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STOCHASTIC FUBINI THEOREM FOR SEMIMARTINGALES IN HILBERT SPACE

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1. Introduction. In this paper we will study the Fubini theorem for stochastic integrals with respect to semimartingales in Hilbert space.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X, \mathcal{B}, \mu)$ a measure space, H and G two Hilbert spaces, $L(H, G)$ the space of bounded linear operators from H into G, Z an H-valued semimartingale relative to a given filtration, and $\varphi : X \times R^+ \times \Omega \longrightarrow L(H, G)$ a function such that for each $t \in R^+$ the iterated integrals

$$Y^{1,t} = \int_{]0,t]} \int_X \varphi(x, r)\mu(dx)\, dZ_r$$
$$Y^{2,t} = \int_X \int_{]0,t]} \varphi(x, r)\, dZ_r \mu(dx)$$

are well-defined (the integrals with respect to μ are Bochner integrals). It is often necessary to have sufficient conditions for the process Y^1 to be a version of the process Y^2 (e.g. [**1**], proof of Theorem 2.11).

In the real case, i.e. Z and φ real-valued, this problem has been studied in several directions. Liptser and Shiryayev [**10**] treated the case where Z is a Wiener process and $(X, \mathcal{B}, \mu)$ is a probability space. Under the assumption that the function $(x, \omega) \longmapsto \int_{]0,t]} \varphi(x, r, \omega) dZ_r$ is $\mathcal{B} \otimes \mathcal{F}$-measurable for each $t \in R^+$, Ikeda and Watanabe [**7**] obtained sufficient conditions when Z is a martingale. Jacod [**8**] considered the existence of a measurable function $Y^3 : X \times R^+ \times \Omega \longrightarrow R$ such that for every $x \in X$ the process $Y^3(x, \cdot)$ is a version of the stochastic integral $\int_{]0,\cdot]} \varphi(x, r)\, dZ_r$ and the integral $\int_X Y^3(x, \cdot)\mu(dx)$ is a version of Y^1. (In part (iii) of Jacod's proof, page 182, it is not clear that the functions defined are elementary functions). Finally, Walsh [**12**] obtained a similar result when Z is a martingale-measure.

In the Hilbert space case, a result similar to that of [**10**] was formulated by Curtain and Falb [**4**], and Chojnowska-Michalik [**2,3**] studied the problem when Z is a semimartingale of a particular class.

Our objective is to give a generalization of the results contained in the references [**2,3,7,8,10**] without any restrictions on the semimartingale Z.

2. Preliminaries. In this section we will recall some basic definitions and facts we shall need.

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Let H and G be two real separable Hilbert spaces with inner products $< \cdot, \cdot >$ and $< \cdot, \cdot >_G$, respectively. The norm in H is written $| \cdot |$, and in G, $| \cdot |_G$. $L(H, G)$ designates the space of bounded linear operators from H into G. The operator norm is denoted by $|| \cdot ||$. The adjoint of an operator A is written A^*.

The space $(L(H, G), || \cdot ||)$ is not in general separable, and several types of measurabilities for functions with values in $L(H, G)$ can be defined, which may not coincide (see [**6**], p. 280).

Definition 2.1. Let $(S, \mathcal{S})$ be a measurable space and $f: S \longrightarrow L(H, G)$.

(i) f is *point-measurable* if and only if for every $h \in H$,

$$f(h): (S, \mathcal{S}) \longrightarrow (G, \mathcal{B}_G),$$

where $\mathcal{B}_G$ is the Borel σ-algebra of G.

(ii) f is *Borel-measurable* if and only if

$$f: (S, \mathcal{S}) \longrightarrow (L(H, G), \mathcal{B}_{L(H,G)}),$$

where $\mathcal{B}_{L(H,G)}$ is the Borel σ-algebra of $(L(H, G), || \cdot ||)$.

(iii) f is *strongly-measurable* if and only if there exists a sequence (f_n) of elementary $L(H, G)$-valued functions defined on S, i.e., f_n has the form $f_n = \sum_{i=1}^N f^i I_{S_i}$ where $f^i \in L(H, G)$ and $S_i \in \mathcal{S}$, such that for each $s \in S$,

$$||f(s) - f_n(s)|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

If $G = R$, the space $(L(H, R), || \cdot ||)$ is separable, and the Pettis theorem implies that the three measurabilities above coincide. In this case f will be simply called measurable.

