THE GIBBS PHENOMENON FOR TAYLOR MEANS AND FOR [F, d_n] MEANS

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1. Introduction. The Gibbs phenomenon may be described, quite generally, as follows. Let a sequence $\{f_n(x)\}$ $(n = 0, 1, 2, \ldots,)$ converge to a function $f(x)$ for x in the interval $x_0 < x < x_0 + h$. We say that $\{f_n(x)\}$ displays the Gibbs phenomenon in a right-hand neighbourhood of the point x_0, if

$$\overline{\lim_{\substack{n\to\infty \\ x\to x_0+0}}} f_n(x) > f(x_0 + 0), \text{ or } \overline{\lim_{\substack{n\to\infty \\ x\to x_0+0}}} f_n(x) < f(x_0 + 0).$$

A similar definition holds for a left-hand neighbourhood. If $\{f_n(x)\}$ displays the Gibbs phenomenon at both sides of x_0, we say simply that $\{f_n(x)\}$ displays Gibbs phenomenon at the point x_0. We define the Gibbs set of the sequence $\{f_n(x)\}$ at the point x_0 to be the union of all numbers η such that $f_n(x) \to \eta$ as $n \to \infty$ and $x \to x_0$ through appropiate values. Here we will be concerned not with the Gibbs phenomenon in general, but with the Gibbs phenomenon as displayed by the sequence of partial sums of a Fourier series. Further, we will restrict ourselves to Fourier series representing functions which satisfy the Dirichlet conditions.

The following is a description of the Gibbs phenomenon for the case we will consider. Suppose the function $f(x)$ satisfies the Dirichlet conditions in the interval $-\pi \leqslant x \leqslant \pi$, and suppose a is a discontinuity of the function $f(x)$. Let $\{s_n(x)\}$ denote the sequence of partial sums of the Fourier series for $f(x)$, then by proper choice of a sequence $\{t_n\}$, which approaches a as n approaches infinity, we can make the sequence $\{s_n(t_n)\}$ approach any number in the closed interval whose endpoints are

$$(1.1)\quad f(a-0) + \frac{f(a+0) - f(a-0)}{\pi} I \quad\text{and}\quad f(a+0) - \frac{f(a+0) - f(a-0)}{\pi} I,$$

where

$$I = \int_\pi^\infty \frac{\sin y}{y}\, dy = -.28 \ldots .$$

Received October 26, 1959. This paper is a condensation of a portion of the author's dissertation written at the University of Kentucky under the direction of Professor V. F. Cowling.

https://doi.org/10.4153/CJM-1960-059-2 Published online by Cambridge University Press

A proof that this conclusion follows from the above hypothesis can be found in Bôcher **(3)** or Carslaw **(4)**. Note that for our case the Gibbs set at the point a is composed of all points having abscissa a and ordinates in the closed interval whose endpoints are given by (1.1).

Let $A = (a_{nk})$ and $\{s_n\}$ $(n, k, = 0, 1, 2, 3, \ldots,)$ be a matrix and a sequence of complex numbers, respectively. Let the members of the sequence $\{\sigma_n\}$ be defined by

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k,$$

then we say $\{\sigma_n\}$ is the A transform of $\{s_n\}$. The matrix $A = (a_{nk})$ is called regular if

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sigma_n,$$

whenever the first limit exists. Necessary and sufficient conditions in order that a matrix $A = (a_{nk})$ be regular are the well known Silverman-Toeplitz conditions:

$$\sum_{k=0}^{\infty} |a_{nk}| \leqslant K \qquad (n = 0, 1, 2, \ldots,), \tag{1.2}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1, \tag{1.3}$$

$$\lim_{n\to\infty} a_{nk} = 0 \qquad (k = 0, 1, 2, \ldots,), \tag{1.4}$$

where K is a constant independent of n.

If $f(x)$ is a function satisfying the Dirichlet conditions and $\{s_n(x)\}$ is the sequence of partial sums of the Fourier series for $f(x)$, then it is well-known that for a given x the sequence $\{s_n(x)\}$ approaches

$$\frac{f(x+0) + f(x-0)}{2}$$

as n approaches infinity. If we transform the sequence $\{s_n(x)\}$ into the sequence $\{R_n(x)\}$ by a regular sequence to sequence matrix A, then it follows that for given x the sequence $\{R_n(x)\}$ also approaches

$$\frac{f(x+0) + f(x-0)}{2}$$

as n approaches infinity. A question then presents itself. Does the sequence $\{R_n(x)\}$ also display the Gibbs phenomenon at every finite discontinuity of $f(x)$? This question has been studied for Césaro means by Cramer **(7)** and Gronwall **(8)**, for Euler means by Szasz **(13)**, for Borel means by Lorch **(12)**, for Hausdorff means by Szasz **(14)**, and for Riesz means by Kuttner **(10)**, to cite only a few cases. Our purpose is to study this question for the Taylor matrix and the $[F, d_n]$ matrix.

