text{and}$$
$$RQ((g \vee \varphi) \circ (f \vee \theta)) \sim 1_{RQ(A\vee B)}.$$

This completes the proof of 2.14.

2.15. Proposition. *If $A > B$ and $B > C$ then $A > C$.*

Proof. Let

$$B \xrightarrow{f} A \xrightarrow{g} B \quad \text{and} \quad C \xrightarrow{\theta} B \xrightarrow{\varphi} C$$

be such that $RQ(g \circ f) \sim 1_{RQ(B)}$ and $RQ(\varphi \circ \theta) \sim 1_{RQ(C)}$. Consider

$$C \xrightarrow{f\theta} A \xrightarrow{\varphi g} C.$$

We have

$$\begin{aligned} RQ((\varphi g) \circ (f\theta)) &= RQ(\varphi \circ (g \circ f) \circ \theta) \\ &\sim RQ(\varphi) \circ RQ(g \circ f) \circ RQ(\theta) \\ &\sim RQ(\varphi) \circ RQ(\theta) \quad (\text{since } RQ(g \circ f) \sim 1_{RQ(B)}) \\ &\sim RQ(\theta \circ \varphi) \\ &\sim 1_{RQ(C)}. \end{aligned}$$

This proves $A > C$.

As in [**6**] we will write $[X, Y]$ for the set $\pi(RQ(X), RQ(Y))$ of homotopy classes of maps of $RQ(X)$ into $RQ(Y)$. Recall that a model category $\mathscr{C}$ is called a pointed model category if the initial object ϕ is isomorphic to the final object *. In particular one can take $\phi = *$. From now on we will be considering a pointed model category $\mathscr{C}$ satisfying axiom W in addition to the axioms $M_1, M_2, \ldots, M_5$ of Quillen.

2.16. *Definition.* An object X of $\mathscr{C}$ is said to be contractible if $[X, X] = 0$.

Given any map $\alpha : RQ(X) \to RQ(Y)$ we denote the homotopy class of α by $[\alpha]$. Given any $f : X \to Y$ we denote the homotopy class of $RQ(f) : RQ(X) \to RQ(Y)$ by $\langle f \rangle$. Clearly $\langle 1_X \rangle = [1_{RQ(X)}]$. The following are trivial to see.

(i) X is contractible if and only if $\langle 1_X \rangle = 0$.

(ii) X is contractible $\Leftrightarrow [X, Y] = 0$ (respectively $[Y, X] = 0$) for every $Y \in \mathscr{C}$.

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(iii) X is contractible $\Leftrightarrow X$ is dominated by $*$.

Let $H \circ \mathscr{C}$ denote the homotopy category of $\mathscr{C}$. In § 2, Chapter I of [**6**] a functor from $(H \circ \mathscr{C})^0 \times (H \circ \mathscr{C})$ to the category of groups, denoted, by $[,]_1$ is constructed and further it is proved that there exist functors Σ (called the suspension functor) and Ω (called the loop functor) from $H \circ \mathscr{C}$ to $H \circ \mathscr{C}$ and canonical isomorphisms $[\Sigma A, B] \simeq [A, B]_1 \simeq [A, \Omega B]$.

2.17. LEMMA. *If A is contractible so are ΣA and ΩA.*

Proof.

$$\begin{aligned} A \text{ contractible} &\Rightarrow [A, B] = 0 \quad \text{for all } B \in \mathscr{C} \\ &\Rightarrow [A, \Omega C] = 0 \quad \text{for all } C \in \mathscr{C} \\ &\Rightarrow [\Sigma A, C] = 0 \quad \text{for all } C \in \mathscr{C} \\ &\Rightarrow \Sigma A \text{ contractible.} \end{aligned}$$

Similarly,

$$\begin{aligned} A \text{ contractible} &\Rightarrow [B, A] = 0 \quad \text{for all } B \in \mathscr{C} \\ &\Rightarrow [\Sigma C, A] = 0 \quad \text{for all } C \in \mathscr{C} \\ &\Rightarrow [C, \Omega A] = 0 \quad \text{for all } C \in \mathscr{C} \\ &\Rightarrow \Omega A \text{ contractible.} \end{aligned}$$

Let $A \in \mathscr{C}_c$ (i.e. A is a cofibrant object in $\mathscr{C}$) and $A \times I$ any cylinder object for A. Let

$$A \vee A \xrightarrow{\partial_0 + \partial_1} A \times I \xrightarrow{\quad \sigma \quad} A$$

be such that $\partial_0 + \partial_1$ is a cofibration, σ a $w.e$ and $\sigma \circ (\partial_0 + \partial_1) = \nabla_A$. Since A is cofibrant the maps $\partial_0 : A \to A \times I$ and $\partial_1 : A \to A \times I$ are cofibrations. The cofibre CA of ∂_0 will be called a cone object for A. Let $v : A \times I \to CA$ be the natural map. Then by the very definition of the cofibre of ∂_0 Diagram 36 below is a push-out diagram.

$$\begin{array}{ccc} A & \longrightarrow & * \\ \Big\downarrow{\scriptstyle \partial_0} & & \Big\downarrow \\ A \times I & \xrightarrow{\quad v \quad} & CA \end{array}$$

Diagram 36

By axiom M_2, $\sigma \to CA$ is a cofibration. Thus $CA \in \mathscr{C}_c$.

2.18. LEMMA. *For any $A \in \mathscr{C}_c$ any cone object CA of A is contractible.*

Proof. The Puppe exact sequence corresponding to the cofibration sequence

$$A \xrightarrow{\partial_0} A \times I \xrightarrow{v} CA$$

yields the following exact sequence

$$[\Sigma(A \times I), B] \xrightarrow{(\Sigma\partial_0)^*} [\Sigma A, B] \xrightarrow{\partial} [CA, B] \xrightarrow{\nu^*} [A \times I, B] \xrightarrow{\partial_0^*} [A, B]$$

for every $B \in \mathscr{C}$. Since ∂_0 is a *w.e.* it follows that ∂_0^* and $(\Sigma\partial_0)^*$ are isomorphisms ($(\Sigma\partial_0)^*$ is an isomorphism of groups and ∂_0^* is an isomorphism of pointed sets). By a standard argument we get $[CA, B] = 0$. Here B is an arbitrary object of $\mathscr{C}$. Hence CA is contractible.

3. Lusternik-Schnirelmann category and cocategory. From now on unless otherwise mentioned $\mathscr{C}$ will denote a pointed model category in the sense of Quillen satisfying further axioms A_1, A_2, A_3, A_4 mentioned in § 1, axiom W mentioned in § 2 and axiom A_5 below.