We will assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space on which is defined an increasing and right-continuous family $(\mathcal{F}_t)_{t \geq 0}$ of complete sub-σ-algebras, and we will denote by $\mathcal{P}$ the predictable σ-algebra.

Let Z be an H-valued semimartingale. An increasing, positive, adapted process A is called a *control process* of Z if for every G, every bounded $L(H, G)$-valued $\mathcal{P}$-$\mathcal{B}_{L(H,G)}$-measurable process X and every stopping time T we have

$$E\left[\sup_{t<T} \left| \int_{]0,t]} X \, dZ \right|_G^2\right] \leq E\left[A_{T-} \int_{]0,T[} ||X||^2 \, dA\right]$$

(see [**11**], Section 26).

3. Stochastic Fubini Theorem. Let μ be a non-negative σ-finite measure on the measurable space $(X, \mathcal{B})$.

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THEOREM 3.1. *Let Z be an H-valued semimartingale and $\varphi: X \times R^+ \times \Omega \longrightarrow L(H, R)$ a $\mathcal{B} \otimes \mathcal{P}$-measurable function, and assume there exists a non-negative $f \in L^1(\mu)$, such that $||\varphi(x, r, \omega)|| \leq f(x)$ for all $(x, r, \omega) \in X \times R^+ \times \Omega$. Then for each $t \in R^+$ there exists a measurable function $Y^t: X \times \Omega \longrightarrow R$ belonging to $L^1(\mu)$ w.p. 1 such that*

$$(3.1) \qquad \int_{]0,t]} \varphi(x, r) dZ_r = Y^t_x \text{ w.p. 1 for almost all } x \in X,$$

and

$$(3.2) \qquad \int_{]0,t]} \int_X \varphi(x, r) \mu(dx) dZ_r = \int_X Y^t_x \mu(dx) \text{ w.p. 1.}$$

Remarks 3.2. (a) The set of probability 1 where the equality in (3.1) holds depends on x.

(b) The Fubini theorem for Bochner integrals and the existence of f imply that the process $\int_X \varphi(x, \cdot)\mu(dx)$ is Z-integrable, hence the left-hand side of (3.2) is well-defined.

(c) If the R-valued function $\int_{]0,t]} \varphi(\cdot, r)\, dZ_r$ is $\mathcal{B} \otimes \mathcal{F}$-measurable, then

$$\int_{]0,t]} \int_X \varphi(x, r) \mu(dx)\, dZ_r = \int_X \int_{]0,t]} \varphi(x, r)\, dZ_r\, \mu(dx) \text{ w.p. 1.}$$

The following consequence is immediate.

COROLLARY 3.3. *If in the assumptions of the Theorem 3.1 we have that φ is an $L(H, G)$-valued $\mathcal{B} \otimes \mathcal{P}$-point-measurable function instead of an $L(H, R)$-valued $\mathcal{B} \otimes \mathcal{P} - \mathcal{B}_{L(H,R)}$-measurable function, then for each $t \in R^+$ and $y \in G$ there exists a measurable map $Y^{t,y}: X \times \Omega \longrightarrow R$ such that*

$$\int_{]0,t]} \langle \varphi^*(x, r)y,\ dZ_r \rangle = Y^{t,y}_x \text{ w.p. 1 for almost all } x \in X,$$

and

$$\int_{]0,t]} \int_X \langle \varphi^*(x, r)y, \cdot \rangle\, \mu(dx)\, dZ_r = \int_X Y^{t,y}_x \mu(dx) \text{ w.p. 1.}$$

Our main result is the following.

THEOREM 3.4. *Let Z be an H-valued semimartingale and $\varphi: X \times R^+ \times \Omega \longrightarrow L(H, G)$ a $\mathcal{B} \otimes \mathcal{P}$-$\mathcal{B}_{L(H,G)}$-measurable function, and assume*

(i) There exists a non-negative function $f \in L^1(\mu)$ such that $||\varphi(x, r, \omega)|| \leq f(x)$ for all $(x, r, \omega) \in X \times R^+ \times \Omega$.