DEFINITION (1.1). *Let $f(x)$ denote any function satisfying the Dirichlet conditions and having a discontinuity at the point a. Let $\{S_n(x)\}$ denote the sequence of partial sums of the Fourier series representing $f(x)$. Let $\{R_n(x)\}$ denote the A transform of $\{S_n(x)\}$. If for every such function $f(x)$ the sequence $\{R_n(x)\}$ displays the Gibbs phenomenon at the point a and has the same Gibbs set at the point a as does $\{S_n(x)\}$, we say that the A transform completely preserves the Gibbs phenomenon for Fourier series.*

In some cases the sequence $\{R_n(x)\}$ displays the Gibbs phenomenon at $x = a$, but does not have the same Gibbs set at a as does the sequence $\{S_n(x)\}$. We use the word completely here to indicate that we are excluding such cases from our consideration. In such cases one might simply say that the A transform preserves the Gibbs phenomenon for Fourier series.

A short calculation shows that the first number in (1.1) can be written as

$$\frac{f(a+0)+f(a-0)}{2}+\frac{f(a+0)-f(a-0)}{\pi}\int_0^{-\pi}\frac{\sin y}{y}\,dy,$$

and the second number as

$$\frac{f(a+0)+f(a-0)}{2}+\frac{f(a+0)-f(a-0)}{\pi}\int_0^{\pi}\frac{\sin y}{y}\,dy.$$

Since

$$\int_0^{\tau}\frac{\sin y}{y}\,dy$$

is a continuous function of τ, it follows that any number in the interval whose end points are given by (1.1) can be written in the form

$$\frac{f(a+0)+f(a-0)}{2}+\frac{f(a+0)-f(a-0)}{\pi}\int_0^{\tau}\frac{\sin y}{y}\,dy \tag{1.2}$$

by proper choice of τ in the interval $-\pi \leqslant \tau \leqslant \pi$. Hence we have the following theorem.

THEOREM (1.1). *Let $f(x)$, A, a, $\{S_n(x)\}$, and $\{R_n(x)\}$ denote the same quantities as in Definition (1.1). If for each τ in $-\pi \leqslant \tau \leqslant \pi$, there is a sequence $\{t_n\}$, with $\lim_{n\to\infty} t_n = a$ and so that*

$$\lim_{n\to\infty} R_n(t_n) = \frac{f(a+0)+f(a-0)}{2}+\frac{f(a+0)-f(a-0)}{\pi}\int_0^{\tau}\frac{\sin y}{y}\,dy,$$

then the A transform completely preserves the Gibbs phenomenon for Fourier series.

2. Taylor and $[F, d_n]$ transforms. The elements a_{nk} of the Taylor matrix T_r are defined by the relation

$$\frac{(1-r)^{n+1}\theta^n}{(1-r\theta)^{n+1}} = \sum_{k=0}^{\infty} a_{nk}\theta^k \qquad (|r\theta| < 1). \tag{2.1}$$

It is shown by Cowling **(5)** that the Taylor matrix satisfies the Silverman-Toeplitz conditions (1.2), (1.3), and (1.4), and thus is regular if and only if $0 \leqslant r < 1$. A short history of the Taylor matrix, and a list of the basic papers concerning it are to be found in a paper by Cowling and Piranian **(6)**.

The elements P_{nk} of the $[F, d_n]$ matrix are defined by the relation

$$(2.2)\qquad P_{00} = 1$$
$$\prod_{j=1}^{n} \frac{\theta + d_j}{1 + d_j} = \sum_{k=0}^{\infty} P_{nk}\theta^k, \qquad d_n \geqslant 0\ (n = 1, 2, 3, \ldots,).$$

This matrix was studied by Jakimovski **(9)**, who shows that it is regular if and only if

$$\sum_{n=1}^{\infty} d_n^{-1} = +\infty.$$

The $[F, d_n]$ matrix is a generalization of the Euler and Lototsky matrices. If we let $d_n = n - 1$ in the $[F, d_n]$ matrix, we get a matrix whose elements a_{nk} are given by

$$\theta(\theta + 1)(\theta + 2) \ldots (\theta + n - 1) = \sum_{k=0}^{n} n! a_{nk}\theta^k.$$

This is the matrix of Lototsky **(11)**. It is shown by Lototsky and Agnew **(2)** that this marix is regular. If we let $d_n = (1 - r)/r$ in the $[F, d_n]$ matrix, we get a matrix whose elements a_{nk} are given by

$$[r\theta + (1 - r)]^n = \sum_{k=0}^{n} a_{nk}\theta^k.$$

This is the well-known Euler matrix. It is shown by Agnew **(1)** that this matrix is regular for $0 < r \leqslant 1$.

3. Preliminary theorem.

We first state the following lemma.

LEMMA (3.1). *Suppose $\{S_n(x)\}$ is a sequence which approaches the function $f(x)$ uniformly in the interval $a \leqslant x \leqslant b$. Further, suppose there exists constants M and $M_n (n = 0, 1, 2, \ldots,)$ such that $|f(x)| \leqslant M$ and $|S_n(x)| \leqslant M_n$ for all x in the interval $a \leqslant x \leqslant b$. If $\{R_n(x)\}$ denotes the transform of the sequence $\{S_n(x)\}$ by any regular sequence to sequence transform A, then the sequence $\{R_n(x)\}$ approaches $f(x)$ uniformly for $a \leqslant x \leqslant b$.*

From the manner in which this lemma is stated, the reader can easily construct the proof using the Silverman-Toeplitz conditions (1.2), (1.3), and (1.4).

