Compositio Mathematica **120:** 227–290, 2000.

Remarks on the Transfer Factors for Unipotent Orbital Integrals in p-adic Classical Split Groups

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(Received: 13 December 1996; accepted in revised form: 8 March 1999)

Abstract. This article studies unipotent orbital integrals on symplectic and orthogonal groups from the point of view of endoscopy. It begins by partitioning stable unipotent classes into packets and goes on to propose a transfer of these packets. It then discusses (in rough form) the associated transfer factors. Some supporting calculations in split odd orthogonal groups are given.

Mathematics Subject Classifications (1991): Primary 22E35, Secondary 22E50.

Key words: unipotent orbital integrals, endoscopy, p-adic groups.

1. Introduction

Let F be a p-adic field of characteristic zero, and $\mathbf{G}$ a connected reductive algebraic group defined over F. A main objective for harmonic analysis on $\mathbf{G}(F)$ is to understand the invariant distributions on $\mathbf{G}(F)$. Orbital integrals and characters of irreducible admissible representations are the two main examples of invariant distributions. The germ expansion of Shalika, as well as the local character expansion of Howe and Harish Chandra, and the recent work of Waldspurger, all point to the importance of the study of unipotent orbital integrals. R. Langlands was the first to recognize the implications, for local and global harmonic analysis, of the difference between conjugacy and stable conjugacy for semi-simple elements of $\mathbf{G}(F)$. His observations were then developed, by himself and others, into the theory of *Endoscopy*. The main objective of endoscopy in local harmonic analysis is to understand the invariant distributions on $\mathbf{G}(F)$ by comparing them to *stable* distributions on the various endoscopic groups, $\mathbf{H}(F)$, associated to $\mathbf{G}$. This comparison is dual to a conjectural map, called *smooth matching*, between smooth and compactly supported functions on $\mathbf{G}(F)$ and $\mathbf{H}(F)$ which satisfies precisely defined identities between their semisimple orbital integrals. The aim of endoscopy in the study of unipotent orbital integrals in then to address the following problems. The first problem is to find an explicit basis for the space of stable distributions supported on the unipotent variety. The second problem is to explicitly describe the endoscopic transfer of a stable distribution supported on the unipotent variety

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of a given endoscopic group. Stated differently, the second problem is to find the *transfer factors* for unipotent orbital integrals (cf. [2]).

An answer to these two questions should also lead to the breaking up of the identity known as the *fundamental lemma* (see [L] and [W]) into an equivalent set of identities which are at least structurally simpler.

The purpose of this article is to discuss the second problem for split classical groups. However, it is clear that this problem cannot be addressed without an answer to the first problem. In [2], we conjectured that every $\mathbf{G}(\overline{F})$-*special* orbit ($\overline{F}$ = algebraic closure of F, and *special* is in the sense of Lusztig (cf. [5])) can be partitioned into disjoint sets (called *packets*), such that an appropriate linear combination (a sum if $\mathbf{G}$ is split) of the integrals over the rational orbits within a given packet (the measures on the rational orbits are assumed to be *related*) is a stable distribution. Moreover, the set of stable distributions associated with all the packets contained in the various $\mathbf{G}(\overline{F})$-special orbits, forms a basis (over $\mathbb{C}$) for the space of stable distributions supported on the unipotent variety.

In Section 1, we define an explicit partitioning of *every* $\mathbf{G}(\overline{F})$-unipotent class (regardless whether it is special or not) into disjoint subsets which we call packets. Here $\mathbf{G}$ is either a special orthogonal group (not necessarily quasi-split) or a symplectic group. For special orbits, this partitioning should be the packet decomposition predicted by the above stated conjecture. The partitioning of the non-special orbits is necessary for the discussion of transfer of packets. Our partitioning is described using the classification of rational unipotent classes via the theory of prehomogeneous spaces associated to $\mathfrak{sl}_2$-triplets. We would like to mention that the packets description given in Section 1 was known to us for some time and was discussed with R. Kottwitz during a visit to the University of Chicago in April/May 1996. A few weeks later, during a visit to the Université of Paris 7, J.-L. Waldspurger kindly informed us about the results he had obtained ([15]). He described to us the packet structure and the stable linear combinations for the unramified unitary groups and the symplectic groups. He also gave an outline of the proof of his results. His description of the packet structure for symplectic groups is via the standard classification of rational orbits, and turns out to be equivalent to the description given in Section 1. We find this encouraging, and expect that our packet description for orthogonal groups will turn out to be equivalent to his packet description in his anticipated work on orthogonal groups.

