BIFURCATION FROM EIGENVALUES IN NON-LINEAR MULTIPARAMETER STURM–LIOUVILLE PROBLEMS

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1. Introduction. This paper continues our study of non-linear multiparameter eigenvalue problems. In recent work [**3**], [**4**], [**5**] we have discussed bifurcation from eigenvalues in both abstract and concrete multiparameter problems. In this note we present further simple conditions which will ensure bifurcation from eigenvalues of a multiparameter Sturm–Liouville problem. We consider the regular case and the asymptotic case of bifurcation from infinity.

The system of differential equations in question is

$$y_r''(x_r)+q_r(x_r)y_r(x_r)+\sum_{s=1}^{k}\lambda_s F_{rs}(x_r, y_r, y_r')=0, \qquad x_r\in[a_r, b_r], \tag{1}$$

$$y_r(a_r)\cos\alpha_r+y_r'(a_r)\sin\alpha_r=0, \qquad 0\le\alpha_r<\pi,$$

$$y_r(b_r)\cos\beta_r+y_r'(b_r)\sin\beta_r=0, \qquad 0<\beta_r\le\pi, \qquad 1\le r\le k.$$

The function q_r is assumed to be real valued and continuous on $[a_r, b_r]$, and the functions F_{rs} will take the form

$$F_{rs}(x_r, \xi_r, \eta_r)=a_{rs}(x_r)\xi_r+H_{rs}(x_r, \xi_r, \eta_r)\xi_r, \tag{2}$$

where a_{rs} is real valued and continuous on $[a_r, b_r]$. Various growth conditions will be placed on H_{rs} subsequently. We also impose the definiteness condition

$$|A|(\mathbf{x})=\det\{a_{rs}(x_r)\}>0, \qquad \forall\mathbf{x}=(x_1,\ldots,x_k)\in[\mathbf{a},\mathbf{b}]=\mathop{\times}_{r=1}^{k}[a_r, b_r].$$

This implies that the linearized version of (1), namely

$$y_r''(x_r)+q_r(x_r)y_r(x_r)+\sum_{s=1}^{k}\lambda_s a_{rs}(x_r)y_r(x_r)=0 \qquad (1\le r\le k), \tag{3}$$

has a countable infinity of simple eigenvalues $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_k)\in\mathbb{R}^k$. The monograph of Sleeman [**7**] is a suitable reference for this and other necessary results from linear multiparameter theory.

The eigenvalues $\boldsymbol{\mu}$ of the linear problem (3) can be indexed with multi-indices $\mathbf{n}=(n_1,\ldots,n_k)$, where each n_i is a non-negative integer, in such a way that the eigenvalue $\boldsymbol{\mu}^{\mathbf{n}}$ has a corresponding decomposable eigenfunction $y(\mathbf{x})=\prod_{r=1}^{k}y_r(x_r)$ with y_r having n_r nodal zeros in (a_r, b_r), $1\le r\le k$.

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Given a multi-index $\mathbf{n}$ and a sign factor k-tuple $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_k)$, where $\sigma_r=\pm 1$ $(1\le r\le k)$, $S(\mathbf{n},\boldsymbol{\sigma})$ will denote the set of decomposable functions $y(\mathbf{x})=\prod_{r=1}^{k} y_r(x_r)$ for which y_r has n_r nodal zeros in (a_r, b_r) and has the sign of σ_r near $x_r=a_r$.

We aim to study bifurcation from the simple eigenvalue $\boldsymbol{\mu}^{\mathbf{n}}$ along the line $\boldsymbol{\lambda}=\boldsymbol{\mu}^{\mathbf{n}}+(t-1)\mathbf{w}$, where $\mathbf{w}\neq\mathbf{0}$, $\mathbf{w}\in\mathbb{R}^k$, $t\in\mathbb{R}$. Thus we shall be interested in conditions which guarantee the existence of non-trivial solutions of (1) of the form $\boldsymbol{\lambda}=\boldsymbol{\mu}+(t-1)\mathbf{w}$, $y(\mathbf{x})\in S(\mathbf{n},\boldsymbol{\sigma})$ with t in a neighbourhood of $t=1$. We shall confine our attention to directions $\mathbf{w}$ for which

$$\sum_{s=1}^{k} w_s a_{rs}(x_r)>0, \qquad x_r\in[a_r, b_r], \qquad 1\le r\le k.$$

The existence of directions $\mathbf{w}$ for which this condition is fulfilled is guaranteed by results of Atkinson [**1**]–see also Binding and Browne [**2**].