THEOREM (3.1). *Let $A = (a_{nk})$ denote a regular sequence to sequence matrix. Define the function $\phi(x)$ by*

$$(3.1)\qquad \phi(x) = \begin{cases} -\pi/2 & -\pi < x < 0 \\ \pi/2 & 0 < x < \pi, \end{cases}$$
$$\phi(-\pi) = \phi(0) = \phi(\pi) = 0, \quad \text{and} \qquad \phi(x) = \phi(x + 2\pi).$$

https://doi.org/10.4153/CJM-1960-059-2 Published online by Cambridge University Press

Let $\{s_n(x)\}$ denote the sequence of partial sums of the Fourier series for $\phi(x)$. Let $\{\sigma_n(x)\}$ denote the A transform of $\{s_n(x)\}$. Now, if $\{\sigma_n(x)\}$ displays the Gibbs phenomenon at zero and has the same Gibbs set as $\{s_n(x)\}$ at zero, then the A transform completely preserves the Gibbs phenomenon for Fourier series.

Proof. Let $f(x)$ be any function with period 2π satisfying the Dirichlet conditions in the interval $-\pi \leqslant x \leqslant \pi$ and having a discontinuity at the point a. Since $f(a)$ does not effect the Fourier series for $f(x)$, let us define $f(a)$ by $f(a) = \frac{1}{2}\{f(a+0) + f(a-0)\}$. Since $f(x)$ satisfies Dirichlet's conditions, it can be represented by a Fourier series whose sequence of partial sums will be denoted by $\{S_n(x)\}$.

Define the function $\Psi(x)$ by

$$(3.2)\quad \Psi(x) = f(x) - \frac{f(a+0)+f(a-0)}{2} - \frac{f(a+0)-f(a-0)}{\pi}\,\phi(x-a).$$

Since $\Psi(x)$ satisfies Dirichlet's conditions, it can be represented by a Fourier series, whose sequence of partial sums will be denoted by $\{\zeta_n(x)\}$.

A short computation shows that $\phi(x-a)$ has the Fourier series expansion

$$\sum_{v=1}^{\infty} [1-(-1)^v]\left[\frac{\cos va}{v}\sin vx - \frac{\sin va}{v}\cos vx\right].$$

Let $\{s_n(x-a)\}$ denote the sequence of partial sums of this Fourier series.

If we replace $f(x)$ by its value from (3.2), compute the coefficients of its Fourier series expansion, and then sum from 1 to n, we get

$$(3.3)\quad S_n(x) = \zeta_n(x) + \frac{f(a+0)+f(a-0)}{2} + \left\{\frac{f(a+0)-f(a-0)}{\pi}\right\} s_n(x-a).$$

Let $\{R_n(x)\}$, $\{\Psi_n(x)\}$, and $\{\sigma_n(x-a)\}$ denote, respectively, the A transforms of the sequences $\{S_n(x)\}$, $\{\zeta_n(x)\}$, and $\{s_n(x-a)\}$. Then applying the A transform to the sequence in both sides of (3.3), we obtain

$$(3.4)\quad R_n(x) = \Psi_n(x) + \frac{f(a+0)+f(a-0)}{2}\sum_{k=0}^{\infty} a_{nk} + \frac{f(a+0)-f(a-0)}{\pi}\,\sigma_n(x-a).$$

From (3.2), $\Psi(a+0) = \Psi(a-0) = \Psi(a) = 0$, and so $\Psi(x)$ is continuous at $x = a$. Since $\Psi(x)$ satisfies the Dirichlet conditions, there exist numbers α and β ($\alpha < a < \beta$) such that $\Psi(x)$ is continuous for $\alpha \leqslant x \leqslant \beta$. From Lemma (3.1) it follows that $\{\Psi_n(x)\}$ approaches $\Psi(x)$ uniformly for $\alpha \leqslant x \leqslant \beta$. Hence given $\epsilon > 0$, there exist an integer n_0 and a number δ such that if $n \geqslant n_0$ and $|x-a| < \delta$, then

$$(3.5)\quad |\Psi_n(x)| < \epsilon.$$

By assumption $\{\sigma_n(x-a)\}$ displays the Gibbs phenomenon at $x = a$ and has the same Gibbs set at $x = a$ as does $\{s_n(x)\}$. Going back to (1.1), this means that by proper choice of a sequence $\{t_n\}$, such that $\lim_{n\to\infty} t_n = a$, we can make the sequence $\{\sigma_n(t_n - a)\}$ approach any number in an interval whose endpoints are

$$\phi(0-) + \frac{\phi(0+) - \phi(0-)}{\pi} \int_\pi^\infty \frac{\sin y}{y}\, dy = \int_0^{-\pi} \frac{\sin y}{y}\, dy$$

and

$$\phi(0+) - \frac{\phi(0+) - \phi(0-)}{\pi} \int_\pi^\infty \frac{\sin y}{y}\, dy = \int_0^{\pi} \frac{\sin y}{y}\, dy.$$