A_5. Let $f : X \to Y$ be any map in $\mathscr{C}$, (E_k, i_k) $(k = 1, 2, \ldots, r)$ a finite number of subobjects of Y and (E, i) = the union of the subobjects (E_k, i_k) $(k = 1, 2, \ldots, r)$, which is well-defined up to an isomorphism as a subobject of Y because of Remark 1.6 and Proposition 1.7.

Let (F_k, j_k) be the inverse image of (E_k, i_k) by f and (F, j) = the union of (F_k, j_k) $(k = 1, 2, \ldots, r)$. Under these conditions axiom A_5 states that (F, j) is the inverse image of (E, i) by f.

All the three categories mentioned in the introduction do satisfy all these axioms.

For any integer $k \geqq 0$ the diagonal map $X \to X^{k+1}$ will be denoted by $\Delta_{k+1,X}$ (on Δ_{k+1} when there is no possibility of confusion). Let $E_{i,k+1}$ for $1 \leqq i \leqq k + 1$ be defined by $E_{i,k+1} = X \times \ldots \times X \times * \times X \times \ldots \times X$ with $*$ at the ith place and X at other places. Let $\mu_{i,k+1,X}$ (or $\mu_{i,k+1}$) be the map of $E_{i,k+1}$ to X^{k+1} given by

$$\mu_{i,k+1} = 1_X \times \ldots \times 1_X \times 0 \times 1_X \times \ldots \times 1_X$$

with 0 at the i-th place and 1_X at all other places. Then $(E_{i,k+1}, \mu_{i+k,1})$ is a subobject of X^{k+1}. Let $(T^k(X), j_{k,X})$ {or $(T^k(X), j)$ when there is no possibility of confusion} be the union of $(E_{i,k+1}, \mu_{i+k,1})$ for $1 \leqq i \leqq k + 1$. Motivated by G. W. Whitehead's definition [7] we introduce the notion of Lusternik-Schnirelmann Category of an object X, denoted by Cat X (or W- cat X) in the following way. (W stands for Whitehead.)

3.1. *Definition.* Let k be any integer $\geqq 0$. We say that Cat $X \leqq k$ if there exists a map $\varphi : X \to T^k(X)$ satisfying $RQ(\Delta_{k+1}) \sim RQ(j \circ \varphi)$.

3.2. LEMMA. Cat $X \leqq k \Leftrightarrow$ *there exist maps* $\alpha_i : X \to X$ *for* $1 \leqq i \leqq k + 1$ *satisfying*

(i) $RQ(\alpha_i) \sim 1_{RQ(X)}$;

(ii) $(\alpha_1, \ldots, \alpha_{k+1}) : X \to X^{k+1}$ *can be written as* $j \circ \varphi$ *for some* $\varphi : X \to T^k(X)$.

Proof. Let Cat $X \leqq k$. Then there exists a map $\varphi : X \to T^k(X)$ such that $RQ(\Delta_{k+1}) \sim RQ(j \circ \varphi)$. Let $\alpha_i = p_i\ (j \circ \varphi) : X\ {}^k\!\!\to X$ where $p_i : X^{k+1} \to X$ is the i-th projection. Then

$$\begin{aligned} RQ(\alpha_i) = RQ(p_i \circ (j \circ \varphi)) &\sim RQ(p_i) \circ RQ(j \circ \varphi) \\ &\sim RQ(p_i) \circ RQ(\Delta_{k+1}) \\ &\sim RQ(p_i \circ \Delta_{k+1}) \\ &\sim RQ(1_X) \\ &\sim 1_{RQ(X)}. \end{aligned}$$

Clearly $\alpha_1, \ldots, \alpha_{k+1}) = j \circ \varphi$.

Conversely, assume there exist maps $\alpha_i : X \to X$ with $RQ(\alpha_i) \sim 1_{RQ(X)}$ and $(\alpha_1, \ldots, \alpha_{k+1}) = j \circ \varphi$ for some $\varphi : X \to T^k(X)$. By Proposition 2.12 (a) we get $RQ(\alpha_1 \times \ldots \times \alpha_{k+1}) \sim RQ(1_X \times \ldots \times 1_X) = RQ(1_{X^{k+1}})$. Hence

$$\begin{aligned} RQ(\alpha_1, \ldots, \alpha_{k+1}) &= RQ((\alpha_1 \times \ldots \times \alpha_{k+1}) \circ \Delta_{k+1}) \\ &\sim RQ(\alpha_1 \times \ldots \times \alpha_{k+1})\ RQ(\Delta_{k+1}) \\ &\sim RQ(\Delta_{k+1}). \end{aligned}$$

This shows that Cat $X \leqq k$.

To make sure that Definition 3.1 makes sense we have to prove the following:

3.3. Lemma. Cat $X \leqq k \Rightarrow$ Cat $X \leqq k + 1$.

Proof. Let $\alpha_i : X \to X$ $(1 \leqq i \leqq k + 1)$ be maps satisfying

(i) $RQ(\alpha_i) \sim 1_{RQ(X)}$ and

(ii) there exists a map $\varphi : X \to T^k(X)$ such that $(\alpha_1, \ldots, \ldots, \alpha_{k+1}) = j_k \circ \varphi$ where we write j_k for $j_{k,X}$.

Let $\lambda_i : E_{i,k+1} \to T^k(X)$ $(1 \leqq i \leqq k + 1)$ be the maps satisfying $j_k \circ \lambda_i = \mu_{i,k+1}$. By Lemma 1.1 the λ_i's are monic. Let (F_i, l_i) be the inverse image of $(E_{i,k+1}, \lambda_i)$ by φ. Then by axiom A_5 the union of (F_i, l_i) for $1 \leqq i \leqq k + 1$ is X. Let $\theta_i : F_i \to E_{i,k+1}$ satisfy $\lambda_i \circ \theta_i = \varphi \circ l_i$. Then $\mu_{i,k+1} \circ \theta_i = j_k \circ \lambda_i \circ \theta_i = j_k \circ \varphi \circ l_i$. Writing ψ for the map $(\alpha_1, \ldots, \alpha_{k+1})$ we see that $\mu_{i,k+1} \circ \theta = \psi \circ l_i$. Define $\alpha_{k+1} = 1_X$. Then $RQ(\alpha_{k+2}) \sim 1_{RQ(X)}$ clearly $E_{i,k+1} \times X = E_{i,r+2}$ and $\mu_{i,k+1} \times 1_X = \mu_{i,r+2}$. If $\epsilon_i : F_i \to E_{i,k+1} \times X$ is the map given by $\epsilon_i = (\theta_i, l_i)$, then $\mu_{i,k+2}\epsilon_i = (\psi, \alpha_{k+2}) \circ l_i$. Hence by Proposition 1.12, there exists a map

$$\epsilon : X \to \Gamma$$

where (Γ, ν) is the union of $(E_{i,k+2}, \mu_{i,k+2})$ for $1 \leqq i \leqq k + 1$ satisfying

$$\nu \circ \epsilon = (\psi, \alpha_{k+2}) = (\alpha_1, \alpha_2, \ldots, \alpha_{k+2}).$$

However $T^{k+1}(X) =$ the union of $E_{i,k+2}$ for $1 \leqq i \leqq k + 2$. Hence there exists a map $\lambda : \Gamma \to T_k{}^{+1}(X)$ satisfying $j_{k+1} \circ \lambda = \nu$. Then $\nu \circ \epsilon = j_{k+1} \circ \lambda \circ \epsilon$. Thus if we define $\bar{\varphi} : X \to T^{k+1}(X)$ by $\bar{\varphi} = \lambda \circ \epsilon$ then

$$(\alpha_1, \alpha_2, \ldots, \alpha_{k+2}) = j_{k+1} \circ \bar{\varphi}.$$

Hence Cat. $X \leqq k + 1$.