When we use $\boldsymbol{\lambda}=\boldsymbol{\mu}^{\mathbf{n}}+(t-1)\mathbf{w}$, equation (1) becomes

$$y_r''(x_r)+q_r(x_r)y_r(x_r)+\left\{\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}-w_s)a_{rs}(x_r)\right.$$
$$\left.+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}(x_r, y_r, y_r')\right\}y_r(x_r)+t\sum_{s=1}^{k} w_s a_{rs}(x_r)y_r(x_r)=0, \quad (4)$$

subject to the boundary conditions as in (1). Each of these equations can now be considered as a one-parameter non-linear Sturm–Liouville problem with spectral parameter t of the type studied by Rabinowitz [**6**, Equations (1.46), (1.52)].

2. The regular case. In this section we work under the following hypotheses on the functions H_{rs}.

H1 *Each $H_{rs}(x_r, \xi_r, \eta_r)\xi_r$ is continuous on $[a_r, b_r]\times\mathbb{R}^2$ and is $o((\xi_r^2+\eta_r^2)^{1/2})$ near $(\xi_r, \eta_r)=(0,0)$.*

Equation (4) is now of the type considered by Rabinowitz [**6**, Equation (1.46)] and we may appeal to Theorem 1.47 and the remark immeditely following that reference to claim that for each $\sigma_r=\pm 1$ the rth equation of (4) has a solution with y_r having n_r zeros in (a_r, b_r) and having the sign of σ_r near $x_r=a_r$, and with t in a neighbourhood of $t=1$; that is each member of (4) has $t=1$ as a bifurcation point. It is our task to ensure linked bifurcation for the entire system (4). We do this by imposing conditions which force the bifurcation of each member of (4) to require $t\le 1$, or equivalently $t\ge 1$ thereby ruling out the possibility that for bifurcation one member of (4) requires $t<1$ while another requires $t>1$. To this end we have the following result.

THEOREM 1. *Let H1 hold. Assume that in a neighbourhood of $t=1$, $\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)\times H_{rs}(x_r, \xi_r, \eta_r)\ge 0$ as a function of (x_r, ξ_r, η_r). Let $\boldsymbol{\sigma}$ be a given k-tuple of sign factors. Then there exists $\delta>0$ so that for $t\in(1-\delta, 1)$ there are solutions $\boldsymbol{\lambda}=\boldsymbol{\mu}+(t-1)\mathbf{w}$, $y(x)\in S(\mathbf{n},\boldsymbol{\sigma})$ of (1). These solutions form a continuum meeting $t=1$, $y(\mathbf{x})=0$.*

REMARK. We say that the solutions (t, y) meet $(1,0)$ in the sense that for each $\varepsilon > 0$ there is a solution (t, y) for which

$$\left[|t-1|^2+\sum_{r=1}^{k}\left(\sup_{x_r\in[a_r,b_r]}|y_r(x_r)|\right)^2\right]^{1/2}<\varepsilon.$$

Proof. As we mentioned above it is sufficient to show that each member of (4) bifurcates in a left hand neighbourhood of $t=1$. Suppose then that u_r satisfies

$$u_r''(x_r)+\left\{q_r(x_r)+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}-w_s)a_{rs}(x_r)\right.$$
$$\left.+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}(x_r,u_r,u_r')\right\}u_r(x_r)+t\sum_{s=1}^{k}w_s a_{rs}(x_r)u_r(x_r)=0.$$

Then the linear problem (for unknown function y_r and spectral parameter τ)

$$y_r''(x_r)+\left\{q_r(x_r)+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}-w_s)a_{rs}(x_r)\right.$$
$$\left.+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}(x_r,u_r,u_r')\right\}y_r(x_r)+\tau\sum_{s=1}^{k}w_s a_{rs}(x_r)y_r(x_r)=0$$

has $\tau_k=t$ as its kth eigenvalue.Thus t can be realized via the minimax principle applied to Rayleigh quotients of the form

$$\left\{\int_{a_r}^{b_r}\sum_{s=1}^{k}w_s a_{rs}(x_r)\,|y_r(x_r)|^2\,dx_r\right\}^{-1}\left\{\int_{a_r}^{b_r}\left[-y_r''(x_r)\overline{y_r(x_r)}-(q_r(x_r)\right.\right.$$
$$\left.\left.+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}-w_s)a_{rs}(x_r)+\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}(x_r,u_r,u_r'))\,|y_r(x_r)|^2\right]dx_r\right\}$$
$$\le\left\{\int_{a_r}^{b_r}\sum_{s=1}^{k}w_s a_{rs}(x_r)\,|y_r(x_r)|^2\,dx_r\right\}^{-1}\left\{\int_{a_r}^{b_r}\left[-y_r''(x_r)y_r(x_r)-q_r(x_r)\,|y_r(x_r)|^2\right.\right.$$
$$\left.\left.-\sum_{s=1}^{k}\mu_s^{\mathbf{n}}a_{rs}(x_r)\,|y_r(x_r)|^2\right]dx_r\right\}+1$$

Note further that the problem $y_r''+q_r y_r+\sum_{s=1}^{k}\mu_s^{\mathbf{n}}a_{rs}y_r+\tau\sum_{s=1}^{k}w_s a_{rs}y_r=0$ has $\tau_k{}'=0$ as its kth eigenvalue, and so, when we apply the minimax principle, the above estimate yields $t\le 1$. Our proof here follows the ideas of [**6**, Lemma 2.1]. The statement of the theorem now follows readily.