Let τ $(-\pi \leqslant \tau \leqslant \pi)$ and $\epsilon > 0$ be given. It now follows that there exists an integer N_1 and a sequence $\{t_n\}$, with $\lim_{n\to\infty} t_n = a$, such that if $n \geqslant N_1$, then

$$\left| \sigma_n(t_n - a) - \int_0^\tau \frac{\sin y}{y}\, dy \right| < \frac{\epsilon\pi}{3|f(a+0) - f(a-0)|}.$$

From (3.5) there exists an integer N_2 such that if $n \geqslant N_2$, then

$$|\Psi_n(t_n)| < \tfrac{1}{3}\epsilon.$$

From (1.3) there exists an integer N_3 such that if $n \geqslant N_3$, then

$$\left| \sum_{k=0}^\infty a_{nk} - 1 \right| < \frac{2\epsilon}{3|f(a+0) + f(a-0)|}.$$

Rearranging (3.4), inserting absolute values, and replacing x by t_n yields

$$\begin{aligned} &\left| R_n(t_n) - \left[\frac{f(a+0) + f(a-0)}{2} + \frac{f(a+0) - f(a-0)}{\pi} \int_0^\tau \frac{\sin y}{y}\, dy \right] \right| \\ &\qquad \leqslant |\Psi_n(t_n)| + \left| \frac{f(a+0) + f(a-0)}{2} \right| \left| \sum_{k=0}^\infty a_{nk} - 1 \right| \\ &\qquad\qquad + \frac{|f(a+0) - f(a-0)|}{\pi} \left| \sigma_n(t_n - a) - \int_0^\tau \frac{\sin y}{y}\, dy \right|. \end{aligned}$$

Let $N = \max(N_1, N_2, N_3)$, then for $n \geqslant N$

$$\left| R_n(t_n) - \left[\frac{f(a+0) + f(a-0)}{2} + \frac{f(a+0) - f(a-0)}{\pi} \int_0^\tau \frac{\sin y}{y}\, dy \right] \right| < \epsilon.$$

The theorem now follows from an application of Theorem (2.1).

4. The two main theorems.

THEOREM (4.1). *The Taylor transform completely preserves the Gibbs phenomenon for Fourier series.*

Proof. A short computation shows that the function $\phi(x)$ given by (3.1) has the Fourier series expansion

$$2 \sum_{v=1}^\infty \frac{\sin(2v-1)x}{2v-1}.$$

Let $\{s_n(x)\}$ denote the sequence of partial sums of this series, then another short computation shows that

$$s_n(x) = \int_0^x \frac{\sin 2nt}{\sin t}\, dt.$$

In the following discussion we consider only values of x in the interval $0 \leqslant x \leqslant \frac{1}{2}\pi$. The Taylor transform $\{\sigma_n(x)\}$ of the sequence $\}s_n(x)\}$ is given by

$$\sigma_n(x) = \sum_{k=n}^{\infty} \int_0^x (1-r)^{n+1} \binom{k}{n} r^{k-n} \frac{\sin 2kt}{\sin t}\, dt. \tag{4.1}$$

Since $0 \leqslant t \leqslant \frac{1}{2}\pi$, we have

$$\frac{\sin 2kt}{\sin t} \leqslant \pi k.$$

Hence,

$$\left| \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} \frac{\sin 2kt}{\sin t} \right| \leqslant \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} \pi k = \frac{\pi(n+r)}{(1-r)},$$

or the series (4.1) is uniformly convergent for $0 \leqslant t \leqslant \pi/2$. Therefore, we may interchange the order of integration and summation in (4.1) which gives us

$$\sigma_n(x) = \int_0^x \frac{(1-r)^{n+1}}{\sin t} \sum_{k=n}^{\infty} \binom{k}{n} r^{k-n} \sin 2kt\, dt$$
$$= \int_0^x \frac{1}{\sin t} \operatorname{Im}\left\{ \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} e^{2kit} \right\} dt.$$

Using (2.1) with $\theta = e^{2it}$ to sum the series, we get

$$\sigma_n(x) = \int_0^x \frac{(1-r)^{n+1}}{\sin t} \operatorname{Im}\left\{ \frac{e^{2nit}}{(1-re^{2it})^{n+1}} \right\} dt. \tag{4.2}$$

Define ρ and θ by the relation

$$\rho e^{-i\theta} = 1 - r\, e^{2it}. \tag{4.3}$$

From (4.3) it follows that

$$\rho \cos\theta = 1 - r\cos 2t, \tag{4.4a}$$

$$\rho \sin\theta = r \sin 2t, \tag{4.4b}$$

$$\rho^2 = 1 - 2r\cos 2t + r^2, \tag{4.4c}$$

$$0 \leqslant \theta \leqslant (\pi/2), \quad \text{and} \tag{4.4d}$$

$$(1-r) \leqslant \rho. \tag{4.4e}$$

Substituting the left-hand member of (4.3) for the right-hand member of (4.3) in (4.2), we obtain

$$\sigma_n(x) = \int_0^x \frac{1}{\sin t} \left(\frac{1-r}{\rho}\right)^{n+1} \sin[(n+1)\theta + 2nt]\, dt. \tag{4.5}$$