3.4. *Definition.* The category of an object X of $\mathscr{C}$ is defined to be k if Cat $X \leqq k$ and it is not true that Cat. $X \leqq k - 1$.

It is clear from the definition that Cat. $X = 0 \Leftrightarrow X$ is contractible.

3.5. LEMMA. *If* $X > A$ *then* Cat $A \leqq$ Cat X.

Proof. Suppose Cat $X \leqq k$. Let

$$A \xrightarrow{f} X \xrightarrow{g} A$$

be such that $RQ(g \circ f) \sim 1_{RQ(A)}$ and $\alpha_i : X \to X$ $(1 \leqq i \leqq k + 1)$ satisfy

(i) $RQ(\alpha_i) \sim 1_{RQ(X)}$ and

(ii) there exists $\varphi : X \to T^i(X)$ with $j_{k,X} \circ \varphi = (\alpha_1, \ldots, \alpha_{k+1})$.

Let $\beta_i : A \to A$ be given by $\beta_i = g \circ \alpha_i \circ f$. Then

$$R\beta_{Bi}) \sim RQ(g) \circ RQ(\alpha_i) \circ RQ(f) \sim RQ(g) \circ RQ(f) \sim 1_{RQ(A)}.$$

Moreover it is clear from Proposition 1.12 that there exists a map $\theta : T^k(X) \to T^k(A)$ satisfying

$$j_{k,A} \circ \theta = (g \times \quad \times g) \circ j_{k,X} \quad (k + 1 \text{ factors } g).$$

It is now easy to see that $j_{k,A} \circ (\theta \circ \varphi \circ f) = (\beta_1, \ldots, \beta_{k+1})$. Hence Cat $A \leqq k$.

3.6. COROLLARY. *For any two objects* X, Y *in*, Cat $X \leqq$ Cat $(X \times Y)$.

Proof. This is because $X \times Y > X$.

3.7. *Definition.* Two objects X and Y of $\mathscr{C}$ are defined to be of the same "homotopy type" if there exist maps $f : X \to Y$, $g : Y \to X$ satisfying $RQ(g \circ f) \sim 1_{RQ(X)}$. $RQ(f \circ g) \sim 1_{RQ(Y)}$.

3.8. *Remarks.* (a) If X and Y are of the same homotopy type with $f : X \to Y$, $g : Y \to X$ satisfying $RQ(g \circ f) \sim 1_{RQ(X)}$, $RQ(f \circ g) \sim 1_{RQ(Y)}$ then $\gamma(f) : X \to Y$ is an isomorphism in $H \circ \mathscr{C}$. However X, Y being isomorphic in $H \circ \mathscr{C}$ in general does not imply that X, Y are of the same homotopy type in the above sense.

(b) The usual definition of homotopy type in the category of based topological spaces differs from the above definition given by us. But for our purposes Definition 3.6 appears to be the best suited.

(c) Suppose $X \in \mathscr{C}_{cf}$. Then clearly X^{k+1} is fibrant. In this case W-Cat $X \leqq k \Leftrightarrow$ there exists a map $\varphi : X \to T^k(X)$ such that $\Delta_{k+1} \sim j_k \circ \varphi$. Actually in this case $Q(X^{k+1})$ is also fibrant and hence $RQ(X^{k+1}) = Q(X^{k+1})$. From the definition of W-Cat X we have W-Cat $X \leqq k \Leftrightarrow$ there exists a map $\varphi : X \to T^k(X)$ such that $RQ(j_k \circ \varphi) \sim RQ(\Delta_{k+1})$. Since $RQ(X^{k+1}) = Q(X^{k+1})$ this condition is equivalent to $Q(j_k \circ \varphi) \sim Q(\Delta_{k+1})$. Since X is cofibrant and both $Q(X^{k+1})$ and X^{k+1} are fibrant and $p_{X^{k+1}} : Q(X^{k+1}) \to X^{k+1}$ is a trivial fibration it follows from Lemma 7, § 1, Chapter I of [**6**] that $Q(j_k \circ \varphi) \sim Q(\Delta_{k+1}) \Leftrightarrow j_k \circ \varphi \sim \Delta_{k+1}$.

3.9. PROPOSITION. *If X and Y are of the same homotopy type then* Cat X = Cat Y.

Proof. This follows immediately from Lemma 3.5.

Following Ganea [**1**; **2**; **3**] we would like to introduce the concept of inductive category of an object $X \in \mathscr{C}$. Before doing this we will recall certain facts. Let A, $Y \in \mathscr{C}_c$ (i.e. A and Y are cofibrant) and $f : A \to Y$ a cofibration. Let $u : Y \to C$ denote the cofibre of f. By the very definition of the cofibre of f Diagram 37 below is a push-out.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \arrow[d, "u"] \\ * \arrow[r] & C \end{tikzcd}$$

Diagram 37

By axiom M_3, $* \to C$ is a cofibration. Thus $C \in \mathscr{C}_c$. Let Diagram 38 below

$$\begin{tikzcd} A \vee A \arrow[r] \arrow[d, "\partial_0 + \partial_1"'] \arrow[dr, phantom, "(a)"] & A \arrow[r, "f"] \arrow[d, "\nu"] \arrow[dr, phantom, "(b)"] & Y \arrow[r] \arrow[d, "\mu"] \arrow[dr, phantom, "(c)"] & * \arrow[d] \\ A \times I \arrow[r, "v"] & L \arrow[r, "j"] & M \arrow[r, "\eta"] & N \end{tikzcd}$$

Diagram 38

be such that each of the squares (a), (b), (c) is a push-out, where $A \times I$ is a given cylinder object for A.

