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BONFERRONI-TYPE INEQUALITIES

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Abstract

We derive the Sobel–Uppuluri and Galambos-type extensions of the Bonferroni bounds, and further extensions of the same nature, as consequences of a single non-probabilistic inequality. The methodology follows that of Galambos.

JORDAN INEQUALITIES; GALAMBOS INEQUALITIES; INVERSION FORMULA; BINOMIAL MOMENTS

1. Unified treatment and extensions

Let w_t, $t = 0, 1, \cdots, n$ be non-negative numbers. Define

$$A_{k,n} = \sum_{t=k}^{n} \binom{t}{k} w_t, \qquad k \geqq 0 \qquad (A_{k,n} = 0,\ k > n). \tag{1}$$

Then, following Galambos [3], pp. 18–20, [2], p. 580,

$$\sum_{k=0}^{a} (-1)^k \binom{k+r}{r} A_{k+r,n} = w_r + (-1)^a \sum_{s=r+a+1}^{n} \binom{s-r-1}{a}\binom{s}{r} w_s \tag{2}$$

for $0 \leqq a \leqq n-r-1$, and

$$\sum_{s=r+a+1}^{n} \binom{s-r-1}{a}\binom{s}{r} w_s \geqq \frac{a+1}{n-r}\binom{a+1+r}{r} A_{a+1+r,n}. \tag{3}$$

Using (3) in (2) yields the bounds for $0 \leqq r \leqq n$, $u \geqq 0$, $(\sum_{s=r}^{r-1} = 0)$:

$$\begin{aligned} &\sum_{s=r}^{r+2u-1} (-1)^{s-r} \binom{s}{r} A_{s,n} + \frac{2u}{n-r}\binom{r+2u}{r} A_{r+2u,n} \leqq w_r \\ &\leqq \sum_{s=r}^{r+2u} (-1)^{s-r} \binom{s}{r} A_{s,n} - \frac{2u+1}{n-r}\binom{r+2u+1}{r} A_{r+2u+1,n} \end{aligned} \tag{4}$$

keeping in mind (for $r + 2u \geqq n$) the inversion of the relation (1):

$$w_r = \sum_{s=r}^{n} (-1)^{s-r} \binom{s}{r} A_{s,n}, \qquad 0 \leqq r \leqq n. \tag{5}$$

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Now let y_t, $t = 1, \cdots, n$, be arbitrary non-negative numbers, and define $S_{k,n}$ by

$$S_{k,n} = \sum_{t=k}^{n} \binom{t-1}{k-1} y_t, \qquad k \geqq 1 \quad (S_{k,n} = 0,\ k > n). \tag{6}$$

Since $\binom{t-1}{k-1} = (k/t)\binom{t}{k}$, it follows that

$$S_{k,n}/k = \sum_{t=k}^{n} \binom{t}{k} (y_t/t), \qquad k \geqq 1.$$

Thus putting $w_0 = 0$ and taking in (1) and (4) for $k \geqq 1$, $A_{k,n} = S_{k,n}/k$, and $w_t = y_t/t$, $1 \leqq t \leqq n$, it follows that for $1 \leqq r \leqq n$, $u \geqq 0$,

$$\begin{aligned} &\sum_{s=r}^{r+2u-1} (-1)^{s-r} \binom{s-1}{r-1} S_{s,n} + \frac{2u}{n-r} \binom{r+2u-1}{r-1} S_{r+2u,n} \leqq y_r \\ &\leqq \sum_{s=r}^{r+2u} (-1)^{s-r} \binom{s-1}{r-1} S_{s,n} - \frac{2u+1}{n-r} \binom{r+2u}{r-1} S_{r+2u+1,n} \end{aligned} \tag{7}$$

again keeping in mind (5).

Turning now to a probabilistic setting, let $A_1, \cdots, A_n$ be a sequence of events on a probability space, let $B_{r,n}$, $0 \leqq r \leqq n$, be the event that exactly r of the A's occur, and let $P_{[r]} = P(B_{r,n})$. Let $S_{k,n} = \sum P(A_{i_1} A_{i_2} \cdots A_{i_k})$ where the sum is over all subscripts satisfying $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n$. Then it is well known (defining $S_{0,n}$ as 1) that:

$$S_{k,n} = \sum_{t=k}^{n} \binom{t}{k} P_{[t]}, \qquad k \geqq 0,$$

so putting $w_t = P_{[t]}$ and $A_{k,n} = S_{k,n}$ in (1), (4) yields the Sobel–Uppuluri–Galambos inequalities ([3], p. 20; [5]), Further, it is well known that if we put

$$P_{(r)} = \sum_{s=r}^{n} P_{[s]},$$

then for $1 \leqq r \leqq n$,

$$S_{k,n} = \sum_{t=k}^{n} \binom{t-1}{k-1} P_{(t)}, \qquad k \geqq 1$$

so putting $y_t = P_{(t)}$ in (6) and (7) yields the Galambos bounds ([2], [6]) for $P_{(r)}$, $r \geqq 1$. The direction of generalisation is now clear. For example, for $n \geqq t \geqq 2$, put

$$P_{\{t\}} = \sum_{s=t}^{n} P_{(s)} = \left(\sum_{h=r}^{n} (h-r+1) P_{[h]} \right).$$