Using (4.4c), it follows that

$$1 - \left(\frac{1-r}{\rho}\right)^2 = \frac{4r}{\rho^2}\sin^2 t.$$

Since $r < 1$ and $\rho > 0$, (4.4c) implies $0 \leqslant (1-r)/\rho \leqslant 1$. Hence we have that

$$0 \leqslant 1 - \left(\frac{1-r}{\rho}\right) \leqslant 1 - \left(\frac{1-r}{\rho}\right)^2 \leqslant \frac{4rt^2}{\rho^2}. \tag{4.6}$$

Applying the inequalities (4.4e) and (4.6), it follows that

$$0 \leqslant 1 - \left(\frac{1-r}{\rho}\right)^{n+1} = \left[1 - \left(\frac{1-r}{\rho}\right)\right]\sum_{v=0}^{n}\left(\frac{1-r}{\rho}\right)^v \leqslant (n+1)\frac{4rt^2}{\rho^2}$$
$$\leqslant \frac{4(n+1)rt^2}{(1-r)^2}.$$

Therefore, we may write

$$\left(\frac{1-r}{\rho}\right)^{n+1} = 1 - \mu\frac{(n+1)rt^2}{(1-r)^2}, \tag{4.7}$$

where $0 \leqslant \mu \leqslant 4$. Note that μ is a function of n, r, and t. Substituting the value of $[(1-r)/\rho]^{n+1}$ from (4.7) into (4.5), we get

$$\sigma_n(x) = \int_0^x \frac{\sin[(n+1)\theta + 2nt]}{\sin t}\,dt - \int_0^x \frac{\mu(n+1)rt^2\sin[(n+1)\theta + 2nt]}{(1-r)^2\sin t}\,dt \equiv I + I'. \tag{4.8}$$

Making use of the well-known inequality $0 \leqslant x - \sin x \leqslant x^3$, valid for $x \geqslant 0$, we obtain the inequality

$$|\rho\theta - 2rt| \leqslant \rho(\theta - \sin\theta) + r(2t - \sin 2t) \leqslant \rho\theta^3 + 8rt^3.$$

It follows from (4.6) that $(\rho + r - 1) \leqslant (4rt^2/\rho)$. Hence

$$|(1-r) - 2rt| \leqslant |\rho\theta - 2rt| + (r + \rho - 1)\,\theta \leqslant \rho\theta^3 + 8rt^3 + (4rt^2\theta/\rho).$$

From (4.4b) and (4.4d), it follows that

$$\theta \leqslant (\pi r/2\rho)\sin 2t \leqslant (\pi rt/\rho).$$

Therefore,

$$|(1-r) - 2rt| \leqslant \frac{\pi^3 r^3 t^3}{\rho^2} + 8rt^3 + \frac{4\pi r^2 t^3}{\rho^2}.$$

Using (4.4e), this inequality becomes

$$\left|\theta - \frac{2rt}{1-r}\right| \leqslant \frac{\pi^3 r^3 t^3}{(1-r)^3} + \frac{8rt^3}{1-r} + \frac{4\pi r^2 t^3}{(1-r)^3}.$$

Hence we have that

$$\theta = \frac{2rt}{1-r} + \lambda t^3 \tag{4.9}$$

where

$$|\lambda| \leqslant \frac{\pi^3 r^3}{(1-r)^3} + \frac{8r}{1-r} + \frac{4\pi r^2}{(1-r)^3} \leqslant \frac{54}{(1-r)^3}.$$

Note that λ is a function of r and t. Upon replacing θ by the right-hand member of (4.9), the integral I in (4.8) becomes

$$I = \int_0^x \frac{\sin L_{nr}t \cos[(n+1)\lambda t^3]}{\sin t}\,dt + \int_0^x \frac{\cos L_{nr}t \sin[(n+1)\lambda t^3]}{\sin t}\,dt,$$

where

$$L_{nr} = (n+1)\left(\frac{2r}{1-r}\right) + 2n.$$

Putting this value of I into (4.8) and adding

$$-\int_0^x t^{-1} \sin L_{nr}t\,dt$$

to both sides of (4.8), we get

$$\begin{aligned} \sigma_n(x) - \int_0^x \frac{\sin L_{nr}t}{t}\,dt = &\\ &-\frac{r(n+1)}{(1-r)^2}\int_0^x \mu t^2 \frac{\sin[(n+1)\theta + 2nt]}{\sin t}\,dt \\ &+ \int_0^x \frac{\cos L_{nr}t \sin[(n+1)\lambda t^3]}{\sin t}\,dt \\ &- \int_0^x \sin L_{nr}t \left\{\frac{1}{t} - \frac{\cos[(n+1)\lambda t^3]}{\sin t}\right\} dt \\ &\equiv T_1 + T_2 + T_3. \end{aligned} \tag{4.10}$$