(ii) For each $(r, \omega) \in R^+ \times \Omega$ the Bochner integral $\int_X \varphi(x, r, \omega)\mu(dx)$ is well-defined.

(iii) The process $\int_X \varphi(x, \cdot)\mu(dx)$ is Z-integrable.

Then for each $t \in R^+$ there exists a measurable function $Y^t: X \times \Omega \longrightarrow G$, belonging to $L^1(\mu)$ w.p.1, such that (3.1) and (3.2) hold.

Remarks. (a) For all $(r, \omega) \in R^+ \times \Omega$ the function $\varphi(\cdot, r, \omega): X \longrightarrow L(H, G)$ is $\mathcal{B}$- $\mathcal{B}_{L(H,G)}$-measurable, but since the space $L(H, G)$ is not in general separable, then $\varphi(\cdot, r, \omega)$ is not necessarily strongly-measurable. Hence the Bochner integral $\int_X \varphi(x, r, \omega)\mu(dx)$ may not be well-defined.

(b) Assumption (ii) implies that for each $(r, \omega) \in R^+ \times \Omega$ there exists a sequence $\left(\varphi_n^{(r,\omega)}\right)$ of elementary $L(H, G)$-valued functions on X such that

$$\left\|\int_X \varphi_n^{(r,\omega)}(x)\mu(dx) - \int_X \varphi(x, r, \omega)\mu(dx)\right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

However, the processes $\int_X \varphi_n^{(\cdot)}(x)\mu(dx)$ may not be predictable. Hence $\int_X \varphi(x, \cdot)\mu(dx)$ is not necessarily a predictable process and therefore it is not necessarily Z-integrable.

(c) For every $x \in X$ the process $\varphi(x, \cdot)$ is predictable. Hence assumption (i) implies that the stochastic integral $\int_{]0,t]} \varphi(x, r)\, dZ_r$, $t \in R^+$, is well-defined.

4. Proofs. We will prove first two lemmas which are needed for the proof of Theorem 3.1.

LEMMA 4.1. *Let $\varphi: X \times R^+ \times \Omega \longrightarrow L(H, R)$ satisfy the assumptions of Theorem 3.1. Then there exists a sequence (φ_n) of elementary $L(H, R)$-valued functions on $X \times R^+ \times \Omega$ such that for every $(x, r, \omega) \in X \times R^+ \times \Omega$,*

$$(4.1) \qquad ||\varphi_n(x, r, \omega) - \varphi(x, r, \omega)|| \longrightarrow 0 \;\; \textit{as} \;\; n \longrightarrow \infty$$

and

$$(4.2) \qquad ||\varphi_n(x, r, \omega)|| \leq f(x) \;\; \textit{for all } n.$$

Remark. The Pettis theorem implies that there is a sequence (φ_n) of elementary $L(H, R)$-valued functions on $X \times R^+ \times \Omega$ such that (4.1) holds, but it may not satisfy (4.2).

Proof. We first note that since H and $L(H,R)$ are separable, φ can be considered as an H-valued measurable function if we identify H with its dual. Let (h_i) be an orthonormal base in H, then for each i there exists a sequence (φ_i^n) of elementary real-valued functions on $X \times R^+ \times \Omega$ such that for every $(x,r,\omega) \in X \times R^+ \times \Omega$, $\varphi_i^n(x,r,\omega) \longrightarrow \langle \varphi(x,r,\omega), h_i \rangle$ as $n \longrightarrow \infty$ and

$$(4.3) \qquad |\varphi_i^n(x,r,\omega)| \leq |\langle \varphi(x,r,\omega), h_i \rangle| \quad \text{for all } n.$$

For each positive integer n, let $\psi^n = \sum_{i=1}^n \varphi_i^n h_i$. We will prove that (ψ^n) is the sequence we are looking for.