Since

$$\begin{tikzcd} A \arrow[r] \arrow[d, "in_1"'] & * \arrow[d] \\ A \vee A \arrow[r, "0 + 1"] & A \end{tikzcd}$$

Diagram 39

is a push-out diagram it follows (from the fact that (a) is push-out) that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\partial_0"] & * \arrow[d] \\ A \times I \arrow[r, "v"] & L \end{tikzcd}$$

Diagram 40

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is a push-out. Hence L is a cone object CA for A. Also

$$\begin{tikzcd} A \vee A \arrow[r] \arrow[d, "\partial_0 + \partial_1"'] & * \arrow[d] \\ A \times I \arrow[r] & N \end{tikzcd}$$

Diagram 41

is a push-out. Hence N is a suspension object ΣA for A. We denote M by $Y \cup_f CA$. We will refer to $Y \cup_f CA$ as got from Y by attaching a cone object of A by means of f. Now, Diagram 38 can be written as follows:

$$\begin{tikzcd} A \vee A \arrow[r, "0+1"] \arrow[d, "\partial_0 + \partial_1"] \arrow[dr, phantom, "\text{(a)}"] & A \arrow[r, "f"] \arrow[d, "\nu"] \arrow[dr, phantom, "\text{(b)}"] & Y \arrow[r] \arrow[d, "\mu"] \arrow[dr, phantom, "\text{(c)}"] & * \arrow[d] \\ A \times I \arrow[r, "v"] & CA \arrow[r, "j"] & Y \cup_f CA \arrow[r, "\eta"] & \Sigma A \end{tikzcd}$$

Diagram 38′

The fact that (b) is a push-out gives a unique map $\varphi : Y \cup_f CA \to C$ making Diagram 42 below commutative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\nu"'] & Y \arrow[d, "\mu"'] \arrow[ddr, "u"] & \\ CA \arrow[r, "j"] \arrow[drr, "O"'] & Y \cup_f CA \arrow[dr, "\varphi"] & \\ & & C \end{tikzcd}$$

Diagram 42

If we write $\langle \varphi \rangle$ for the homotopy class of $RQ(\varphi)$ in $\mathrm{Hom}_{H0\mathscr{C}}(Y \cup_f CA, C) = \pi(RQ(Y \cup_f CA), RQ(C)) = [Y \cup_f CA, C]$, then the dual of Proposition 3 in § 3, Chapter I of [**6**] can be summarized as follows:

Let $\partial : C \to \Sigma A$ denote the composition of the maps

$$C \xrightarrow{n} C \vee \Sigma A \xrightarrow{0+1} \Sigma A$$

where $n : C \to C \vee \Sigma A$ is the right co-action of ΣA on C constructed in § 3, Chapter I of [**6**]. Then

$$\begin{tikzcd}
Y \arrow[r, "\langle u \rangle"] \arrow[d, equal] & C \arrow[r, "\langle \partial \rangle"] & A \arrow[d, equal] \\
Y \arrow[r, "\langle \mu \rangle"] & Y \cup_f CA \arrow[u, dash, "\langle \varphi \rangle"'] \arrow[r, "\langle \eta \rangle"] & A
\end{tikzcd}$$

Diagram 43

is a commutative diagram in $H \circ \mathscr{C}$ with $\langle \varphi \rangle$ an isomorphism in $H \circ \mathscr{C}$.

The inductive category of an object X in $\mathscr{C}$, denoted by Ind Cat X is defined inductively as follows.

3.10. *Definition.* Ind Cat $X = 0$ if and only if X is contractible. Ind Cat $X \leqq k$ if there exists a cofibration $A \xrightarrow{f} Y$ with A, $Y \in \mathscr{C}_c$ satisfying the following conditions:

(i) Ind Cat $Y \leqq k - 1$ and

(ii) The cofibre C of f dominates X.

3.11. *Remarks.* (1) Since (a) and (b) are push-outs in Diagram 38′ it follows that ν and μ are cofibrations. Since A, Y lie in $\mathscr{C}_c$ it follows that CA and $Y \cup_f CA$ also lie in $\mathscr{C}_c$.

(2) Suppose in the category $\mathscr{C}$ every object is fibrant. Then $\varphi : Y \cup_f CA \to C$ is a homotopy equivalence. In fact, in this case $Y \cup_f CA$ and C both lie in $\mathscr{C}_{cf}$. Hence $\langle \varphi \rangle^{-1}$ is represented by a map $\theta : C \to Y \cup_f CA$. Then $\theta \circ \varphi = RQ(\theta \circ \varphi) \sim 1_{Y \cup_f CA}$ and $\varphi \circ \theta = RQ(\varphi \circ \theta) \sim 1_C$. Also in this case, condition (ii) in Definition 3.10 can be replaced by (ii)′ below:

(ii)′ $Y \cup_f CA$ dominates X.

Incidentally, it also follows in this case that (ii)′ is independent of the cylinder object $A \times I$ chosen.

(3) Let $\mathscr{T}_*$ denote the pointed model category of based topological spaces and $C_+(\wedge)$ the category of chain complexes over $\wedge$ which are bounded below, where $\wedge$ is a given ring. The zero chain complex is the initial and the final object in $C_+(\wedge)$. Serre fibrations are by definition the fibrations in $\mathscr{T}_*$ and epimorphisms are by definition the fibrations in $C_+(\wedge)$ [**6**]. Thus in $\mathscr{T}_*$ and $C_+(\wedge)$ all the objects are fibrant.

In the category of based semi-simplicial complexes the fibrations are by definition those maps which are also Kan fibrations. Thus not all objects are fibrant in this category.

3.12. LEMMA. *If* $X > Z$ *then* Ind Cat $Z \leqq$ Ind Cat X.

Proof. This is an immediate consequence of Proposition 2.15.

3.13. PROPOSITION. *If in the category* $\mathscr{C}$ *every object is fibrant then* W-Cat $X \leqq$ Ind Cat X *for all* $X \in \mathscr{C}$.

Proof. The proposition is trivially true when Ind Cat $X = 0$. Assume the proposition valid whenever Ind Cat $X \leqq k - 1$.

Let Ind Cat $X = k$. Then there exist a cofibration $f : A \to Y$ with $A, Y \in \mathscr{C}_c$, Ind Cat $Y \leqq k - 1$ and $Y \cup_f CA > X$.

By the inductive assumption W-Cat $Y \leqq k - 1$. Hence there exist maps $\alpha_i : Y \to Y$ for $1 \leqq i \leqq k$ satisfying (i) and (ii) below:

(i) $\alpha_i \sim 1_Y$ (observe that $RQ(Y) = Y$)

(ii) there exists a map $\varphi : Y \to T^{k-1}(Y)$ satisfying $j_{k-1,Y} \circ \varphi = (\alpha_1, \ldots, \alpha_k)$.