Since $\sin t \geqslant (2t/\pi)$ for $0 \leqslant t \leqslant \frac{1}{2}\pi$, we have that

$$|T_1| \leqslant \frac{r(n+1)}{(1-r)^2}\int_0^x \frac{\mu\pi t^2}{2t}\,dt \leqslant \frac{r(n+1)\pi x^2}{(1-r)^2}$$

and

$$|T_2| \leqslant (n+1)\int_0^x \frac{\pi\lambda t^3}{2t} \leqslant \frac{9\pi(n+1)x^3}{(1-r)^3}.$$

After expanding $\sin t$ and $t\cos[(n+1)\lambda t^3]$ in series, it follows from a well-known theorem for convergent alternating series that

$$|\sin t - t\cos[(n+1)\lambda t^3]| \leqslant t^3 + (n+1)^2\lambda^2 t^7.$$

Applying this inequality, we get

$$|T_3| \leqslant \int_0^x \frac{\pi[t^3 + (n+1)^2\lambda^2 t^7]}{2t^2}\,dt \leqslant x^2 + \frac{243\pi(n+1)^2 x^6}{(1-r)^6}.$$

Inserting absolute values on both sides of (4.10) and using the above inequalities for $|T_1|$, $|T_2|$, and $|T_3|$, we have

$$(4.11)\qquad \left|\sigma_n(x) - \int_0^x \frac{\sin L_{nr}t}{t}\,dt\right| \leqslant \frac{r\pi(n+1)x^2}{(1-r)^2} + \frac{9\pi(n+1)x^3}{(1-r)^3} + x^2 + \frac{243\pi(n+1)^2 x^6}{(1-r)^6}.$$

Let τ such that $0 \leqslant \tau \leqslant \pi$ be given, and define the sequence $\{t_n\}$ by $t_n = \tau/L_{nr}$, then for $n \geqslant 1$ we have $t_n \leqslant \pi/2$. Therefore, if $n \geqslant 1$, we may replace x by t_n in (4.11). This gives

$$\left|\sigma_n(t_n) - \int_0^{t_n} \frac{\sin L_{nr}t}{t}\,dt\right| \leqslant \frac{r\pi(n+1)\tau^2}{(1-r)^2 L_{nr}^2} + \frac{9\pi(n+1)\tau^3}{(1-r)^3 L_{nr}^3} + \frac{\tau^2}{L_{nr}^2} + \frac{243\pi(n+1)^2\tau^6}{(1-r)^6 L_{nr}^6},$$

where $r(0 \leqslant r \leqslant 1)$ is fixed. Upon setting $y = L_{nr}\,t$, it follows that given $\epsilon > 0$ there exists an integer N $(N \geqslant 1)$ such that if $n \geqslant N$, then

$$\left|\sigma_n(t_n) - \int_0^{\tau} \frac{\sin y}{y}\,dy\right| < \epsilon.$$

Let $\epsilon > 0$ and τ such that $-\pi \leqslant \tau \leqslant 0$ be given. Since $-\tau$ is in the interval $0 \leqslant -\tau \leqslant \pi$, we have just shown the existence of a sequence $\{-t_n\}$, with $-t_n = -\pi/L_{nr}$, and an integer N such that if $n \geqslant N$, then

$$\left|\sigma_n(-t_n) - \int_0^{-\tau} \frac{\sin y}{y}\,dy\right| < \epsilon.$$

It follows from (4.1) that $\sigma_n(x) = -\sigma_n(x)$. Substituting $-\sigma_n(t_n)$ for $\sigma_n(-t_n)$ and $-y$ for y, this inequality becomes

$$\left|\sigma_n(t_n) - \int_0^{\tau} \frac{\sin y}{y}\,dy\right| < \epsilon.$$

Since

$$\frac{\phi(0+) + \phi(0-)}{2} = 0 \quad\text{and}\quad \frac{\phi(0+) - \phi(0-)}{\pi} = 1,$$

it now follows from (1.2) that $\{\sigma_n(x)\}$ displays the Gibbs phenomenon at $x = 0$. Note we have also shown that $\{\sigma_n(x)\}$ has the same Gibbs set at zero as does the sequence $\{s_n(x)\}$. Theorem (4.1) now follows from an application of Theorem (3.1).

THEOREM (4.2). *The $[F, d_n]$ transform completely preserves the Gibbs phenomenon for Fourier series.*

Proof. Let $\{s_n(x)\}$ denote the sequence of partial sums of the Fourier series representing the function $\phi(x)$ as given by (3.1). As in Theorem (4.1),

$$s_n(x) = \int_0^x \frac{\sin 2nt}{\sin t}\, dt.$$

In the following discussion we consider only values of x in the interval $0 \leqslant x \leqslant \pi/4$. Let $\{\sigma_n(x)\}$ denote that $[F, d_n]$ transform of $\{s_n(x)\}$, then

$$\sigma_n(x) = \sum_{k=0}^{n} P_{nk} \int_0^x \frac{\sin 2kt}{\sin t}\, dt \tag{4.12}$$

$$= \int_0^x \frac{1}{\sin t} \sum_{k=0}^{n} P_{nk} \sin 2kt\, dt,$$

where the numbers P_{nk} are defined by (2.2). Using (2.2), we can write the last sum in (4.12) as

$$\sum_{k=0}^{n} P_{nk} \operatorname{Im} e^{2kit} = \operatorname{Im}\left\{\sum_{k=0}^{n} P_{nk}\, e^{2kit}\right\} = \operatorname{Im}\left\{\prod_{j=1}^{n} \left(\frac{e^{2it} + d_j}{1 + d_j}\right)\right\}.$$