Using the same arguments we can establish the following result.

THEOREM 2. *Let $H1$ hold. Assume that in a neighbourhood of $t=1$, $\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}(x_r,\xi_r,\eta_r)\le 0$ as a function of (x_r,ξ_r,η_r). Let $\boldsymbol{\sigma}$ be a given k-tuple of sign factors. Then there exists $\delta>0$ so that for $t\in(1,1+\delta)$ there are solutions $\boldsymbol{\lambda}=\boldsymbol{\mu}+(t-1)\mathbf{w}$, $y(\mathbf{x})\in S(\mathbf{n},\boldsymbol{\sigma})$ of (1). These solutions form a continuum meeting $t=1$, $y(\mathbf{x})=0$.*

THEOREM 3. *Let $H1$ hold. Assume that in a neighbourhood of* $t = \sum_{s=1}^{k} (\mu_s^{\mathbf{n}} + (t-1)w_s) H_{rs}(x_r, \xi_r, \eta_r) \geq 0$ *for some values of r and* ≤ 0 *for the remaining values r. Then no non-trivial solutions* $y(\mathbf{x}) \in S(\mathbf{n}, \boldsymbol{\sigma})$ *of* (1) *can be found.*

We now consider bifurcation for a somewhat different class of non-linearities. He the results are described in terms of the following function spaces.

C_r^s will denote the space $C^s[a_r, b_r]$ of real valued functions on $[a_r, b_r]$ with continuous derivatives, normed by

$$\|y\|_{s,r} = \sup_{0 \leq i \leq s} \sup_{a_r \leq x_r \leq b_r} |y^{(i)}(x_r)|.$$

H_r will denote the subspace of C_r^s consisting of those functions satisfying the Sturn Liouville boundary conditions accompanying (1) for $1 \leq r \leq k$. We also introduce the op ball in H_r

$$B_\rho^r = \{y \in H_r \mid \|y\|_{1,r} < \rho\}.$$

Once again we set $\boldsymbol{\lambda} = \boldsymbol{\mu}^{\mathbf{n}} + (t-1)\mathbf{w}$ in equation (1), but instead of using equation (4) work with the rearranged form

$$-y_r''(x_r) - q_r(x_r)y_r(x_r) - \sum_{s=1}^{k} (\mu_s^{\mathbf{n}} - w_s) a_{rs}(x_r) y_r(x_r)$$
$$= t\left\{\sum_{s=1}^{k} w_s a_{rs}(x_r) y_r(x_r) + \sum_{s=1}^{k} \frac{\mu_s^{\mathbf{n}} + (t-1)w_s}{t} H_{rs}(x_r, y_r, y_r')\right\}$$

subject to the boundary conditions as in (1). Each of these equation can now considered as a one-parameter non-linear Sturm–Liouville problem with spectral param ter t of the type studied by Turner [**10**, Equation (3.2)].

We introduce the notation

$$N_r(x_r, \xi_r, \eta_r, t) = \sum_{s=1}^{k} \frac{\mu_s^{\mathbf{n}} + (t-1)w_s}{t} H_{rs}(x_r, \xi_r, \eta_r)$$

and assume the following hypotheses.

For each r, N_r is a continuous map from $[a_r, b_r] \times \mathbb{R}^2 \times (0, \infty)$ into $\mathbb{R}$ which satisfies tl following conditions.

H2 $\xi_r^{-1} N_r(x_r, \xi_r, \eta_r, t) \to \infty$ *as* $\xi_r \to \pm\infty$ *uniformly for* $(x_r, \eta_r) \in [a_r, b_r] \times \mathbb{R}$ *and t in a compact subset of* $(0, \infty)$.

H3 $\eta_r^{-2} N_r(x_r, \xi_r, \eta_r, t) \to 0$ *as* $\eta_r \to \pm\infty$ *uniformly for* $x_r \in [a_r, b_r]$, ξ_r *in a bounded interval* $\mathbb{R}$ *and t in any compact subset of* $(0, \infty)$.

Note that the assumed continuity of N_r implies that the map $(y_r, t) \in C_r^1 \times (0, \infty)$ - $N_r(\cdot, y_r(\cdot), y_r'(\cdot), t) \in C_r^0$ is continuous.