Since $\mu : Y \to Y \cup_f CA$ is a cofibration and $Y \cup_f CA$ is fibrant, by the homotopy extension theorem (dual to the corollary of Lemma 2, § 1, Chapter I of [**6**]) we get maps $\beta_i : Y \cup_f CA \to Y \cup_f CA$ for $1 \leqq i \leqq k$ satisfying (iii) and (iv) below:

(iii) $\beta_i \sim 1_{Y \cup_f CA}$

(iv) $\beta_i \circ \mu = \mu \circ \alpha_i$.

Also $j : CA \to Y \cup_f CA$ is a cofibration (since (b) is a push-out in Diagram 38′). Since CA is contractible we have $j \sim 0$. By the homotopy extension theorem there exists a map $\beta : Y \cup_f CA \to Y \cup_f CA$ such that $\beta \sim 1_{Y \cup_f CA}$ and $\beta \circ j = 0$.

Define $\beta_{k+1} = \beta$. Consider the map

$$(\beta_1, \ldots, \beta_{k+1}) : Y \cup_f CA \to (Y \cup_f CA)^{k+1}.$$

Since $\beta_{k+1} \circ j = \beta \circ j = 0$ it follows that there exists a map $\theta : CA \to T^k(Y \cup_f CA)$ such that Diagram 44 below is commutative.

$$\begin{tikzcd}
CA \arrow[rr, "{(\beta_1 \circ j, \ldots, \beta_{k+1} \circ j)}"] \arrow[drr, "\theta"'] & & (Y \cup_f CA)^{k+1} \\
& & T^k(Y \cup_f CA) \arrow[u, "{j_{k,\, Y \cup_f CA}}"']
\end{tikzcd}$$

Diagram 44

Also $(\beta_1 \circ \mu, \ldots, \beta_{k+1} \circ \mu) : Y \cup_f CA \to (Y \cup_f CA)^{k+1}$ is the same as the composition of the maps

$$Y \xrightarrow{(\alpha_1, \ldots, \alpha_k, 1_Y)} Y^{k+1} \xrightarrow{\mu \times \ldots \times \mu \times (\beta_{k+1} \circ \mu)} (Y \cup_f CA)^{k+1}.$$

From the proof of Lemma 3.3 it follows that there exists a map $\varphi : Y \to T^k(Y)$ such that

$$(\alpha_1, \ldots, \alpha_k, 1_Y) = j_{k,Y} \circ \varphi.$$

Also from Proposition 1.12 it follows immediately that there exists a map $\theta : T^k(Y) \to T^k(Y \cup_f CA)$ such that $\{\mu \times \ldots \times \mu \times (\beta_{k+1} \circ \mu)\} \circ j_{k,Y} = j_{k,Y \cup_f CA} \circ \bar{\theta}$. In other words, Diagram 45 below is commutative. Define $\theta' = \bar{\theta} \circ \varphi$. Then $j_{k,Y \cup_f CA} \circ \theta' = (\beta_1 \circ \mu, \ldots, \beta_{k+1} \circ \mu)$.

$$\begin{tikzcd}
 & Y \arrow[dl, "\theta'"'] \arrow[d, "\varphi"] \arrow[dr, "{(\alpha_1, \ldots, \alpha_k, 1_Y)}"] \arrow[drr, "{(\beta_1 \circ \mu, \ldots, \beta_{k+1} \circ \mu)}"] & & \\
 & T^k(Y) \arrow[r, "j_{k,Y}"'] \arrow[d, "\theta"] & Y^{k+1} \arrow[d, "\mu \times \ldots \times \mu \times \beta_{k+1} \circ \mu"] & (Y \cup_f CA)^{k+1} \\
T^k(Y \cup_f CA) \arrow[r, equal] & T^k(Y \cup_f CA) \arrow[r, "j_{k,Y \cup_f CA}"'] & (Y \cup_f CA)^{k+1} \arrow[ur, equal] &
\end{tikzcd}$$

Diagram 45

Let the inverse image of $(T^k(Y \cup_f CA), j_{k,Y \cup_f CA})$ by the map

$$(\beta_1, \ldots, \beta_{k+1}) : Y \cup_f CA \to (Y \cup_f CA)^{k+1}$$

be denoted by (Γ, i). Since

$$(\beta_1, \ldots, \ldots, \beta_{k+1}) \circ \mu = (\beta_1 \circ \mu, \ldots, \beta_{k+1} \circ \mu) = j_{k,Y \cup_f CA} \circ \theta'$$

it follows that the inverse image of

$$(T^k(Y \cup_f CA), j_{k,Y \cup_f CA}) \text{ by } (\beta_1, \ldots, \beta_{k+1}) \circ \mu$$

is $(Y, 1_Y)$. Hence by Lemma 1.13, the inverse image of (Γ, i) by μ is $(Y, 1_Y)$. Similarly it follows that the inverse image of (Γ, i) by $j : CA \to Y \cup_f CA$ is $(CA, 1_{CA})$. It follows that there exists maps $\epsilon_1 : Y \to \Gamma$, $\epsilon_2 : CA \to \Gamma$ such that $i \circ \epsilon_1 = \mu$ and $i \circ \epsilon_2 = j$.

Let $(L_1, i_1) = \operatorname{Im} \mu$ and $(L_2, i_2) = \operatorname{Im} j$. Because of the existence of ϵ_1 and ϵ_2 satisfying $i \circ \epsilon_1 = \mu$ (respectively $i \circ \epsilon_2 = j$) it follows that there exists a map $\lambda_1 : L_1 \to \Gamma$ (respectively $\lambda_2 : L_2 \to \Gamma$) satisfying $i \circ \lambda_1 = i_1$ (respectively

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$i \circ \lambda_2 = i_2$). Since

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\nu"'] & Y \arrow[d, "\mu"] \\ CA \arrow[r, "j"] & Y \cup_f CA \end{tikzcd}$$

Diagram 46

is a push-out, it follows from Theorem 1.8 that $(Y \cup_f CA, 1_{Y \cup_f CA}) = (L_1, i_1) \cup (L_2, i_2)$. (up to an isomorphism). The existence of λ_1 and λ_2 satisfying $i \circ \lambda_1 = i_1, i \circ \lambda_2 = i_2$ now gives a map $\lambda : Y \cup_f CA \to \Gamma$ such that

$$\begin{tikzcd} Y \cup_f CA \arrow[r, "\lambda"] \arrow[dr, equal] & \Gamma \arrow[d, "i"] \\ & Y \cup_f CA \end{tikzcd}$$

Diagram 47

is commutative. Thus $i \circ \lambda = 1_{Y \cup_f CA}$. Also $i \circ (\lambda \circ i) = (i \circ \lambda) \circ i = (1_{Y \cup_f CA}) \circ i = i = i \circ 1_\Gamma$. Since i is injective $\lambda \circ i = 1_\Gamma$. Thus $i : \Gamma \to Y \cup_f CA$ is an isomorphism. This shows that the inverse image of $(T^k(Y \cup_f CA), j_{k, Y \cup_f CA})$ by $(\beta_1, \ldots, \beta_{k+1})$ is $(Y \cup_f CA, 1_{Y \cup_f CA})$. Hence there exists a map $\gamma : Y \cup_f CA \to T^k(Y \cup_f CA)$ satisfying $(\beta_1, \ldots, \beta_{k+1}) = j_{k, Y \cup_f CA}$. Thus W-Cat $Y \cup_f CA \leqq k$. Since $Y \cup_f CA$ dominates X it follows from Lemma 3.5 that W-Cat $X \leqq k$. This completes the proof of Proposition 3.13.