Next, with an answer to the first problem, we are ready to discuss the transfer problem. Now, the transfer of a stable distribution associated to some packet of special rational orbits in an endoscopic group $\mathbf{H}$ of $\mathbf{G}$, ought to be a linear combination of integrals over some set of rational orbits in $\mathbf{G}(F)$. Thus the transfer problem can be divided into two parts. The first part is to understand the set of rational orbits in $\mathbf{G}(F)$ involved. Is there any structure to it? The second part is to understand the nature of the coefficients appearing in that linear combination. In [1] we used the map (initially introduced by Lusztig (cf. [9]), and later elaborated on by Spaltenstein (cf. [13])), called *endoscopic induction*, to obtain identities between unipotent orbital integrals on complex semisimple groups and their 'endoscopic' counterparts. In fact, endoscopic induction can be characterized as the 'unique' map defined on the set of special orbits which produces matching relations (see Proposition 5.2.2. in [2]). In Section 2, we introduce a rational refinement of endoscopic induction for a classical split group $\mathbf{G}$. To a packet of special orbits in an elliptic endoscopic group $\mathbf{H}(F)$, we attach a union of packets in $\mathbf{G}(F)$. Note that since endoscopic induction of a special orbit may not be special, it is necessary to introduce the notion of a packet for nonspecial orbits (as we did in Section 1). It can be shown, moreover, using a descent argument analogous to Lemma 2.5.10, that it is always possible to reduce to the situation where every single packet transfers to a single packet. Our description of transfer of packets is presented in three steps. The first and most critical step makes use of the fact that for special orbits whose dual orbits (in the sense of Spaltenstein [13]) are *even*, endoscopic induction, when viewed from the Langlands dual group side, simply becomes inclusion (This is an observation of Barbasch and Vogan (cf. [4]).) See Section 2.5 (step 1) for the precise type of orbits involved. In order to define the transfer of packets in step 1, two intermediary correspondences of packets between special orbits and their duals (both the order preserving and reversing ones, see Section 2.3.) are defined. We believe that the correspondence of packets between special orbits and their order preserving duals will play a role in the study of twisted endoscopy. The second and third step treats sets of orbits of increasing generality, with each step reducing to the preceding one via a formal argument (see Lemmas 2.5.10, and 2.5.11). Here we mention that the idea of trying to bring in duality in some fashion was suggested to us by R. Kottwitz. Next, we pay attention to the second part of the transfer problem, namely the nature (and the precise definition) of the transfer factors. The first real glimpses concerning the nature of the transfer factors came from the long calculations done in [2]. In these calculations, which dealt with symplectic groups, the transfer factors (for the cases considered there) turned out to be character values of certain finite Abelian 2-groups. It should be noted that in [2], only orbits which are endoscopically induced from the trivial orbit were considered. This is due to the fact that no other 'collection' of rational orbits giving rise to a stable distribution were known at the time (except, of course, Richardson orbits). In fact, it was the transfer relations obtained in [2], which suggested to us the notion of a 'packet' of unipotent orbits. Thus a more elaborate test was needed, where the transfer factors are obtained from transferring stable distributions associated to 'nontrivial' packets. In Section 3, we present a transfer calculation where $\mathbf{G} = \mathbf{SO}(2n+1)$, and O_G is the orbit corresponding to the partition 331^{2n-5}, $n \geqslant 3$. These orbits are all special and contain only one packet. We consider the situation where $\mathbf{H} = \mathbf{SO}(2n-3) \times \mathbf{SO}(5)$, and $O_H = (\mathbf{1}, O_{\text{sub}})$, where $\mathbf{1}$ denotes the trivial orbits in $\mathbf{SO}(2n-3)$, and O_{sub} denote the subregular orbit in $\mathbf{SO}(5)$. Assuming that the residual characteristic of F is not equal to 2, $O^{\text{st}}_{\text{sub}}$ (the $\mathbf{SO}(5, \overline{F})$ subregular orbit) breaks up into four rational classes each forming a packet (this is now a special case of a general result of Waldspurger ([15]) but can also be deduced from the Shalika germs calculations of T. Hales (see [2], Proposition 5.5.1)). Now for $n \geqslant 4$, the rational orbits within the $\mathbf{G}(\overline{F})$-orbit O_G^{st} are classified by the equivalence classes of quadratic forms of rank 2. The transfer calculation alluded to above involves three pairs (f, f^H) of matching spherical functions. These three pairs satisfy the property that the dimension of the space obtained by restricting the integrals over the rational orbits within O_G^{st} (resp. O_H^{st}) to the three-dimensional space spanned by the functions f (resp. f^H), is equal to three. The transfer factors emerging from this calculation turn out to be the four characters of the group of square classes of $F^{\times}$. In Section 4, we discuss some examples which exhibit various aspects of the transfer of packets and the transfer factors. Our discussions are based on the calculations done in Section 3, in [2], and various descent arguments. In all these examples, the transfer factors turn out to be character values of Abelian 2-groups. Motivated by all these calculations, we present a conjecture describing a rough form for the transfer factors in the 'critical cases'. By the critical cases we mean the following. Given a $\mathbf{G}(\overline{F})$-unipotent orbit O_G^{st}, special or not, one would like to characterize a set of pairs $(\mathbf{H}, O_H)$, where $\mathbf{H}$ is an elliptic endoscopic group, and O_H is a special orbit in $\mathbf{H}$ which endoscopically induces to O_G, such that the set of identities obtained from transferring all the stable distributions associated to the various packets within the $\mathbf{H}(\overline{F})$-orbits O_H^{st} forms an invertible linear system. In Section 4.1, we introduce the notion of an *elliptic unipotent endoscopic datum* relative to a given $\mathbf{G}(\overline{F})$-orbit O_G^{st}. It consists of a pair $(\mathbf{H}, O_H)$ satisfying in addition to the above stated properties, the following condition: If $\mathbf{G}$ is special odd orthogonal, then $\overline{A}(O_H) \cong C(O_G)$, and if $\mathbf{G}$ is special even orthogonal or symplectric, then $\overline{A}(O_H) \times \mathbb{Z}/2\mathbb{Z} \cong C(O_G)$. Here, for a unipotent orbit O in a reductive group, $C(O)$ denotes the group of connected components of the centralizer of some $u \in O$ (the center is not being divided out). $\overline{A}(O)$ denotes the quotient group introduced by Lusztig in [9]. We shall prove, somewhere else, that the number of such pairs is (properly counted) equal to $2^{\eta(O_G^{\mathrm{st}})}$, where $\eta(O_G^{\mathrm{st}})$, the η-*index* of O_G^{st}, is a certain integer associated with O_G^{st} which we introduce in Section 1.3.3. We predict that the set of elliptic unipotent endoscopic data relative to O_G will lead to an invertible linear system allowing for the expression of the integral over any rational orbit within O_G^{st} as a linear combination of stable distributions on various endoscopic groups. A critical case for us is then a case involving an elliptic unipotent endoscopic datum relative to some orbit O_G. Given a $\mathbf{G}(\overline{F})$-orbit O_G^{st}, every packet within O_G^{st} can be embedded (as becomes clear from the prehomogeneous vector space classification of rational orbits) into a common group which is a product of several copies of $F^{\times}/(F^{\times})^2$ and several copies of $\mathbb{Z}/2\mathbb{Z}$. The conjectured transfer factors are then a product of three factors. The first two factors are restrictions to the transferred packet, of characters of the appropriate powers of $F^{\times}/(F^{\times})^2$ and $\mathbb{Z}/2\mathbb{Z}$, respectively. The third factor is a constant which is independent of the packets within O_H^{st} and O_G^{st}. Unfortunately, we do not describe the precise characters which occur, in the general situation. However, for orbits O_G in which the packets are determined by the square classes of the discriminants of all the quadratic forms which classify the rational orbits within O_G^{st}, we give the precise transfer factors (in this case the first factor is always trivial).

Now, we give a detailed description of the contents of each section. In Section 1, we first present the classification of rational unipotent orbits for special orthogonal and symplectic groups, via the theory of prehomogeneous spaces associated to $\mathfrak{sl}_2$-triplets, and indicate its relationship with the more standard classification (Lemmas 1.2.5. and 1.2.9). In Section 1.3, we use the given prehomogeneous classification to explicitly describe the packets for any unipotent orbit O. This description is also valid in the nonquasi-split case. We also introduce the η-exponent, $\eta(O^{\mathrm{st}})$. In Section 2.1. we review the Springer correspondence and use it to classify the unipotent orbits (over $\overline{F}$) into families. In Section 2.2, we discuss the quotient group $\overline{A}(O)$ of Lusztig and indicate its relationship to packets. In Section 2.3, we discuss two duality maps (one is order preserving and the other is order reversing) between the lattices of special orbits in a classical group and its Langlands dual group (which we take to be defined over F, and not over $\mathbb{C}$, as in usually the case). These maps are discussed in Spaltenstein's book [13]. We give explicit formulas for these maps which will be useful in other parts of this paper. We also define correspondences of packets between a special orbit and its two duals. This will be needed when discussing the transfer of packets from endoscopic groups. In Section 2.4, we review endoscopic induction and give a direct description of it in terms of partitions (see Lemma 2.4.4). This description is essentially due to Spaltenstein. In Section 2.5, we define the transfer of packets of special orbits in an elliptic endoscopic group. Our definition may be viewed as a rational generalization of endoscopic induction. Our definition proceeds in three steps, each step treats a larger class of orbits than the preceding step and reduces to it by a formal argument. The first and critical step treats a class of special orbits whose (order reversing) dual orbits are even. Here the correspondence of packets discussed in Section 2.3, is used. Section 3 is devoted to the transfer calculation alluded to above. In Section 3.2, we use some results of Igusa to calculate some p-adic integrals needed for the orbital integral calculations. In Section 3.3, we use Macdonald's formulae for the Satake transform and the spherical Plancherel measure to give a more practical formula for the endoscopic transfer, f^H, of a spherical function f. Our goal is to apply this formula to three particular functions. In Section 3.4, we introduce three auxiliary spherical functions, and compute their endoscopic transfer via the formula given in Section 3.3. In Section 3.5, we use these results to calculate the transfers of the three given functions. In Section 3.6, we put the results of the preceeding sections together to obtain the transfer factors. In Section 3.7, we give a prediction based on our previous calculations. In Section 4.1 we introduce the notion of an elliptic unipotent endoscopic datum. In Section 4.2, we present several examples which illustrate various aspects of the transfer factors. Each example ends with a prediction about the precise form of the transfer factors. In Section 4.3, we describe our general (but not completely explicit) conjecture regarding the transfer factor. We do, however, present a precise conjecture for a broad class of orbits.