Replacing the sum in the last member of (4.12) by this product, we have

$$\sigma_n(x) = \int_0^x \frac{1}{\sin t} \operatorname{Im}\left\{\prod_{j=1}^{n} \left(\frac{e^{2it} + d_j}{1 + d_j}\right)\right\}. \tag{4.13}$$

Let us define ρ_j and θ_j $(j = 1, 2, 3, \ldots ,)$ by

$$\rho_j\, e^{i\theta_j} = e^{2it} + d_j. \tag{4.14}$$

From (4.14) it follows that

$$\rho_j \cos \theta_j = \cos 2t + d_j, \tag{4.15a}$$

$$\rho_j \sin \theta_j = \sin 2t, \tag{4.15b}$$

$$\rho_j^2 = 1 + 2d_j \cos 2t + d_j^2, \tag{4.15c}$$

$$\rho_j \leqslant 1 + d_j, \tag{4.15d}$$

$$0 \leqslant \theta_j \leqslant 2t \leqslant \pi/2. \tag{4.15e}$$

Substituting the left-hand side of (4.14) for the right-hand side of (4.14) in (4.13), we get

$$\sigma_n(x) = \int_0^x \frac{1}{\sin t} \prod_{j=1}^{n} \left(\frac{\rho_j}{1 + d_j}\right) \sin\left(\sum_{j=1}^{n} \theta_j\right) dt. \tag{4.16}$$

From (4.15c) and (4.15d), it follows that

$$0 \leqslant 1 - \left(\frac{\rho_j}{1 + d_j}\right) \leqslant 1 - \left(\frac{\rho_j}{1 + d_j}\right)^2 = \frac{4d_j \sin^2 t}{(1 + d_j)^2} \leqslant \frac{4d_j t^2}{(1 + d_j)^2} \leqslant \frac{4t^2}{1 + d_j},$$

https://doi.org/10.4153/CJM-1960-059-2 Published online by Cambridge University Press

or that

$$1 - \left(\frac{\rho_j}{1 + d_j}\right) \leqslant \frac{4t^2}{1 + d_j}. \tag{4.17}$$

Hence in view of (4.15d), we have

$$\begin{aligned} 0 \leqslant 1 - \prod_{j=1}^{n} \left(\frac{\rho_j}{1 + d_j}\right) &= \left(1 - \frac{\rho_1}{1 + d_1}\right) + \frac{\rho_1}{1 + d_1}\left(1 - \frac{\rho_2}{1 + d_2}\right) \\ &\quad + \ldots + \frac{\rho_1}{1 + d_1}\cdot\frac{\rho_2}{1 + d_2}\cdots\frac{\rho_{n-1}}{1 + d_{n-1}}\left(1 - \frac{\rho_n}{1 + d_n}\right) \\ &\leqslant \sum_{j=1}^{n}\left(1 - \frac{\rho_j}{1 + d_j}\right) \leqslant \sum_{j=1}^{n} \frac{4t^2}{1 + d_j}. \end{aligned}$$

Therefore, we are able to write

$$1 - \prod_{j=1}^{n} \left(\frac{\rho_j}{1 + d_j}\right) = \lambda H_n t^2, \tag{4.18}$$

where $0 \leqslant \lambda \leqslant 4$ and $H_n \equiv \sum_{j=1}^{n} (1 + d_j)^{-1}$. Note that λ is a function of n and t. Substituting in (4.16) the value of

$$\prod_{j=1}^{n} \left(\frac{\rho_j}{1 + d_j}\right)$$

as given in (4.18), we get

$$\begin{aligned} \sigma_n(x) &= \int_0^x \csc t \sin\left(\sum_{j=1}^{n} \theta_j\right) dt - H_n \int_0^x \lambda t^2 \csc t \sin\left(\sum_{j=1}^{n} \theta_j\right) dt \\ &\equiv I + I'. \end{aligned} \tag{4.19}$$

Since $\theta_j \leqslant \pi/2$, the inequality $(2\theta_j/\pi) \leqslant \sin\theta_j$ holds. This plus the well-known inequality $\sin x \leqslant x$, valid for $x \geqslant 0$, applied to (4.15b) yields the inequality

$$\theta_j \leqslant (\pi t/\rho_j) \quad (j = 1, 2, 3, \ldots,).$$

Making use of the well-known inequality $0 \leqslant x - \sin x \leqslant x^3$, valid for $x \geqslant 0$, and the inequality (4.17), we have

$$\begin{aligned} \left|\theta_j - \frac{2t}{1 + d_j}\right| &\leqslant \frac{\rho_j}{1 + d_j}(\theta_j - \sin\theta_j) + \frac{1}{1 + d_j}(2t - \sin 2t) \\ &\qquad + \left(1 - \frac{\rho_j}{1 + d_j}\right)\theta_j \\ &\leqslant \frac{\rho_j \theta_j^3}{1 + d_j} + \frac{8t^3}{1 + d_j} + \frac{4t^2\theta_j}{1 + d_j}. \end{aligned}$$