3.14. LEMMA. Ind Cat $\Sigma X \leqq 1$ *for any* $X \in \mathscr{C}_c$.

Proof. $\nu : A \to CA$ is a cofibration with cofibre ΣA (refer to Diagram 38′). Since CA is contractible we get Ind Cat $\Sigma A \leqq 1$.

The definition of inductive cocategory (or simply the cocategory) of an object X in $\mathscr{C}$ is exactly dual to the definition of inductive category. The cocategory of X denoted by Cocat X is defined as follows:

3.15. *Definition.* Cocat $X = 0$ *if and only if X is contractible.* Cocat $X \leqq k$ *if there exists a fibration $E^p \to B$ with E, B in $\mathscr{C}_f$ satisfying the following conditions*
(i) Cocat $E \leqq k - 1$;
(ii) $F > X$ *where* $i : F \to E$ *is the fibre of* p.

3.16. LEMMA. *If $X > Z$ then* Cocat $Z \leqq$ Cocat X. *If X and Y are of the same homotopy type then* Cocat $X =$ Cocat Y.

Proof. This is immediate consequence of Proposition 2.15.

3.17. *Remark.* For defining the invariants Ind Cat X and Co-cat X for an object X in $\mathscr{C}$ it is not necessary that the pointed model category $\mathscr{C}$ satisfy axioms A_1 to A_5 introduced by us and 3.10, 3.12, 3.15 and 3.16 are all valid for pointed model categories satisfying axiom W.

4. The invariants Nil X **and** Conil X. In this section $\mathscr{C}$ will denote a pointed model category in the sense of Quillen. We do not assume anything more ($\mathscr{C}$ need not satisfy axioms A_1 to A_5 and W). Following Ganea we will introduce two invariants nil X and conil X. These will be invariants associated to an object X in $H \circ \mathscr{C}$ and will depend only on the isomorphism class of X in $H \circ \mathscr{C}$. We denote the product of two objects A and B in $H \circ \mathscr{C}$ by $A \times_H B$. All the maps and diagrams in this section will be in $H \circ \mathscr{C}$.

For any $X \in H \circ \mathscr{C}$ it is known that ΩX is a group object in $H \circ \mathscr{C}$. For any object A in $H \circ \mathscr{C}$ we write A^2 for $A \times_H A$ and for any map $f : A \to B$ in $H \circ \mathscr{C}$ we write f^2 for $f \times_H f$. Let $\varphi : (\Omega X)^2 \to \Omega X$ be the composition of the maps

$$(\Omega X)^2 \xrightarrow{\Delta(\Omega X)^2} (\Omega X)^2 \times (\Omega X)^2 \xrightarrow{1^2 \times \nu^2} (\Omega X)^2 \times_H (\Omega X)^2$$
$$\xrightarrow{\mu \times \mu} \Omega X \times_H \Omega X \xrightarrow{\quad \mu \quad} \Omega X$$

where $\mu : \Omega X \times \Omega X \to \Omega X$ is the "multiplication" and $\nu : \Omega X \to \Omega X$ the "inversion" under which ΩX is a group object in $H \circ \mathscr{C}$.

Write 0×1 for the map

$$\Omega X \times_H \Omega X \xrightarrow{0 \times 1} \Omega X \times_H \Omega X = (\Omega X)^2$$

which is 0 on the first factor and $1_{\Omega X}$ on the second factor.

4.1. LEMMA. $\varphi \circ (0 \times 1) = 0\ [\Omega X \times_H \Omega X, \Omega X]$.

Proof. Since $\Delta_{(\Omega X)} = (1^2, 1^2)$ where $1 = 1_{\Omega X}$ we have

$$\varphi = \mu \circ (\mu \times \mu) \circ (1^2 \times \nu^2) \circ (1^2, 1^2).$$

Also

$$\begin{aligned}(\mu \times \mu) &\circ (1^2 \times \nu^2) \circ (1^2, 1^2) \circ (0 \times 1)\\ &= (\mu \circ (1 \times 1) \circ (1 \times 1) \circ (0 \times 1),\\ &\qquad\qquad\qquad \mu \circ (\nu \times \nu) \circ (1 \times 1) \circ (0 \times 1))\\ &= (\mu \circ (0 \times 1), \mu \circ (0 \times \nu)).\end{aligned}$$

$\mu \circ (0 \times 1) : \Omega X \times_H \Omega X \to \Omega X$ is the same as p_2, the second projection in

$$[\Omega X \times \Omega X, \Omega X] = \mathrm{Hom}_{H \circ \mathscr{C}}(\Omega X \times_H \Omega X, \Omega X)$$

and

$$\mu \circ (0 \times \nu) = \mu \circ (0 \times 1) \circ (1 \times \nu) = p_2 \circ (1 \times \nu) = \nu \circ p_2.$$

Hence $\varphi \circ (0 \times 1) = \mu \circ (p_2, \nu \circ p_2) = \mu \circ (1, \times) \circ p_2$. But $\mu \circ (1, \nu) = 0$ (μ is the "multiplication" in ΩX and ν the inversion and 0 the neutral element for the group object ΩX). Hence $\varphi \circ (0 \times 1) = 0$.

The commutator map $\varphi_k : (\Omega X)^k \to \Omega X$ of weight k (for any integer $k \geqq 1$) is defined by induction on k as follows.

4.2. *Definition.* The commutator map φ_1 of weight 1 is defined to be the identity map $1_{\Omega X}$ of ΩX. For $k \geqq 2$ the commutator map φ_k of weight k is the composite

$$(\Omega X) = (\Omega X)^{k-1} \times \Omega X \xrightarrow{\varphi_{k-1} \times 1} \Omega X \times \Omega X \xrightarrow{\quad \varphi \quad} \Omega X$$

where φ is the map in (11).

4.3. LEMMA. *If* $\varphi_k = 0$ *in* $[(\Omega X)^k, \Omega X]$ *then* $\varphi_{k+1} = 0$ *in* $[(\Omega X)^{k+1}, \Omega X]$.