Preface (by R. Kottwitz, testamentary editor)

This article was submitted to *Compositio Mathematica* shortly before the tragically early death of Magdy Assem. It contains his ideas on packets and transfer factors for unipotent orbits, all in the context of orthogonal and symplectic groups. The article omits many proofs and has to be used with considerable caution. For example, Lemma 4.3.2 seems to be incorrect. Moreover, as Waldspurger has observed, Lemma 2.4.4 is incorrect, at least for **G** of type **C** and **D**. The best way to use the paper is as a source of ideas (some of which are clearly valuable), but it is not safe to quote results from the paper without checking them first for oneself.

One key point (see Lemma 2.2.4) that Assem emphasized in conversations with me is that that for classical groups Lusztig's quotient group $\overline{A}(\lambda)$ can be read off naturally from the prehomogeneous vector space associated to the nilpotent orbit. Assem's intuition was that transfer factors for unipotent orbital integrals should also be naturally associated to the corresponding prehomogeneous vector spaces. Possibly there is an as yet undiscovered theory of endoscopy for prehomogeneous vector spaces.

Following the suggestions of the referee, I have corrected a number of misprints and other minor errors in the manuscript. I have also added some footnotes, often in response to the referee's comments, as well as some additional references (these are labeled by letters rather than numbers in the list of references at the end of the paper). But in all essential aspects the paper is the same as the original manuscript.

Notation and Review of Some Definitions

- Throughout this manuscript, F is a p-adic field of characteristic zero with odd residual characteristic. O_F will denote the ring of integers, and P_F its maximal ideal. The order of the residue field O_F/P_F is equal to q. We let π denote a uniformizer of O_F, and ε a Teichmüller representative of a non-square in the residue field; thus $1, \pi, \varepsilon, \pi\varepsilon$ are a set of representatives for the square classes in the multiplicative group of F. The absolute value function $|\cdot|$ is normalized such that $|\pi| = q^{-1}$.
- For a connected reductive algebraic group $\mathbf{G}$ over F, we use $\mathbf{G}(F)$ to denote the group of F-rational points equipped with the p-adic topology. Given $x \in \mathbf{G}(F)$, let $x = us = su$ be its Jordon decomposition, where u is unipotent and s semi-simple. The *stable* class of x, denoted $O^{\mathrm{st}}(x)$, or just O^{st} if x is understood, is by definition, (following Kottwitz) equal to $\{g^{-1}xg : g \in \mathbf{G}(\overline{F})$ and $g^{-1}g^{\sigma} \in \mathbf{G}_s^{\circ}(\overline{F})$ for all $\sigma \in \mathrm{Gal}\,(\overline{F}/F)\} \cap \mathbf{G}(F)$, where $\mathbf{G}_s^{\circ}$ denotes the identity connected component of the centralizer of s in $\mathbf{G}$. In particular, if x is unipotent, then the stable orbit of x is simply its $\mathbf{G}(\overline{F})$-orbit. Given a stable unipotent orbit in $\mathbf{G}(\overline{F})$, we say that the measures on the rational orbits within it are *related* if they are obtained from a single $\mathbf{G}(\overline{F})$-invariant volume form, defined over F, on the $G(\overline{F})$-orbit. We shall always assume that the measures on

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the rational orbit within a given stable class are related. Given a unipotent orbit O, we shall denote the integral over O (with respect to a given measure) by: $\int_O$.

- Finally, recall that an invariant distribution D on $\mathbf{G}(F)$ is *stable* (in the sense of Langlands) if the following condition is satisfied:

$$\forall f \in C_c^\infty(\mathbf{G}(F))[\int_{O^{\mathrm{st}}(x)} f = 0 \quad , \quad \forall x \quad \text{semi-simple} \;\Rightarrow D(f) = 0] .$$

1. Packets of Unipotent Orbits and Prehomogeneous Spaces

1.1. PARTITIONS AND UNIPOTENT ORBITS

Let N denote a positive integer. A partition of N is a sequence of integers $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, and $\sum_{i=1}^r \lambda_i = N$. The elements λ_i are called *parts*. Sometimes, we also write $\boldsymbol{\lambda} = (\lambda_1^{a_1}, \ldots, \lambda_t^{a_t})$, where $a_i > 0$ denotes the multiplicity of the part λ_i in $\boldsymbol{\lambda}$. The set of all partitions of N will be denoted by $\mathbf{P}(N)$. Given $\boldsymbol{\lambda} = (\lambda_1, \ldots \lambda_p), \mu \in \mathbf{P}(N)$, we write $\boldsymbol{\lambda} \leqslant \mu$ if $\lambda_1 \leqslant \mu_1$, $\lambda_1 + \lambda_2 \leqslant \mu_1 + \mu_2$, $\lambda_1 + \lambda_2 + \lambda_3 \leqslant \mu_1 + \mu_2 + \mu_3 \ldots$, etc. Then $(\mathbf{P}(N), \leqslant)$ is a partially ordered set. Let $\boldsymbol{\lambda} \in \mathbf{P}(N)$ and $A \subseteq \mathbf{P}(N)$. We say that $\inf_A \boldsymbol{\lambda}$ exists if there exists a unique $\mu \in A$ satisfying

(i) $\boldsymbol{\mu} \leqslant \boldsymbol{\lambda}$, and
(ii) $\forall \boldsymbol{v} \in A[\boldsymbol{v} \leqslant \boldsymbol{\lambda} \Rightarrow \boldsymbol{v} \leqslant \boldsymbol{\mu}]$. In this case we set $\mu =: \inf_A \boldsymbol{\lambda}$.

Given $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbf{P}(N_1)$, $\mu = (\mu_1, \ldots, \mu_m) \in \mathbf{P}(N_2)$, with $n \leqslant m$, we define $\boldsymbol{\lambda} + \boldsymbol{\mu} := \lambda_1 + \mu_1, \ldots, \lambda_n + \mu_n, \mu_{n+1}, \ldots, \mu_m) \in \mathbf{P}(N_1 + N_2)$. We also define $\boldsymbol{\lambda} \cup \boldsymbol{\mu} \in \mathbf{P}(N_1 + N_2)$ to be the partition whose set of parts is $\{\lambda_1, \ldots \lambda_n, \mu_1, \ldots, \mu_m\}$. For each $N \in \mathbb{Z}^+$, and $\boldsymbol{\lambda} \in \mathbf{P}(N)$, we denote by ${}^t\boldsymbol{\lambda}$ the transpose of $\boldsymbol{\lambda}$. It is given as follows. If $\boldsymbol{\lambda} = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r})$, then

$${}^t\boldsymbol{\lambda} = \left((a_1 + \cdots + a_r)^{\lambda_r}, (a_1 + \cdots + a_{r-1})^{\lambda_{r-1}-\lambda_r}, \ldots, (a_1 + a_2)^{\lambda_2 - \lambda_3}, a_1^{\lambda_1 - \lambda_2}\right) .$$

It is clear that if $\boldsymbol{\lambda} \in \mathbf{P}(N_1)$, $\mu \in \mathbf{P}(N_2)$, then ${}^t(\boldsymbol{\lambda} + \boldsymbol{\mu}) = {}^t\boldsymbol{\lambda} \cup {}^t\boldsymbol{\mu}$.