Since we have just shown $\theta_j \leqslant (\pi t/\rho_j)$, it follows that

$$\left|\theta_j - \frac{2t}{1+d_j}\right| \leqslant \frac{\pi^3 t^3}{1+d_j} + \frac{8t^3}{1+d_j} + \frac{4\pi t^3}{1+d_j},$$

for from (4.15c) $\rho_j \geqslant 1$. Consequently, we may write

$$\theta_j = \frac{2t}{1+d_j} + \frac{\mu_j t^3}{1+d_j} \qquad (j = 1, 2, 3, \ldots,), \tag{4.20}$$

where $|\mu_j| \leqslant \pi^3 + 8 + 4\pi \leqslant 54$. Note that μ_j is a function of d_j and t.

Summing both sides of (4.20) over j, we get

$$\sum_{j=1}^{n} \theta_j = 2tH_n + (\mu_n H_n)t^3, \text{ where } (\mu_n H_n) = \sum_{j=1}^{n} \frac{\mu_j}{1+d_j} \text{ and } |(\mu_n H_n)| \leqslant 54H_n.$$

Substituting for $\sum_{j=1}^{n} \theta_j$ in the integral I in (4.19), and adding

$$-\int_0^x t^{-1} \sin 2H_n t \, dt$$

to both sides of (4.19), we have

$$\begin{aligned} \sigma_n(x) - \int_0^x \frac{\sin 2H_n t}{t} dt &= -\int_0^x \lambda H_n t^2 \csc t \sin\left(\sum_{j=1}^{n} \theta_j\right) dt \\ &+ \int_0^x \csc t \sin (\mu_n H_n) t^3 \cos 2H_n t \, dt \\ &- \int_0^x \sin 2H_n t \left\{\frac{1}{t} - \frac{\cos (\mu_n H_n) t^3}{\sin t}\right\} dt \equiv T_1 + T_2 + T_3. \end{aligned} \tag{4.21}$$

Since $t \leqslant \pi/4$, $\csc t \leqslant \pi/2t$. Hence

$$|T_1| \leqslant (\pi/2) H_n \int_0^x \lambda t dt \leqslant \pi H_n x^2,$$

and

$$|T_2| \leqslant (\pi/2) \int_0^x (\mu_n H_n) t^2 \, dt < 9\pi H_n x^3.$$

After expanding $\sin t$ and $t \cos (\mu_n H_n) t^3$ in series, it follows from a well-known theorem for convergent alternating series that

$$|\sin t - t \cos (\mu_n H_n) t| \leqslant t^3 + (\mu_n H_n)^2 t^7.$$

Applying this inequality and the inequality $(2t/\pi) \leqslant \sin t \leqslant t$, valid for $0 \leqslant t \leqslant \pi/4$, we get

$$|T_3| \leqslant (\pi/2) \int_0^x [t + (\mu_n H_n)^2 t^5] \, dt < x^2 + 243\pi H_n^2 x^6.$$

Now inserting absolute values on both sides of (4.21) and making use of the above inequalities for $|T_1|$, $|T_2|$, and $|T_3|$, we have that

$$\left|\sigma_n(x) - \int_0^x \frac{\sin 2H_n t}{t} dt\right| \leqslant H_n \pi x^2 + 9\pi H_n x^3 + x^2 + 243\pi H_n^2 x^6. \tag{4.22}$$

Let τ such that $0 \leqslant \tau \leqslant \pi$ be given. Since $\sum_{n=1}^{\infty} d_n^{-1} = +\infty$, it follows that $H_n \to \infty$ as $n \to \infty$. Define the sequence $\{t_n\}$ by $t_n = \tau/2H_n$, then for n greater than or equal to some fixed integer n_0 the numbers t_n are in the interval $0 \leqslant t_n \leqslant \frac{1}{4}\pi$. Replacing x by t_n in (4.22), we have for $n \geqslant n_0$ that

$$\left| \sigma_n(t_n) - \int_0^{\tau_n} \frac{\sin 2H_n t}{t}\,dt \right| < \frac{\pi\tau^2}{4H_n} + \frac{9\pi\tau^3}{8H_n^2} + \frac{\tau^2}{4H_n^2} + \frac{243\pi\tau^6}{64H_n^4}.$$

Upon setting $y = 2H_n t$, it follows that given $\epsilon > 0$ there exists an integer $N (N \geqslant n_0)$ such that if $n \geqslant N$, then

$$\left| \sigma_n(t_n) - \int_0^{\tau} \frac{\sin y}{y}\,dy \right| < \epsilon.$$

The remainder of the proof of this theorem follows almost exactly the last two paragraphs in the proof of Theorem (4.1).

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