H4 *If* E_r *is a bounded set in* C_r^1 *then for t ranging through a compact subset of* $(0, \infty)$ *and through* E_r, $N_r(\cdot, y_r(\cdot), y_r'(\cdot), t)$ *lies in a bounded set in* C_r^0.

H5 $|N_r(x_r, \xi_r, \eta_r, t)| = o(|\xi_r| + |\eta_r|)$ *as* $(|\xi_r + |\eta_r|) \to 0$ *for* $(x_r, t) \in [a_r, b_r] \times (0, \infty)$.

These hypotheses parallel those of Turner [**10**]. The bifurcation theorem can now be stated as follows.

THEOREM 4. *Let H2–H5 be satisfied. Let* $\boldsymbol{\sigma}$ *be a given k-tuple of sign factors and* $\mathbf{n}$ *a given multi-index. Then for* $0<t<1$ *there are solutions* $\boldsymbol{\lambda}=\boldsymbol{\mu}^{\mathbf{n}}+(t-1)\boldsymbol{\omega}$, $y\in S(\mathbf{n},\boldsymbol{\sigma})$ *of* (1). *These solutions form a continuum meeting* $t=1$, $y=0$ *which can be characterised for each r as follows.*

Given $0<\alpha_r<\beta_r<1$, *there are constants* $M>\varepsilon>0$ *so that for* $\Omega_r=(B^r-B^r_\varepsilon)\cap S_r(n_r,\sigma_r)$, *the problem* (1), *for a given r, has solutions* $\boldsymbol{\lambda}=\boldsymbol{\mu}^{\mathbf{n}}+(t-1)\boldsymbol{\omega}$, $y_r(x_r)$ *which form a continuum in* $[\alpha_r,\beta_r]\times\Omega_r$. *This continuum intersects* $\{\alpha_r\}\times\Omega_r$ *and* $\{\beta_r\}\times\Omega_r$. *Here* $S_r(n_r,\sigma_r)$ *denotes the set of functions which have* n_r *zeros in* (a_r,b_r) *and the sign of* σ_r *near* $x_r=a_r$.

Proof. The proof follows immediately from [**10**], Theorem 3.3] and the above constructions.

3. Bifurcation from Infinity. In this paragraph we use the hypothesis

H6 *Each* $H_{rs}(x_r,\xi_r,\eta_r)\xi_r$ *is continuous on* $[a_r,b_r]\times\mathbb{R}^2$ *and is* $o((\xi_r^2+\eta_r^2)^{1/2})$ *as* $(\xi_r^2+\eta_r^2)^{1/2}\to\infty$.

This hypothesis is of the normal type used for bifurcation from infinity and we cite the works of Stuart [**8**] and Toland [**9**] for reference to the standard Sturm–Liouville case.

We begin by considering the functions

$$K_{rs}(x_r,\xi_r,\eta_r)=(\xi_r^2+\eta_r^2)H_{rs}(x_r,\xi_r/(\xi_r^2+\eta_r^2),\eta_r/(\xi_r^2+\eta_r^2)).$$

Note that $K_{rs}(x_r,\xi_r,\eta_r)=o((\xi_r^2+\eta_r^2)^{1/2})$ as $(\xi_r,\eta_r)\to(0,0)$. We now use the following result. (Cf Toland [**9**, Theorem 2.1]).

LEMMA 1. *Suppose* u_r *satisfies* (4) *with* H_{rs} *replaced by* K_{rs} *and that* $|||u_r|||^2=\|u_r\|^2+\|u_r'\|^2\neq 0$. *Then* $v_r=u_r/|||u_r|||^2$ *satisfies* (4).

Proof. This is a straightforward verification.

Note that K_{rs} satisfies H1, so that we may appeal to our results of the previous section and use Lemma 1 to claim the following results.

THEOREM 5. *Let H6 hold. Assume that in a neighbourhood of* $t=1$, $\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}(x_r,\xi_r,\eta_r)\geq 0$ *(respectively* ≤ 0*) as a function of* (x_r,ξ_r,η_r). *Let* $\boldsymbol{\sigma}$ *be a given k-tuple of sign factors. Then there exists* $\delta>0$ *so that for* $t\in(1-\delta,1)$ *(respectively* $(1,1+\delta)$*) there are solutions* $\boldsymbol{\lambda}=\boldsymbol{\mu}+(t-1)\mathbf{w}$, $y(\mathbf{x})\in S(\mathbf{n},\boldsymbol{\sigma})$ *of* (1). *Further there is a continuum of such solutions meeting* $(1,\infty)$.

If in a neighbourhood of $t=1$, $\sum_{s=1}^{k}(\mu_s^{\mathbf{n}}+(t-1)w_s)H_{rs}\geq 0$ *for some values of r and* ≤ 0 *for the remaining values of r then no solutions of* (1) *can be found.*

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