For a partition $\boldsymbol{\lambda} = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r})$, let

$$|\boldsymbol{\lambda}| := \sum_{i=1}^r a_i \lambda_i \quad \text{and} \quad \ell(\boldsymbol{\lambda}) := \sum_{i=1}^r a_i.$$

Next, for an integer, $n \geqslant 1$, we define

$$\begin{aligned}
\mathbf{P}(\mathbf{B}_n) :&= \{\lambda = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r}) \in \mathbf{P}(2n+1) : \\
&\qquad [\lambda_i \text{ even } \Rightarrow a_i \text{ even }],\ 1 \leqslant i \leqslant r\}, \\
\mathbf{P}(\mathbf{C}_n) :&= \{\lambda = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r}) \in \mathbf{P}(2n) : [\lambda_i \text{ odd } \Rightarrow a_i \text{ even }],\ 1 \leqslant i \leqslant r\}, \\
\mathbf{P}(\mathbf{D}_n) :&= \{\lambda = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r}) \in \mathbf{P}(2n) : [\lambda_i \text{ even } \Rightarrow a_i \text{ even }],\ 1 \leqslant i \leqslant r\}.
\end{aligned}$$

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It is well known that the set of unipotent orbits over $\overline{F}$ in groups of type $\mathbf{T}_n$ is in natural bijection with $\mathbf{P}(\mathbf{T}_n)$ where $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}\}$. When $\mathbf{T} = \mathbf{D}$, then each $\lambda \in \mathbf{P}(\mathbf{D}_n)$ with at least one odd part corresponds to one unipotent orbit in a group of type $\mathbf{D}_n$, while every $\lambda \in \mathbf{P}(\mathbf{D}_n)$ with only even parts corresponds to exactly two unipotent orbits in such a group. For these facts see 13.3 in [5].

LEMMA 1.1.1. *Let* $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$ *and set* $N_{T_n} := 2n+1$ *if* $\mathbf{T}_n = \mathbf{B}_n$ *and* $N_{T_n} := 2n$ *if* $\mathbf{T} \in \{\mathbf{C}, \mathbf{D}\}$. *Then* $\forall \lambda \in \mathbf{P}(N_{T_n}) : \inf_{\mathbf{P}(\mathbf{T}_n)} \lambda$ *exists.*

Proof. See Lemme 3.6 in Ch. III of [13]. □

1.2. RATIONAL ORBITS AND PREHOMOGENEOUS SPACES

Let $\mathbf{G}$ denote a connected reductive algebraic group defined over F, and $\mathfrak{g} := \mathrm{Lie}\,(\mathbf{G})$. Let $u \in \mathbf{G}(F)$ be a unipotent element. Let $X \in \mathfrak{g}(F)$ such that $u = \exp X$. Let $\{X, H, Y\}$ denote an $\mathfrak{sl}_2$-triplet with $\mathrm{ad}\,H$ semi-simple. For $i \in \mathbb{Z}$, set $\mathfrak{g}_i := \{Z \in \mathfrak{g} : [H, Z] = iZ\}$. Then $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-grading, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, $\forall\, i, j \in \mathbb{Z}$.

Set $\mathbf{M} := (\mathbf{Z}_{\mathbf{G}}(H))^0$. $\mathbf{M}$ acts via Ad on each $\mathfrak{g}_i$. Moreover, a result of Vinberg [V] states that each triple $(\mathbf{M}, \mathrm{Ad}_{|\mathbf{M}}, \mathfrak{g}_i)$ for which $\mathfrak{g}_i \neq (0)$ is a *Prehomogeneous vector space* defined over F. Recall that a prehomogeneous vector space defined over F is a triple $(\mathbf{H}, \rho, V)$ where $\mathbf{H}$ is a connected algebraic group, V a finite-dimensional vector space, and ρ a rational representation of $\mathbf{H}$ on V, all defined over F, such that V contains a Zariski dense open $\mathbf{G}$-orbit. An element $v \in V$ such that $\rho(\mathbf{G}) \cdot v$ is dense is called *generic*. If $\mathbf{H}$ is reductive, then the Prehomogeneous vector space (PVS for short) $(\mathbf{H}, \rho, V)$ is called *regular* if the stabilizer of a generic point is reductive. A result of Kostant⋆ states that the PVS $(\mathbf{M}, \mathrm{Ad}_{|M}, \mathfrak{g}_2)$ is regular and that $\mathrm{Ad}\,\mathbf{M}(F) \cdot X$ is open (in the p-adic topology) in $\mathfrak{g}_2(F)$. A Lemma of Ranga Rao (cf. [11]) shows that

$$O(u) = \exp\left[\mathrm{Ad}\,K(\mathrm{Ad}\,\mathbf{M}(F) \cdot X) + \bigoplus_{i>2} \mathfrak{g}_i(F)\right],$$

where K is a 'good' maximal compact subgroup of $\mathbf{G}(F)$ in the sense of Bruhat–Tits. This lemma also implies that the conjugacy classes within $O^{\mathrm{st}}(u)$ are in one-to-one correspondence⋆⋆ with the $\mathbf{M}(F)$-open orbits in $\mathfrak{g}_2(F)$. On the other hand, a general result in Galois cohomology ([12]) implies that the set of $\mathbf{M}(F)$-open orbits in

⋆The point is not only that the centralizer $\mathbf{M}_X$ of X in $\mathbf{M}$ is reductive, but that it is a Levi component of the centralizer $\mathbf{G}_X$ of X in $\mathbf{G}$. For this see Proposition 2.4 in [4], supplemented by Corollary 3.5 of [K], which shows that $\mathbf{M}_X$ coincides with the centralizer in $\mathbf{G}$ of the entire $\mathfrak{sl}_2$-triplet.

⋆⋆The reasoning seems unclear, but the statement is correct. One can use the result stated in the previous footnote, which gives a bijection between the first Galois cohomology of $\mathbf{M}_X$ and that of $\mathbf{G}_X$, together with the injectivity of the canonical map from the first Galois cohomology of $\mathbf{M}$ to that of $\mathbf{G}$.

$\mathfrak{g}_2(F)$ is in one-to-one correspondence with the set

$$\mathrm{Ker}\,[H^1(F, \mathbf{M}_v) \longrightarrow H^1(F, \mathbf{M})],$$

where v denotes a generic point of $\mathfrak{g}_2$, and $\mathbf{M}_v$ is the stabilizer of v in $\mathbf{M}$. The arrow indicates the morphism between first Galois cohomology sets induced by the inclusion $\mathbf{M}_v \hookrightarrow \mathbf{M}$.

Next, let $\mathbf{G}$ denote a symplectic group or a special orthogonal group of a quadratic space. Each such group is equipped with an F-structure which induces an F-structure on its Lie algebra and the various PVS, associated to the unipotent orbits. If $\mathbf{G}$ is of rank n and type $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$, then we shall often write $\mathbf{T}_n$ instead of $\mathbf{G}$.