Fix $\varepsilon > 0$ and $(x,r,\omega) \in X \times R^+ \times \Omega$. Then there exists N large enough so that for all $n \geq N$,

$$(4.4) \qquad \Big|\sum_{i=1}^n \langle \varphi(x,r,\omega), h_i \rangle h_i - \varphi(x,r,\omega)\Big| \leq \varepsilon/6,$$

and there exists $N_0 > N$ such that for all $i \in \{1, 2, \ldots, N\}$ and $n \geq N_0$,

$$(4.5) \qquad |\varphi_i^n(x,r,\omega) - \langle \varphi(x,r,\omega), h_i \rangle| \leq \varepsilon/3N.$$

From (4.3), (4.4) and (4.5) we have for all $n \geq N_0$,

$$\begin{aligned} |\varphi(x,r,\omega) - \psi^n(x,r,\omega)| &\leq \left|\varphi(x,r,\omega) - \sum_{i=1}^N \langle \varphi(x,r,\omega), h_i \rangle h_i\right| \\ &\quad + \left|\sum_{i=1}^N \Big(\langle \varphi(x,r,\omega), h_i \rangle h_i - \varphi_i^n(x,r,\omega) h_i\Big)\right| \\ &\quad + \left|\sum_{i=N+1}^n \varphi_i^n(x,r,\omega) h_i\right| \\ &\leq \varepsilon/6 + \varepsilon/3 + \varepsilon/3 \\ &\leq \varepsilon. \end{aligned}$$

Finally, by (4.3), for every positive integer n,

$$|\psi^n(x,r,\omega)|^2 \leq \sum_{i=1}^n \langle \varphi(x,r,\omega), h_i \rangle^2 \leq |\varphi(x,r,\omega)|^2 \leq f^2(x),$$

and as this holds for all $(x,r,\omega) \in X \times R^+ \times \Omega$, the proof is complete.

LEMMA 4.2. *Let Z be an H-valued semimartingale and $\varphi: X \times R^+ \times \Omega \longrightarrow L(H,R)$. Let $f \in L^1(\mu)$ and (φ_n) a sequence of $\mathcal{B} \otimes \mathcal{P}$-measurable $L(H,R)$-valued functions on $X \times R^+ \times \Omega$ for which Theorem 3.1 holds. If for every $(x,r,\omega) \in X \times R^+ \times \Omega$,*

$$(4.6) \qquad \|\varphi_n(x,r,\omega) - \varphi(x,r,\omega)\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

and

$$(4.7)\qquad \|\varphi_n(x,r,\omega)\| \le f(x) \text{ for all } n,$$

then for each $t \in R^+$ there exists a measurable funciton $Y^t : X \times \Omega \longrightarrow R$ belonging to $L^1(\mu)$ w.p. 1 such that (3.1) and (3.2) hold.

Proof. Note that (4.6) and (4.7) imply that the integral $\int_{]0,t]} \varphi(x,r)\, dZ_r$ is well-defined. Fix $t \in R^+$. The proof is divided into two steps.

Step 1. Let us assume that there exist a control process A of Z and a stopping time T such that $E(A_{T-}^2) < \infty$.

By hypothesis there is a sequence $(Y^{t,n})$ of measurable real-valued functions on $X \times \Omega$ such that

$$(4.8)\qquad \int_{]0,t]} \varphi_n(x,r)\, dZ_r = Y_x^{t,n} \quad \text{w.p. 1 for almost all } x \in X,$$

and

$$(4.9)\qquad \int_{]0,t]} \int_X \varphi_n(x,r)\mu(dx)\, dZ_r = \int_X Y_x^{t,n} \mu(dx) \quad \text{w.p. 1}.$$

Fix $x \in X$ such that (4.8) holds. Since A is a control process of Z, then, using (4.7) and (4.8), for all n, m, $n < m$, we have

$$E\Big[I_{]0,T[}(t)\,|Y_x^{t,n} - Y_x^{t,m}|\Big]$$
$$\le \left\{ E\Big[\sup_{t'<T} \Big| \int_{]0,t']} \big(\varphi_n(x,r) - \varphi_m(x,r)\big)\, dZ_r \Big|^2\Big]\right\}^{1/2}$$
$$(4.10)\qquad \le \left\{ E\Big[A_{T-} \int_{]0,T[} \|\varphi_n(x,r) - \varphi_m(x,r)\|^2\, dA_r\Big]\right\}^{1/2}$$
$$(4.11)\qquad \le 2\Big(E(A_{T-}^2)\Big)^{1/2} f(x).$$