Substituting for $P_{(s)}$ from the inversion formula (5), i.e.

$$P_{(s)} = \sum_{k=s}^{n} (-1)^{k-s} \binom{k-1}{k-s} S_{k,n}, \qquad 1 \leqq s \leqq n,$$

and using a combinatorial identity we obtain

$$P_{\{t\}}=\sum_{k=t}^{n}(-1)^{k-t}\binom{k-2}{k-t}S_{k,n},$$

i.e.

$$P_{\{t\}}\Big/(t(t-1)=\sum_{k=t}^{n}(-1)^{k-t}\binom{k}{t}S_{k,n}/k(k-1),$$

whence by the inversion formula to (5), viz. (1),

$$S_{k,n}=\sum_{r=k}^{n}\binom{r-2}{k-2}P_{\{r\}},\ k\geqq 2.$$

Hence for $2\leqq r\leqq n,\ u\geqq 0,$

$$\sum_{s=r}^{r+2u-1}(-1)^{s-r}\binom{s-2}{r-2}S_{s,n}+\frac{2u}{n-r}\binom{r+2u-2}{r-2}S_{r+2u,n}\leqq P_{\{r\}}$$
$$\leqq\sum_{s=r}^{r+2u}(-1)^{s-r}\binom{s-2}{r-2}S_{s,n}-\frac{2u+1}{n-r}\binom{r+2u}{r-1}S_{r+2u+1,n}.$$

2. The method of polynomials

In the general vein of Galambos' conceptualization of the Bonferroni inequalities ([1]), the following deductions can be made from extending slightly the argument in [4]. Suppose for all p, $0\leqq p\leqq 1$ and all integers $m\geqq 0$,

$$(1-p)^m\leqq\sum_{k=0}^{m}c_k(m)\binom{m}{k}p^k\quad(\text{where } c_0(0)=1). \tag{8}$$

Then with the notation of the above probabilistic setting

$$P_{[r]}=P(B_{r,n})\leqq\sum_{z=r}^{n}c_{z-r}(n-r)\binom{z}{r}S_{z,n}. \tag{9}$$

(If the inequality in the supposition (8) is reversed, it is reversed in (9).)

Thus Theorem 4 in [4] can be written down directly from its Lemma 2. Thus, once the inequalities (in the form of Taylor expansion with remainder):

$$\sum_{k=0}^{2u-1}\binom{m}{k}(-p)^k+\binom{m}{2u}(-p)^{2u}\leqq(1-p)^m$$
$$\leqq\sum_{k=0}^{2u}\binom{m}{k}(-p)^k+\frac{2u+1}{m}\binom{m}{2u+1}(-p)^{2u+1}$$

have been obtained for $m\geqq 0$, $0\leqq p\leqq 1$, the Sobel–Uppuluri-type bounds for $P_{[r]}$ follow immediately.

3. An application

We give in Table 1 below some data on performance of first-year students in the Faculty of Science at the University of Sydney in the 1980 annual examinations. The intention is to illustrate use of the Sobel–Uppuluri–Galambos bounds (4) in a real situation where the events $A_1,\cdots,A_n$ are not exchangeable, and effectiveness of prediction of $P_{[0]}$ via partial knowledge is the object.

TABLE 1

	Proportions passing			
	Mathematics	Physics	Chemistry	Remaining subject
Mathematics	0·86			
Physics	0·72	0·74		
Chemistry	0·65	0·61	0·67	
Remaining subject	0·72	0·65	0·62	0·79
Proportions passing:				
Mathematics, Physics, Chemistry				0·60
Mathematics, Physics, Remaining subject				0·64
Mathematics, Chemistry, Remaining subject				0·60
Physics, Chemistry, Remaining subject				0·57
Mathematics, Physics, Chemistry, Remaining subject				0·57

If we take A_1 = fail mathematics, A_2 = fail physics, A_3 = fail chemistry, A_4 = fail remaining subject, we wish to bound the probability of passing all subjects, $P_{[0]} = 0{\cdot}57$.

If the Sobel–Uppuluri–Galambos bounds are used in conjunction with 'ordinary' Bonferroni bounds 'of the same order' in the manner of [3], p. 27, then the results are:

	$u = 0$	$u = 1$	$u = 2$
Lower bound	0·06	**0·46**	0·50
Upper bound	**0·77**	0·85	**0·59**

where bounds of form (4) with $r = 0$, $n = 4$ are shown in bold type. Thus even in the case $u = 0$, where only the $P(A_i)$, $i = 1, \cdots, 4$ are used, the upper bound is a useful approximation in our setting, although the upper bound $\min_i P(\bar{A}_2) = 0.67$ is better.

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