Let $\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{T}_n)$ and write $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 \cup \boldsymbol{\lambda}^e$ where $\boldsymbol{\lambda}^0$ consists of the odd parts of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^e$ consists of the even parts of $\boldsymbol{\lambda}$. We introduce prehomogeneous vector spaces $(\mathbf{M}(\boldsymbol{\lambda}^*), \mathfrak{g}_2(\boldsymbol{\lambda}^*))$, where $* \in \{0, e\}$ as follows. First note that we may (and do) write ${}^t(\boldsymbol{\lambda}^0) =: (\mu_1, \mu_2^2, \ldots, \mu_r^2)$, where $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$, and where μ_1 is odd if $\mathbf{T} = \mathbf{B}$ and is even if $\mathbf{T} = \mathbf{C}$ or $\mathbf{D}$; ${}^t(\boldsymbol{\lambda}^e) =: (\nu_1^2, \cdots, \nu_p^2)$, where $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_p$, and where ν_1 is even if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$. Let $\mathbf{S}_\mathbf{T}(\nu_1)$ denote the space of $\nu_1 \times \nu_1$ skew symmetric matrices if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, and let it denote the space of $\nu_1 \times \nu_1$ symmetric matrices if $\mathbf{T} = \mathbf{C}$. Define

$$\mathbf{M}(\lambda^0) := \prod_{k=1}^{r-1} \mathbf{GL}(\mu_{r-k+1}) \times \mathbf{T}_{[\mu_1/2]}\ ,$$

$$\mathfrak{g}_2(\lambda^0) := \bigoplus_{j=1}^{r-1} \mathbf{Mat}\,(\mu_{r-j+1}, \mu_{r-j})\ ,$$

$$\mathbf{M}(\lambda^e) := \prod_{j=1}^{p} \mathbf{GL}(\nu_{p-j+1}),$$

$$\mathfrak{g}_2(\lambda^e) := \bigoplus_{j=1}^{p-1} \mathbf{Mat}\,(\nu_{p-j+1}, \nu_{p-j}) \ \oplus\ \mathbf{S}_\mathbf{T}(\nu_1).$$

$\mathbf{M}(\boldsymbol{\lambda}^0)$ acts on $\mathfrak{g}_2(\boldsymbol{\lambda}^0)$ by

$$(g_1, g_2, \ldots, g_{r-1}, h) \cdot (X_1, X_2, \ldots, X_{r-1}) := (g_1 X_0 g_2^{-1}, \ldots, g_{r-1} X_{r-1} h^{-1}),$$

$$g_k \in \mathbf{GL}(\mu_{r-k+1}), \quad X_k \in \mathbf{Mat}\,(\mu_{r-k+1}, \mu_{r-k}), \quad 1 \leqslant k \leqslant r-1, h \in \mathbf{T}_{[\mu_1/2]};$$

$\mathbf{M}(\boldsymbol{\lambda}^e)$ acts on $\mathfrak{g}_2(\boldsymbol{\lambda}^e)$ by

$$(g_1, g_2, \ldots, g_p) \cdot (X_1, X_2, \ldots, X_{p-1}, Y) := (g_1 X_1 g_2^{-1}, \ldots, g_{p-1} X_{p-1} g_p^{-1}, g_p Y {}^t g_p),$$

$g_j \in \mathbf{GL}(\nu_{p-j+1})$, $X_j \in \mathbf{Mat}\,(\nu_{p-j+1}, \nu_{p-j})$, $1 \leqslant j \leqslant p$, $Y \in \mathbf{S}_\mathbf{T}(\nu_1)$. Here, $\mathbf{Mat}\,(n, m)$ denotes the space of $n \times m$ matrices, and $[x] :=$ integer part of x.

https://doi.org/10.1023/A:1001839511409 Published online by Cambridge University Press

LEMMA 1.2.1. Let $\lambda \in \mathbf{P}(\mathbf{T}_n)$, $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$. Let $(\mathbf{M}(\lambda), \mathfrak{g}_2(\lambda))$ denote the PVS corresponding to λ. Then

$$\mathbf{M}(\lambda) \cong \mathbf{M}(\lambda^0) \times \mathbf{M}(\lambda^e), \qquad \mathfrak{g}(\lambda) \cong \mathfrak{g}_2(\lambda^0) \oplus \mathfrak{g}_2(\lambda^e)\ .$$

The action of $\mathbf{M}(\lambda)$ on $\mathfrak{g}(\lambda)$ is given by the actions of $\mathbf{M}(\lambda^*)$ on $\mathfrak{g}_2(\lambda^*)$, where $* \in \{o, e\}$.

Proof. Omitted. □

Next, we determine the fundamental relative invariants, the stabilizers of generic points, and the $\mathbf{M}(\lambda^*)(F)$-open orbit in $\mathfrak{g}_2(\lambda^*)(F)$. In what follows we shall denote by J_{T,μ_1} the matrix representing the form used to define the group $\mathbf{T}_{[\mu_1/2]}$. We start with the PVS $(\mathbf{M}(\lambda^0), \mathfrak{g}_2(\lambda^0))$. Let s denote the number of distinct parts occurring in ${}^t\lambda^0$ (or λ^0). We inductively define a subset $\{j_1, j_2, \ldots, j_s\} \subseteq \{1, \ldots, r-1\}$ as follows. Set $j_1 = 1$. For $1 < \ell \leqslant s$, let j_ℓ denote the smallest integer k larger than $j_{\ell-1}$ such that $\mu_{r-k+1} < \mu_{r-k}$. Define the following functions on $\mathfrak{g}_2(\lambda^0)$. For $1 \leqslant k \leqslant s-1$, set $Q_k := X_{j_k} X_{j_k+1} \cdots X_{r-1} J_{T,\mu_1} {}^tX_{r-1} \cdots {}^tX_{j_k+1} {}^tX_{j_k}$, where $X_i \in \mathbf{Mat}\,(\mu_{r-i+1}, \mu_{r-i})$ for $k \leqslant i \leqslant r-1$. Q_k is then a $\mu_{r-k+1} \times \mu_{r-k+1}$ matrix.

Set

$$f_k := \left\{ \begin{array}{ll} \det(Q_k), & \text{if } \mathbf{T} = \mathbf{B} \text{ or } \mathbf{D} \\ \operatorname{Pff}(Q_k), & \text{if } \mathbf{T} = \mathbf{C}, \end{array} \right\} 1 \leqslant k \leqslant s-1,$$

where Pff denotes the Pfaffian. Recall that a regular function φ on a PVS $(\mathbf{H}, \rho, V)$ is said to be a *relative invariant*⋆ if there exists a non-trivial rational character χ of $\mathbf{H}$ such that $\varphi(\rho(h) \cdot v) = \chi(h)\varphi(v)$, $\forall h \in \mathbf{H}$, $\forall v \in V$.