On the other hand, (4.6), (4.7) and the dominated convergence theorem imply

$$\int_{]0,T[} \|\varphi_n(x,r) - \varphi_m(x,r)\|^2\, dA_r \longrightarrow 0 \quad \text{as} \quad n, m \longrightarrow \infty \quad \text{for all } \omega \in \Omega.$$

Hence, (4.7), (4.10), the dominated convergence theorem and the fact that $E\big(A_{T-}^2\big) < \infty$ imply

$$E\Big[I_{]0,T[}(t)\,|Y_x^{t,n} - Y_x^{t,m}|\Big] \longrightarrow 0 \quad \text{as} \quad n, m \longrightarrow \infty.$$

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Since inequality (4.11) holds for almost all $x \in X$, by the dominated convergence theorem again we have that $\left(I_{]0,T[}(t)Y_{\cdot}^{t,n}\right)$ is a Cauchy sequence in $L^1(\mu \times P)$. Hence there is a function $Y^{T,t}: X \times \Omega \longrightarrow R$ belonging to $L^1(\mu \times P)$ such that

$$(4.12) \quad \int_X E|I_{]0,T[}(t)Y_x^{t,n} - Y_x^{T,t}|\ \mu(dx) \longrightarrow 0 \ \text{ as } \ n, m \longrightarrow \infty,$$

and there is a subsequence (n_k) such that

$$(4.13) \quad I_{]0,T[}(t)Y_x^{t,n_k} \longrightarrow Y_x^{T,t} \ \text{ as } \ k \longrightarrow \infty \text{ w.p. 1 for almost all } x \in X.$$

The Fubini theorem implies that $Y_{\cdot}^{T,t}$ belongs to $L^1(\mu)$ w.p.1. Therefore (4.8), (4.13) and the fact that

$$E\left[I_{]0,T[}(t)\left|\int_{]0,t]}\left(\varphi(x,r) - \varphi_n(x,r)\right) dZ_r\right|\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

which follows from (4.6) and (4.7), yield

$$Y_x^{T,t} = I_{]0,T[}(t)\int_{]0,t]} \varphi(x,r)\, dZ_r \text{ w.p. 1 for almost all } x \in X,$$

and the definition of $Y^{T,t}$ and (4.12) imply that there exists a subsequence $(n_{k'})$ of (n_k) such that

$$(4.14) \quad I_{]0,T[}(t)\int_X Y_x^{t,n_{k'}}\ \mu(dx) \longrightarrow \int_X Y_x^{T,t}\mu(dx) \ \text{ as } \ k' \longrightarrow \infty \ \text{ w.p. 1.}$$

Proceeding similarly, there exists a subsequence $(n_{k''})$ of $(n_{k'})$ such that

$$\begin{aligned} &I_{]0,T[}(t)\int_{]0,t]}\int_X \varphi_{n_{k''}}(x,r)\mu(dx)\, dZ_r \\ &\qquad \longrightarrow I_{]0,T[}(t)\int_{]0,t]}\int_X \varphi(x,r)\mu(dx)\, dZ_r \ \text{ as } \ k'' \longrightarrow \infty \ \text{ w.p. 1} \end{aligned}$$

Therefore (4.9) and (4.14) imply

$$I_{]0,T[}(t)\int_{]0,t]}\int_X \varphi(x,r)\mu(dx)\, dZ_r \int_X \ Y_x^{T,t}\mu(dx) \ \text{ w.p. 1.}$$

Step 2. By [**11**] (Section 26) there exist a control process A of Z and a sequence (T_n) of stopping times such that $T_n \nearrow \infty$ and $E(A^2_{T_n-}) < \infty$ for all n. Then by Step 1 there is a sequence $(Y^{T_n,t})$ of measurable real-valued functions on $X \times \Omega$ such that

$$(4.15) \quad Y_x^{T_n,t} = I_{]0,T_n[}(t)\int_{]0,t]} \varphi(x,r)\, dZ_r \text{ w.p. 1 for almost all } x \in X,$$

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and

$$I_{]0,T_n[}(t)\int_{]0,t]}\int_X \varphi(x,r)\mu(dx)\,dZ_r = \int_X Y_x^{T_n,t}\mu(dx) \text{ w.p. } 1.$$