Proof. When $\varphi_k = 0$ we have

$$\varphi_{k+1} = \varphi \circ (\varphi_k \times 1) = \varphi \circ (0 \times 1) = 0 \quad \text{by Lemma 4.1.}$$

4.4. *Definition.* For any $X \in H \circ \mathscr{C}$ we define nil X to be the smallest integer $k \geqq 0$ such that $\varphi_{k+1} = 0$ (if such an integer exists). If no such integer exists nil X is defined to be ∞.

Lemma 4.3 is needed to see that Definition 4.4 makes sense.

The definiton of conil X is completely dual to the definition of nil X. We omit it.

For any group π let nil π denote the index of nilpotence of the group π.

4.5. PROPOSITION. *For any* $Y \in H \circ \mathscr{C}$ *we have*

$$\text{nil } Y = \operatorname*{Sup}_{X \in H \circ \mathscr{C}} \text{nil } [\Sigma X, Y].$$

Proof. We have $[\Sigma X, Y] \simeq [X, \Omega Y]$ (as groups) and the commutator of any k elements $f_1, \ldots, f_k$ in $[X, \Omega Y]$ is given by the composite

$$X \xrightarrow{\quad \Delta_k \quad} X^k \xrightarrow{f_1 \times \ldots \times f_k} (\Omega Y)^k \xrightarrow{\quad \varphi_k \quad} \Omega Y.$$

This immediately gives nil $[\Sigma X, Y] \leqq$ nil Y.

Conversely, suppose nil $[X, \Omega Y] \leqq k - 1$ for all $X \in H \circ \mathscr{C}$. In particular nil $[(\Omega Y)^k, \Omega Y] \leqq k - 1$. Let $p_i : (\Omega Y)^k \to \Omega Y$ be the i-th projection. ($1 \leqq i \leqq k$). Then the composite

$$(\Omega Y)^k \xrightarrow{\Delta_k,\, (\Omega Y)^k} ((\Omega Y)^k)^k \xrightarrow{p_1 \times \ldots \times p_k} (\Omega Y)^k$$

is clearly the identity element in $[(\Omega Y)^k, (\Omega Y)^k]$. Hence

$$\varphi_k = \varphi_k \circ (p \times \ldots \times p_k) \circ \Delta_{k,(\Omega Y)}k \quad \text{in } [(\Omega Y)^k, \Omega Y].$$

Since φ_k $(p_1 \times \ldots \times p_k) \circ \Delta_{k,(\Omega Y)}k$ denotes the commutator of the elements $p_1, \ldots, p_k$ in $[(\Omega Y)^k, \Omega Y]$ we have $\varphi_k \circ (p_1 \times \ldots \times p_k) \circ \Delta_{k,(\Omega Y)}k = 0$. Hence $\varphi_k = 0$. Hence nil $Y \leqq k - 1$. This completes the proof of Proposition 4.5.

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4.6. PROPOSITION. *For any* $X \in H \circ \mathscr{C}$ *we have*

$$\operatorname{Conil} X = \sup_{Y \in H \circ \mathscr{C}} \operatorname{nil}\,[X, \Omega Y].$$

Proof. This is the dual of Proposition 4.5.

It is clear that nil X and conil X depend only on the isomorphism class of X in $H \circ \mathscr{C}$.

For any two objects A, B in $H \circ \mathscr{C}$ we denote their union in $H \circ \mathscr{C}$ by $A \vee_H B$. (Actually $R(RQ()A \vee RQ(B))$ where $\vee$ denotes the union in $\mathscr{C}$ gives the union of A and B in $H \circ \mathscr{C}$). It is known that for any $X \in H \circ \mathscr{C}$, ΣX is a cogroup object in $H \circ \mathscr{C}$. We write $\mu' : \Sigma X \to \Sigma X \vee_H \Sigma X$ for the comultiplication in ΣX.

4.7. LEMMA. *Let* $\alpha : \Sigma X \to \Sigma X \times_H \Sigma X$ *and* $\beta : \Sigma X \to \Sigma X \times_H \Sigma X$ *denote the maps* $(1_{\Sigma X}, 0)$ *and* $(0, 1_{\Sigma X})$ *respectively. Then in* $[\Sigma X, \Sigma X \times_H \Sigma X]$ *we have* $\alpha \cdot \beta = \beta \cdot \alpha$.

Proof. $\alpha \cdot \beta$ is given by the composite

$$\Sigma X \xrightarrow{\mu'} \Sigma X \vee_H \Sigma X \xrightarrow{\alpha + \beta} \Sigma X \times_H \Sigma X$$

and $\beta \cdot \alpha$ is given by the composite

$$\Sigma X \xrightarrow{\mu'} \Sigma X \vee_H \Sigma X \xrightarrow{\beta + \alpha} \Sigma X \times_H \Sigma X.$$

Let $p_1 : \Sigma X \times_H \Sigma X \to \Sigma X$ and $p_2 : \Sigma X \times_H \Sigma X \to \Sigma X$ be the first and second projections. Then

$$\begin{aligned} p_1 \circ (\alpha \cdot \beta) &= p_1 \circ ((1_{\Sigma X}, 0) + (0, 1_{\Sigma X})) \circ \mu' \\ &= (1_{\Sigma X} + 0) \circ \mu' = 1_{\Sigma X} \quad \text{since } \mu' : \Sigma X \to \Sigma X \vee_H \Sigma X \end{aligned}$$

is the comultiplication for the co-group structure on ΣX in $H \circ \mathscr{C}$. Similarly,

$$\begin{aligned} p_2 \circ (\alpha \cdot \beta) &= p_2 \circ ((1_{\Sigma X}, 0) + (0, 1_{\Sigma X})) \circ \mu' = (0 + 1_{\Sigma X}) \circ \mu' = 1_{\Sigma X} \\ p_1 \circ (\beta \cdot \alpha) &= p_1 \circ ((0, 1_{\Sigma X}) + (1_{\Sigma X}, 0)) \circ \mu' = (0 + 1_{\Sigma X}) \circ \mu' = 1_{\Sigma X} \\ p_2 \circ (\beta \cdot \alpha) &= p_2 \circ ((0, 1_{\Sigma X}) + (1_{\Sigma X}, 0)) \circ \mu' = (1_{\Sigma X} + 0) \circ \mu' = 1_{\Sigma X}. \end{aligned}$$

Since $(p_{1*}, p_{2*}) : [\Sigma X, \Sigma X \times_H \Sigma X] \to [\Sigma X, \Sigma X] \times [\Sigma X, \Sigma X]$ is an isomorphism we get $\alpha \cdot \beta = \beta \cdot \alpha$.