LEMMA 1.2.2. *The fundamental relative invariants for the PVS* $(\mathbf{M}(\lambda^0), \mathfrak{g}_2(\lambda^0))$ *are* $f_1, \ldots, f_{s-1}$. □

The set of generic points of $(\mathbf{M}(\lambda^0), \mathfrak{g}_2(\lambda^0))$ is the set of all $v \in \mathfrak{g}_2(\lambda^0)$ such that $f_i(v) \neq 0$, $1 \leqslant i \leqslant s-1$. Consider the following generic point of $\mathfrak{g}_2(\lambda^0)$: $v_0 := \bigoplus_{j=1}^{r-1} [I_{\mu_{r-j+1}}, 0]$. Here, if $0 < m < n$, then $[I_m, 0]$ denotes the $m \times n$ matrix, where $I_m =$ identity $m \times m$ matrix and the last $n-m$ columns are all zero. The stabilizer of v_0 in $\mathbf{M}(\lambda^0)$ is given as follows. If $\mathbf{T} \in \{\mathbf{B}, \mathbf{D}\}$, then

$$\operatorname{stab}_{\mathbf{M}(\lambda^0)}(v_0) \cong \{(h_1, \ldots, h_r) \in \mathbf{O}(\mu_r) \times \prod_{j=1}^{r-1} \mathbf{O}(\mu_{r-j} - \mu_{r-j+1}) : \prod_{i=1}^{r} \det h_i = 1\};$$

and if $\mathbf{T} = \mathbf{C}$, then

$$\operatorname{stab}_{\mathbf{M}(\lambda^0)}(v_0) \cong \mathbf{Sp}(\mu_r) \times \prod_{j=1}^{r-1} \mathbf{Sp}(\mu_{r-j} - \mu_{r-j+1})$$

(note that all the μ's are even in this case).

⋆Recall also that the *fundamental* relative invariants are the irreducible polynomials on the PVS whose zero-sets give the irreducible components of codimension 1 of the complement of the open orbit in the PVS

Remarks. (1) Since $H^1(F, \mathbf{Sp}(2m)) = \langle 1 \rangle$, we immediately see that if $\mathbf{T} = \mathbf{C}$, then there is only one $\mathbf{M}(\lambda^0)(F)$-open orbit in $\mathfrak{g}_2(\lambda^0)(F)$.

(2) For $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, write $\lambda^0 = (\lambda_1^{a_1}, \ldots, \lambda_s^{a_s})$, $a_i \geqslant 1$, $1 \leqslant i \leqslant s$. Then $\mathrm{stab}_{\mathbf{M}(\lambda^0)}(v_0)$ is of type $\prod_{a_i \text{ odd}} \mathbf{B}_{(a_i-1)/2} \times \prod_{a_i \text{ even}} \mathbf{D}_{a_i/2}$.

(3) If $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, then the group $\mathrm{stab}_{\mathbf{M}(\lambda^0)}(v_0)/(\mathrm{stab}_{\mathbf{M}(\lambda^0)}(v_0))^0$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{s-1}$. Moreover, using the description of $\mathrm{stab}_{\mathbf{M}(\lambda^0)}(v_0)$ given above one can show that, for $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, the set

$$\mathrm{Ker}\,[H^1(F, \mathrm{Stab}_{M(\lambda^0)}(v_0)) \longrightarrow H^1(F, \mathbf{M}(\lambda^0))]$$

is in one-to-one correspondence with the set of equivalence classes of quadratic forms $(q_i)_{1 \leqslant i \leqslant s}$, where q_i is a nondegenerate quadratic form of rank a_i such that $\bigoplus_{i=1}^s q_i$ has the same anisotropic kernel as the form used to define the orthogonal group factor of $\mathbf{M}(\lambda^0)$.

Next, we give another description of the $\mathbf{M}(\lambda^0)(F)$-open orbits in $\mathfrak{g}_2(\lambda^0)(F)$ which is more closely connected with the geometry of the PVS $(\mathbf{M}(\lambda^0), \mathfrak{g}_2(\lambda^0))$. For each generic $v \in \mathbf{M}(\lambda^0)(F)$, and each $1 \leqslant k \leqslant s-1$, $Q_k(v)$ is a $\mu_{r-k+1} \times \mu_{r-k+1}$ non-degenerate symmetric matrix which we may think of as a nondegenerate quadratic form of rank μ_{r-k+1}. Thus, to each generic point $v \in \mathfrak{g}_2(\lambda^0)(F)$ we may attach quadratic forms $(Q_k(v))_{1 \leqslant k \leqslant s-1}$.

LEMMA 1.2.3. *For each generic v_1, $v_2 \in \mathfrak{g}_2(\lambda^0)(F)$, v_1 and v_2 belong to the same $\mathbf{M}(\lambda^0)(F)$-open orbit iff $Q_k(v_1)$ is equivalent to $Q_k(v_2)$ for all k, $1 \leqslant k \leqslant s-1$.* □

Remark 1.2.4. Not every $(s-1)$-tuple $(Q_i)_{1 \leqslant i \leqslant s-1}$ of quadratic forms with rank $Q_i = \mu_{r-i+1}$ does correspond to a generic point. The relationship between the two classifications of $\mathbf{M}(\lambda^0)$-open orbits discussed above is given by the following lemma (with the same notation as in the 'Remarks').

LEMMA 1.2.5. *Let $v \in \mathfrak{g}_2(\lambda^0)(F)$ be a generic point. Suppose that the $\mathbf{M}(\lambda^0)(F)$-open orbit containing v corresponds to an s-tuple of equivalence classes of quadratic forms $(q_i)_{1 \leqslant i \leqslant s}$, with rank $q_i = a_i$. Then for $1 \leqslant k \leqslant s-1$, we have $Q_k(v) \cong \bigoplus_{i=1}^k q_i$.* □

Next, we treat the PVS $(\mathbf{M}(\lambda^e), \mathfrak{g}_2(\lambda^e))$. This time, let s denote the number of distinct parts occuring in ${}^t\lambda^e$ (or λ^e). Inductively define $\{j_1, j_2, \ldots, j_s\} \subset \{1, 2, \ldots, p-1\}$ as follows. Set $j_1 = 1$, for $1 \leqslant \ell \leqslant s$, let j_ℓ denote the smallest integer k larger than $j_{\ell-1}$ such that $\mu_{r-k+1} < M_{r-k}$. Define the following functions on $\mathfrak{g}_2(\lambda^e)$. For $1 \leqslant k \leqslant s$, set $Q_k := X_{j_k} X_{j_k+1} \cdots X_{p-1} Y {}^tX_{p-1} \cdots {}^tX_{j_k+1} {}^tX_{j_k}$, where $X_i \in \mathrm{Mat}\,(v_{p-i+1}, v_{p-i})$ for $k \leqslant i \leqslant p-1$, $Y \in \mathbf{S}_{\mathbf{T}}(v_1)$. Q_k is then a $v_{p-k+1} \times v_{p-k}$ matrix for $1 \leqslant k \leqslant s$. Set

$$f_k := \begin{cases} \det(Q_k), & \text{if } \mathbf{T} = \mathbf{C}, \\ \mathrm{Pff}\,(Q_k), & \text{if } \mathbf{T} = \mathbf{B} \text{ or } \mathbf{D}, \end{cases} \qquad 1 \leqslant k \leqslant s.$$

LEMMA 1.2.6. *The set* $\{f_1, \ldots, f_s\}$ *is a set of fundamental relative invariants for the* PVS $(\mathbf{M}(\lambda^0), \mathfrak{g}_2(\lambda^e))$. □

Set $v_0 = (\bigoplus_{j=0}^{p-1}[I_{v_{p-j+1}}, 0], I_T)$, where $I_T := v_1 \times v_1$ identity matrix if $\mathbf{T} = \mathbf{C}$, and is equal to the $v_1 \times v_1$ skew symmetric matrix

$$\begin{bmatrix} 0 & I_{v_1/2} \\ -I_{v_1/2} & 0 \end{bmatrix}$$

if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$. If $\mathbf{T} = \mathbf{C}$, then $\mathrm{stab}_{\mathbf{M}(\lambda^e)}(v_0) \cong \mathbf{O}(v_p) \times \prod_{j=0}^{p-1} \mathbf{O}(v_{p-j} - v_{p-j+1})$, and if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, then $\mathrm{stab}_{\mathbf{M}(\lambda^e)}(v_0) \cong \mathbf{Sp}(v_p) \times \prod_{j=1}^{p-1} \mathbf{Sp}(v_{p-j} - v_{p-j+1})$.