We now define the following measurable real-valued function on $X \times \Omega$:

$$Y^t = \begin{cases} Y^{T_1,t} & \text{on } X \times [t < T_1], & n = 1, \\ Y^{T_n,t} & \text{on } X \times [T_{n-1} \leq t < T_n], & n \geq 2, \end{cases}$$

where $[T_{n-1} \leq t < T_n] = \{\omega \in \Omega: T_{n-1}(\omega) \leq t < T_n(\omega)\}$. Thus, by Step 1, to finish the proof we have only to show that for every n,

$$\int_X Y_x^{T_n,t}\mu(dx) = \int_X Y_x^t\mu(dx) \text{ w.p. 1 on } [t < T_n]. \tag{4.16}$$

Fix n and $x \in X$ such that (4.15) holds. Then, by (4.15),

$$\begin{aligned}
&E\big[I_{[t<T_n]}\,|Y_x^t - Y_x^{T_n,t}|\big] \\
&\quad \leq E\big[I_{[t<T_1]}\,|Y_x^t - Y_x^{T_n,t}|\big] + \sum_{i=2}^n E\big[I_{[T_{i-1}\leq t<T_i]}\,|Y_x^t - Y_x^{T_n,t}|\big] \\
&\quad = E\big[I_{[t<T_1]}\,|Y_x^{T_1,t} - Y_x^{T_n,t}|\big] + \sum_{i=2}^n E\big[I_{[T_{i-1}\leq t<T_i]}\,|Y_x^{T_i,t} - Y_x^{T_n,t}|\big] \\
&\quad \leq E\big[I_{[t<T_1]}\,|Y_x^{T_1,t} - Y_x^{T_n,t}|\big] + \sum_{i=2}^n E\big[I_{[t<T_i]}\,|Y_x^{T_i,t} - Y_x^{T_n,t}|\big] \\
&\quad = 0.
\end{aligned}$$

Finally, the measurability of the functions Y^t and $Y^{T_n,t}$ and Fubini's theorem yield (4.16).

Proof of Theorem 3.1. Fix $t \in R^+$. Let $\psi \in L(H,R)$, n a positive integer and $A \in \mathcal{B} \otimes \mathcal{P}$, and consider $\varphi: X \times R^+ \times \Omega \longrightarrow L(H,R)$ of the form

$$\varphi(x,r,\omega) = \psi \cdot \big(I_A(x,r,\omega) \wedge nf(x)\big).$$

The family $\mathcal{H} = \{B \times F \subset X \times R^+ \times \Omega: B \in \mathcal{B} \text{ and } F \in \mathcal{P}\}$ is closed under intersection and the theorem is trivially true for $A \in \mathcal{H}$, and in this case

$$Y^t_\cdot \equiv \int_{]0,t]} \varphi(\cdot,r)\,dZ_r$$

(see Remark 3.2 (c)). Then Lemma 4.2 and the monotone class theorem imply that there exists a measurable function $Y^t: X \times \Omega \longrightarrow R$ belonging to $L^1(\mu)$ w.p.1 such that (3.1) and (3.2) hold for such φ.

On the other hand, let $A \in \mathcal{B} \otimes \mathcal{P}$ such that for every $(x, r, \omega) \in X \times R^+ \times \Omega$,

$$\|\psi I_A(x, r, \omega)\| \leq f(x),$$

then for each $(x, r, \omega) \in X \times R^+ \times \Omega$,

$$\|\psi \cdot \big(I_A(x, r, \omega) \wedge nf(x)\big)\| \leq f(x).$$

Hence, letting $n \longrightarrow \infty$, Lemma 4.2 implies that the result holds for $\varphi = \psi I_A$.

Finally, by Lemmas 4.1 and 4.2 the proof is complete.