4.8. PROPOSITION. *Let*

$$(12) \quad F \xrightarrow{i} E \xrightarrow{p} B, \quad F \times_H \Omega B \xrightarrow{m} F$$

be a fibration sequence in $H \circ \mathscr{C}$. *Let* $A \in H \circ \mathscr{C}$ *and*

$$\ldots \to [\Sigma A, \Omega B] \xrightarrow{\partial} [\Sigma A, F] \xrightarrow{i_*} [\Sigma A, E] \to \ldots$$

be part of the "Eckmann-Hilton" exact sequence corresponding to the fibration sequence (12). *Then the image of* $\partial : [\Sigma A, \Omega B] \to (\Sigma A, F]$ *lies in the centre of* $[\Sigma A, F]$.

Proof. Let $g \in [\Sigma A, F]$ and $\lambda \in [\Sigma A, \Omega B]$ be artitrary. Then $\partial\lambda \in [\Sigma A, F]$ is given by the composite

$$\Sigma A \xrightarrow{\lambda} \Omega B \xrightarrow{(0,\, 1_{\Omega B})} F \times_H \Omega B \xrightarrow{m} F.$$

Let $\tau : \Sigma A \to \Sigma A$ denote the inversion in the co-group object ΣA. We want to prove that the commutator of g and $\partial\lambda$ in the group $[\Sigma A, F]$ is the neutral element $0 \in [A, F]$. The commutator of g and $\partial\lambda$ is given by the composite

$$\Sigma A \xrightarrow{\psi} \Sigma A \vee_H \Sigma A \xrightarrow{g + \partial\lambda} F$$

where ψ is the co-commutator map given by the composition of

$$\Sigma A \xrightarrow{\mu'} \Sigma A \vee_H \Sigma A \xrightarrow{\mu' \vee \mu'}$$
$$(\Sigma A \vee_H \Sigma A) \vee_H (\Sigma A \vee_H \Sigma A) \xrightarrow{(\tau \vee \tau) \vee (\tau \vee \tau)}$$
$$(\Sigma A \vee_H \Sigma A) \vee_H (\Sigma A \vee_H \Sigma A) \xrightarrow{\nabla} \Sigma A \vee_H \Sigma A.$$

Consider the following diagram

$$\Sigma A \vee_H \Sigma A \xrightarrow{\alpha + \beta} \Sigma A \times_H \Sigma A \xrightarrow{g \times 1_{\Sigma A}} F \times_H \Sigma A \xrightarrow{1_F \times \lambda} F \times_H \Omega B \xrightarrow{m} F \tag{13}$$

where $\alpha = (1_{\Sigma A}, 0)$ and $\beta = (0, 1_{\Sigma A})$. We have

$$\begin{aligned} m \circ (1_F \times \lambda) \circ (g \times 1_{\Sigma A}) \circ \alpha &= m \circ (1_F \times \lambda) \circ (g \times 1_{\Sigma A}) \circ (1_{\Sigma A}, 0) \\ &= m \circ (g, 0) \\ &= g \end{aligned}$$

since m is a right action of the group object ΩB on F.

Also

$$\begin{aligned} m \circ (1_F \times \lambda) \circ (g \times 1_{\Sigma A}) \circ_B &= m \circ (1_F \times \lambda) \circ (g \times 1_{\Sigma A}) \circ (0, 1_{\Sigma A}) \\ &= m \circ (0, \lambda) \\ &= m \circ (0, 1_{\Omega B}) \circ \lambda \\ &= \partial\lambda. \end{aligned}$$

Thus the composition of the maps in (13) is $g + \partial\lambda$. Denoting $m \circ (1_F \times \lambda) \circ (g \times 1_{\Sigma A})$ by γ we have $g + \partial\lambda = \gamma \circ (\alpha + \beta)$.

Hence $(g + \partial\lambda) \circ \psi = \gamma \circ (\alpha + \beta) \circ \psi$. The composite $(\alpha + \beta) \circ \psi : \Sigma A \to \Sigma A \times_H \Sigma A$ gives the commutator of α and β in $[\Sigma A, \Sigma A \times_H \Sigma A]$. By Lemma 4.7 this is the neutral element $0 \in [\Sigma A, \Sigma A \times_H \Sigma A]$. Hence $(g + \partial\lambda) \circ \psi = \gamma \circ 0 = 0$ in $[\Sigma A, F]$.

This completes the proof of Proposition 4.8.

4.9. *Remark*. In the case of $\mathscr{T}_*$ Proposition 4.8 has been proved by Hilton. A proof can be found in [**4**].

4.10. PROPOSITION. *Let*

(14) $F \xrightarrow{i} E \xrightarrow{p} B, \quad F \times_H \Omega B \xrightarrow{m} F$

be a fibration sequence in $H \circ \mathscr{C}$. *Then*

$$\text{nil } F \leqq 1 + \text{nil } E.$$

Proof. Consider the part

$$\ldots \to [\Sigma X, \Omega B] \xrightarrow{\partial} [\Sigma X, F] \xrightarrow{i_*} [\Sigma X, E] \to \ldots$$

of the "Eckmann-Hilton" exact sequence corresponding to the fibration sequence (14) where $X \in H \circ$ is arbitrary. From Proposition 4.8 we see that $\partial[\Sigma X, \Omega B] \subset$ centre of the group $[\Sigma X, F]$. It follows immediately that

$$\text{nil } [\Sigma X, F] \leqq 1 + \text{nil } [\Sigma X, E].$$

Since

$$\text{nil } F = \sup_{X \in H \circ \mathscr{C}} \text{nil } [\Sigma X, F]$$

we get

$$\text{nil } F \leqq 1 + \text{nil } E.$$

4.11. PROPOSITION. *Let*

(15) $A \xrightarrow{f} X \xrightarrow{v} C, \quad n : C \to C \vee \Sigma A$

be a cofibration sequence in $H \circ \mathscr{C}$. *Then*

$$\text{Conil } C \leqq 1 + \text{Conil } X.$$

Proof. This is the dual of Proposition 4.10.

If $\mathscr{C}$ is a pointed model category satisfying axiom W in addition to the axioms of Quillen we have already observed (Remark 3.17) that for every $X \in \mathscr{C}$ we can define Ind Cat X and Cocat X (= the inductive cocategory of X). Since X is also an object of $H \circ \mathscr{C}$ nil X, conil X are defined.

4.13. THEOREM. *Let* $\mathscr{C}$ *be a pointed model category satisfying axiom* W. *Then for any* $X \in \mathscr{C}$,

(16) nil $X \leqq$ Cocat X, and
Conil $X \leqq$ Ind Cat X.

Proof. If X is contractible nil $X = 0 =$ conil X and so the inequalities (16) are trivially valid. The general case follows from the definition of Cocat X (respectively Ind Cat X) and Proposition 4.10 (respectively Proposition 4.11).

Finally we want to comment that all the results in § 4 of this paper are generalizations of results obtained by Ganea [**3**]. Ganea dealt with pointed CW-complexes. Our results are valid in the general set-up of Quillen.

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