Remarks 1.2.7. (1) If $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, then there is only one $\mathbf{M}(\lambda^e)(F)$-open orbit in $\mathfrak{g}_2(\lambda^e)(F)$.

(2) If $\mathbf{T} = \mathbf{C}$, then the group $\mathrm{stab}_{M(\lambda^e)}(v_0)/(\mathrm{stab}_{M(\lambda^e)}(v_0))^0$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^s$.

(3) Since $H^1(F, \mathbf{Sp}(2m)) = \langle 1 \rangle$, we see that the set of $\mathbf{M}(\lambda^e)(F)$-open orbits in $\mathfrak{g}_2(\lambda^e)(F)$, assuming $\mathbf{T} = \mathbf{C}$, is in one-to-one correspondence with the set $H^1(F, \mathrm{stab}_{\mathbf{M}(\lambda^e)}(v_0))$ which in turn is in one-to-one correspondence with the set of equivalence classes of non-degenerate quadratic forms $(q_i)_{1 \leqslant i \leqslant s}$, where $\mathrm{rank}\, q_i = b_i$. Here $\lambda^e := (\lambda_1^{b_1}, \ldots, \lambda_s^{b_s})$.

We also have the following description of the $\mathbf{M}(\lambda^e)(F)$-open orbits on $\mathfrak{g}_2(\lambda^e)(F)$. Assume $\mathbf{T} = \mathbf{C}$.

LEMMA 1.2.8. *For each generic* $v_1, v_2 \in \mathfrak{g}_2(\lambda^e)(F)$, v_1 *and* v_2 *belong to the same* $\mathbf{M}(\lambda^e)(F)$*-open orbit iff* $Q_k(v_1)$ *is equivalent to* $Q_k(v_2)$ *for all* k, $1 \leqslant k \leqslant s$. □

The relationship between the two above classifications is given by

LEMMA 1.2.9. *Let* $v \in \mathfrak{g}_2(\lambda^e)(F)$ *be a given generic element. Suppose that the* $\mathbf{M}(\lambda^e)(F)$*-open orbit containing* v *corresponds to an* s*-tuple of equivalence classes of quadratic forms* $(q_i)_{1 \leqslant i \leqslant s}$ *with rank* $q_i = b_i$. *Then for* $1 \leqslant k \leqslant s$, *we have* $Q_k(v) \cong \bigoplus_{i=1}^{k} q_i$. □

Remark 1.2.10. From the previous lemmas and remarks we find that the set of $\mathbf{G}(F)$-orbits within the stable orbit with corresponding partition λ, correspond bijectively to the $\mathbf{M}(\lambda^*)$-open orbits in $\mathfrak{g}_2(\lambda^*)(F)$ where $* = o$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$ and $* = e$ if $\mathbf{T} = \mathbf{C}$. We may, thus, parametrize these orbits using the quadratic forms Q_k. Recall that the equivalence class of quadratic form q is determined by its discriminant $\Delta(q) \in F^\times/(F^\times)^2$, and its Hasse-invariant $\eta(q) \in \{\pm 1\}$.

1.3. THE EXPONENT $\eta(O^{\mathrm{st}})$ AND THE DEFINITION OF PACKETS OF ORBITS

NOTATION 1.3.1. Let $\lambda \in \mathbf{P}(\mathbf{T})$. The stable unipotent orbit corresponding to λ will be denoted by O_λ^{st}. Let $\lambda = \lambda^0 \cup \lambda^e$. Write $\lambda^* = (\lambda_1^{a_1}, \ldots, \lambda_s^{a_s})$ where $a_i \geqslant 1$, for $1 \leqslant i \leqslant s$. Here $* = o$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$ and $* = e$ if $\mathbf{T} = \mathbf{C}$. Define $Q_1, \ldots, Q_t$ as before, where $t = s - 1$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$ and $t = s$ if $\mathbf{T} = \mathbf{C}$. Let $O \subseteq O_\lambda^{\mathrm{st}}$ denote a rational orbit corresponding to some $\mathbf{M}(\boldsymbol{\lambda}^*)(F)$-open orbit in $\mathfrak{g}_2(\boldsymbol{\lambda}^*)(F)$. Let v denote a generic point belonging to that orbit. Set $\Delta_i := \Delta(Q_i(v))$, $\eta_i := \eta(Q_i(v))$, $1 \leqslant i \leqslant t$. We shall, then, label O by

$$O_\lambda(\Delta_1, \eta_1; \Delta_2, \eta_2; \dots; \Delta_t, \eta_t) \; .$$

Thus, the set of rational orbits within O_λ^{st} may be parametrized by a subset of the group $[F^\times/(F^\times)^2]^t \times (\mathbb{Z}/2\mathbb{Z})^t$. Next, we determine the adjoint classes* within O_λ^{st}, $\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{T})$, where $\mathbf{T} = \mathbf{C}$ or $\mathbf{D}$, and is split. We keep the above notation.

LEMMA 1.3.2. *Let $\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{T})$. Let O_1, $O_2 \subseteq O^{\mathrm{st}}$ be two orbits. Let v_1, v_2 be two generic elements contained in the $\mathbf{M}(\boldsymbol{\lambda}^*)(F)$-open orbits in $\mathfrak{g}_2(\boldsymbol{\lambda}^*)(F)$ corresponding to O_1, O_2 respectively, where $* = o$ if $\mathbf{T} = \mathbf{D}$ and $* = e$ if $\mathbf{T} = \mathbf{C}$. Then O_1 is conjugate to O_2 under the adjoint group iff $Q_k(v_2)$ is equivalent to $Q_k(\sigma v_1)$ $\forall k$, $1 \leqslant k \leqslant t$, $\forall \sigma \in F^\times/(F^\times)^2$. (Here σv is obtained from v by multiplying every entry of v by σ.)*

Proof. We use the fact that O_1 is conjugate to O_2 under the adjoint group iff there exists $(h_\varphi) \in H^1(F, \mathbf{Z})$, where $\mathbf{Z} =$ center of $\mathbf{G}$, such that $O_2 = \mathrm{Ad}\, h_\varphi(O_1)$ $\forall \varphi \in \mathrm{Gal}\,(\overline{F}/F)$. We realize $\mathbf{G}$ as the special isometry group of the form

$$\begin{bmatrix} & I_n \\ \varepsilon I_n & \end{bmatrix},$$

where $\varepsilon = 1$ if $\mathbf{G}$ is orthogonal and $\varepsilon = -1$ if $\mathbf{G}$ is symplectic. Let $\tau \in \{1, \epsilon, \pi, \epsilon\pi\}$, and set $E_\tau := F(\sqrt{\tau})$. For $\varphi \in \mathrm{Gal}\,(\overline{F}/F)$, let φ_τ denote its restriction to E_τ. Define $g_\tau := \mathrm{diag}\,(\sqrt{\tau}, \dots, \sqrt{\tau}, \sqrt{\tau^{-1}}, \dots, \sqrt{\tau^{-1}}) \in \mathbf{G}$. Then $\varphi \mapsto g_\tau^{\varphi_\tau} g_\tau^{-1}$ is in $H^1(F, \mathbf{Z})$. Now, a typical element of $\mathfrak{g}_2(F)$ has the form

$$X = \begin{bmatrix} A & B \\ 0 & -{}^tA \end{bmatrix},$$

where B is symmetric if $\mathbf{G}$ is symplectic, and is skew symmetric is $\mathbf{G}$ is orthogonal. Then

$$\mathrm{Ad}\,(g_\tau)X = \begin{bmatrix} A & \tau B \\ 0 & -{}^tA \end{bmatrix}.$$

The statement of the lemma can then be easily deduced. □

We now define the η-exponent of a rational** unipotent orbit O and we also introduce the concept of a packet of unipotent orbits.