Proof of Theorem 3.4. Let $t \in R^+$ and $y \in G$. By Corollary 3.3 there exists a measurable function $Y^{t,y}\colon X \times \Omega \longrightarrow R$ belonging to $L^1(\mu)$ w.p.1 such that

$$(4.17) \qquad Y_x^{t,y} = \Big\langle y, \int_{]0,t]} \varphi(x, r)\, dZ_r \Big\rangle_G \text{ w.p. 1 for almost all } x \in X,$$

and

$$\int_{]0,t]} \int_X \langle \varphi^*(x, r)y, \cdot \rangle\, \mu(dx)\, dZ_r = \int_X Y_x^{t,y} \mu(dx) \text{ w.p.1.}$$

Assumptions (ii) and (iii) imply that

$$\int_{]0,t]} \int_X \langle \varphi^*(x, r)y, \cdot \rangle\, \mu(dx)\, dZ_r = \Big\langle y, \int_{]0,t]} \int_X \varphi(x, r) \mu(dx) dZ_r \Big\rangle_G \text{ w.p.1,}$$

and since G is a separable space, then by Fubini's theorem, to finish the proof of the theorem we have only to show that there is a measurable function $Y^t\colon X \times \Omega \longrightarrow G$ which is Bochner integrable with respect to μ w.p.1 and such that (3.1) holds.

Let (g_i) be an orthonormal base in G, $Y^n = \sum_{i=1}^n Y^{t,g_i} g_i$, and A and T satisfying the assumptions of Step 1 in the proof of Lemma 4.2. Then assumption (i) and (4.17) imply that for almost all $x \in X$ and $n < m$,

$$\begin{aligned}
(4.18) \qquad & E\big[I_{]0,T[}(t)\, |Y_x^n - Y_x^m|_G\big] \\
&= E\Big[I_{]0,T[}(t)\, \Big| \sum_{i=n+1}^{m} \langle g_i, \int_{]0,t]} \varphi(x, r)\, dZ_r \rangle_G g_i \Big|_G\Big] \\
&\leq E\Big[I_{]0,T[}(t)\, \Big| \int_{]0,t]} \varphi(x, r)\, dZ_r \Big|_G\Big] \\
&\leq \Big\{E\Big[A_{T-} \int_{]0,T[} \|\varphi(x, r)\|^2\, dA_r\Big]\Big\}^{1/2} \\
&\leq \big\{E(A_{T-}^2)\big\}^{1/2} f(x).
\end{aligned}$$

Since $\big|\sum_{i=n+1}^m \langle g_i, \int_{]0,t]} \varphi(x, r)\, dZ_r \rangle_G g_i\big|_G$ converges to 0 as $n, m \longrightarrow \infty$ and is bounded by $\big| \int_{]0,t]} \varphi(x, r)\, dZ_r \big|_G$, then, as in the proof of Lemma 4.2 (Step 1), the dominated convergence theorem, (4.18) and the fact that $E(A_{T-}^2) < \infty$ imply that there exists a measurable function $Y^{T,t}: X \times \Omega \longrightarrow G$ such that $|Y^{T,t}|_G \in L^1(\mu)$ w.p.1 and

$$Y_x^{T,t} = I_{]0,T[}(t) \int_{]0,t]} \varphi(x,r)\, dZ_r \text{ w.p. 1 for almost all } x \in X.$$

Finally, as in the proof of Lemma 4.2 (Step 2), we can construct a measurable function $Y^t: X \times \Omega \longrightarrow G$ which is Bochner integrable with respect to μ w.p.1 and (3.1) holds.

5. An application. Consider the stochastic evolution equation for the process $X \equiv \{X_t, 0 \leq t \leq T\}$ formally written as

$$\begin{aligned} dX_t &= A(t)X_t\, dt + dZ(X)_t,\ 0 \leq s \leq t \leq T, \\ X_s &= \xi, \end{aligned} \tag{5.1}$$

where ξ is an H-valued $\mathcal{F}_s$-measurable random variable, $\{A(t), 0 \leq t \leq T\}$ is a family of closed linear operators on H which generates an evolution system $\{U(t,s), 0 \leq s \leq t \leq T\}$ on H and $Z(X)$ is an H-valued semimartingale which may depend on X.

Equation (5.1) is a symbolic expression and it may be interpreted in various different ways. León [**9**], and Gorostiza and León [**5**] defined several types of solutions of (5.1) and investigated relationships between them. In particular, under the assumptions that there exists a real separable Hilbert space W, densely and continuously embedded in H, such that $W \subset \bigcap_{t\in[0,T]} \mathcal{D}(A^*(t))$, where $\mathcal{D}(A^*(t))$ is the domain of $A^*(t)$, and $\{U^*(t,s)\}$ is a weak forward adjoint evolution operator (WFA), i.e. $\int_s^t \langle x, U^*(r,s)A^*(r)y\rangle\, dr = \langle x, U^*(t,s)y\rangle - \langle x, y\rangle$ for all $s < t$, $x \in H$ and $y \in W$, León [**9**] proved the following result (see [**9**] for full details).

THEOREM 5.1. *Let X be a weak evolution solution of (5.1), i.e. X is a progressively measurable process such that*

$$\langle X_t, y\rangle = \langle \xi, U^*(t,s)y\rangle + \int_{]s,t]} \langle U^*(t,r)y, dZ(X)_r\rangle \text{ w.p. 1}$$

for each $t \in [s,T]$ and $y \in H$, and assume

(i) $A^*(\cdot)y \in L^1([0,T], H)$ for all $y \in W$.

(ii) The function $U^*(t,\cdot)y$ is of bounded variation on $[0,t]$ for all $t \in [s,T]$ and $y \in W$.

(iii) The function $U^*(t,\cdot)A^*(t)y$ is of bounded variation on $[0,t]$ for all $y \in W$ and it has the same control process V^y for all $t \in [s,T]$.

Then X is also a W-solution of (5.1), i.e.

$$\langle X_t, y\rangle = \langle \xi, y\rangle + \int_s^t \langle X_r, A^*(r)y\rangle\, dr + \langle Z(X)_t - Z(X)_s, y\rangle \text{ w.p. 1}$$

for each $t \in [s, T]$ *and* $y \in W$.

Assumptions (ii) and (iii) were needed in order to apply the integration by parts formula, which was our basic tool. On the other hand, the stochastic Fubini theorem can be used instead of the integration by parts formula, and then assumptions (ii) and (iii) of Theorem 5.1 can be dropped. Indeed, fix $y \in W$ and $t \in [s, T]$. Then Theorem 3.1 implies that there exists a measurable function $Y^t \colon [s, t] \times \Omega \longrightarrow R$, Lebesgue integrable w.p. 1, such that

$$(5.2) \qquad \int_{]0,t]} I_{]s,r]}(r') \langle U^*(r, r')A^*(r)y, dZ(X)_{r'} \rangle = Y_r^t \text{ w.p. 1 for almost all } r \in [s, t]$$

and

$$(5.3) \qquad \int_{]s,t]} \int_{r'}^{t} \langle U^*(r, r')A^*(r)y, \cdot \rangle \, dr \, dZ(X)_{r'} = \int_s^t Y_r^t \, dr \quad \text{w.p. 1}$$

Since X is a weak evolution solution, then (5.2) implies

$$\langle X_r, A^*(r)y \rangle = \langle \xi, U^*(r, s)A^*(r)y \rangle + Y_r^t \quad \text{w.p. 1 for almost all } r \in [s, t].$$

The measurability of each term and Fubini's theorem imply that the last expression is integrable in $r \in [s, t]$ w.p. 1. Then by (5.3) we have

$$\begin{aligned}\int_s^t \langle X_r, A^*(r)y \rangle \, dr = &\int_s^t \langle \xi, U^*(r, s)A^*(r)y \rangle \, dr \\ &+ \int_{]s,t]} \int_{r'}^{t} \langle U^*(r, r')A^*(r)y, \cdot \rangle \, dr \, dZ(X)_{r'} \quad \text{w.p. 1.}\end{aligned}$$

The WFA property implies

$$\begin{aligned}\int_s^t \langle X_r, A^*(r)y \rangle \, dr &= \langle \xi, U^*(t, s)y - y \rangle \\ &\quad + \int_{]s,t]} \langle U^*(t, r)y - y, \ dZ(X)_r \rangle \\ &= \langle \xi, U^*(t, s)y - y \rangle \\ &\quad + \int_{]s,t]} \langle U^*(t, r)y, \ dZ(X)_r \rangle \\ &\qquad - \langle Z(X)_t - Z(X)_s, y \rangle \quad \text{w.p. 1.}\end{aligned}$$

Finally, since X is a weak evolution solution, it is also a W-solution.

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