1.3.3. *Definition of* $\eta(O^{\mathrm{st}})$. Let $\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{T})$. Following the notation in 1.2, let $O = O_\lambda(\Delta_1, \eta_1; \; \dots; \Delta_t, \eta_t)$ denote a rational orbit within O_λ^{st}. We define the *η-exponent* of O^{st}, denoted $\eta(O^{\mathrm{st}})$ as follows: Let $\boldsymbol{\lambda}^* = (\lambda_1^{a_1}, \cdots, \lambda_s^{a_s})$.

*The adjoint class of an element in $\mathbf{G}(F)$ is its orbit under the F-points of the adjoint group of $\mathbf{G}$.

**The η-exponent only depends on the stable orbit. Its definition (see 1.3.3) involves a partition $\boldsymbol{\lambda}^*$; presumably $\boldsymbol{\lambda}^*$ is the partition defined in 1.3.1.

$\eta(O^{\mathrm{st}}) := \#\{k, 1 \leqslant k \leqslant s$, such that $a_k > 1\} - \delta$; if there exists at least one j, $1 \leqslant j \leqslant s$ such that $a_j > 1$. Here $\delta = 1$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, and $\delta = 0$ if $\mathbf{T} = \mathbf{C}$.
If $a_j = 1$ for every $0 \leqslant j \leqslant t$, then we set $\eta(O^{\mathrm{st}}) = 0$.

1.3.4. *Definition of packets.* Let $\lambda \in P(\mathbf{T})$. Let $I(\lambda) := \{1, \dots, t\}$, write $I(\lambda) = I_0(\lambda) \cup I_e(\lambda)$,where

$$I_0(\lambda) := \{i \in I(\lambda) : \text{rank } Q_i \text{ is odd }\}, \qquad I_e(\lambda) := \{i \in I(\lambda) : \text{rank } Q_i \text{ is even }\}.$$

Let

$$I_*(\lambda) = \begin{cases} I_0(\lambda), & \text{if } \mathbf{T} = \mathbf{B}, \\ I_e(\lambda), & \text{if } \mathbf{T} = \mathbf{C} \text{ or } \mathbf{D}. \end{cases}$$

Let $\psi : I_* \longrightarrow F^\times/(F^\times)^2$. We associate a *packet* $\prod(\lambda, \psi)$ of unipotent orbits within O_λ^{st} as follows:

$$\prod(\lambda, \psi) := \{O_\lambda(\Delta_1, \eta_1; \cdots; \Delta_t, \eta_t) \subseteq O_\lambda^{\mathrm{st}} : \Delta_\alpha = \psi(\alpha) \text{ if } \alpha \in I_*(\lambda)\}$$

Remark 1.3.5. For split even orthogonal groups and symplectic groups, each packet is a union of adjoint orbits.

2. The Springer Correspondence and Transfer of Packets

2.1. THE SPRINGER CORRESPONDENCE

Let $\mathbf{G}$ denote a connected reductive algebraic group over F. Let W denote the abstract Weyl group of $\mathbf{G}$. Let $u \in \mathbf{G}$ be a unipotent element and let $\mathcal{B}_u$ denote the variety of Borel subgroups of $\mathbf{G}$ containing u. Set $e(u) := \dim \mathcal{B}_u$, and define the group $A(u) := \mathbf{Z}_{\mathbf{G}}(u)/(\mathbf{Z}_{\mathbf{G}}(u))^0\mathbf{Z}(\mathbf{G})$, where $\mathbf{Z}_{\mathbf{G}}(u)$ is the centralizer of u in $\mathbf{G}$ and $(\mathbf{Z}_{\mathbf{G}}(u))^0$ denotes the identity connected component of $\mathbf{Z}_{\mathbf{G}}(u)$, and $\mathbf{Z}(\mathbf{G}) :=$ center of $\mathbf{G}$. The group $A(u)$ acts naturally on the set of irreducible components of $\mathcal{B}_u$, and hence on the étale cohomology space $H^*(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)$, $\ell \neq p$. Springer has defined a representation of W on $H^{2e(u)}(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell)$ which commutes with the action of $A(u)$. For every irreducible representation ϕ of $A(u)$, let $E_{u,\phi} := \mathrm{Hom}_{A(u)}(\phi, H^{2e(u)}(\mathcal{B}_u, \overline{\mathbb{Q}}_\ell))$ regarded as a W-module. Springer has shown that $E_{u,\phi}$ is either (0) or is an irreducible module of W, and that every irreducible W-module is obtained in this way. Moreover, $E_{u_1,\phi_1} \cong E_{u_2,\phi_2}$ iff (u_1, ϕ_1) and (u_2, ϕ_2) are conjugated in $\mathbf{G}$. Thus one obtains an injection, called the *Springer correspondence*, between the set of irreducible representations of W and the set of pairs (O, φ) where O is a unipotent conjugacy class and $\varphi \in \widehat{A(u)}$, where $u \in O$. The pairs $(O, \mathbf{1})$, where $\mathbf{1}$ denotes the trivial character, are always in the image of the Springer correspondence. Assume now that $\mathbf{G}$ is of type $\mathbf{B}_n$ or $\mathbf{C}_n$. The irreducible representations of W can then (following Lusztig) be parameterized by *symbols* of *rank n and defect* 1, i.e. tableaux

https://doi.org/10.1023/A:1001839511409 Published online by Cambridge University Press

of the form

$$\Lambda = \begin{pmatrix} \alpha_1 \alpha_2 \cdots \alpha_{m+1} \\ \beta_1 \beta_2 \cdots \beta_m \end{pmatrix},$$

where

$$0 \leqslant \alpha_1 \leqslant \alpha_2 < \ldots < \alpha_{m+1}, \qquad 0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_m$$

are all integers with $\sum \alpha_i + \sum \beta_i = n + m^2$. An irreducible representation is *special* if its corresponding symbol satisfies the conditions

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_m \leqslant \alpha_{m+1} .$$

If $\mathbf{G}$ is of type $\mathbf{D}_n$, then the irreducible representations of W can be (again following Lusztig) parametrized using symbols of *rank n and defect* 0 (in which the first and second rows can be interchanged), i.e. tableaux of the form

$$\Lambda = \begin{pmatrix} \alpha_1 \alpha_2 \cdots \alpha_m \\ \beta_1 \beta_2 \cdots \beta_m \end{pmatrix} = \begin{pmatrix} \beta_1 \beta_2 \cdots \beta_m \\ \alpha_1 \alpha_2 \cdots \alpha_m \end{pmatrix},$$

where

$$0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_m, \qquad 0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_m$$

are integers and $\sum \alpha_i + \sum \beta_i = n + m^2 - m$. If $\{\alpha_1, \ldots, \alpha_m\} \neq \{\beta_1, \ldots, \beta_m\}$, then Λ corresponds to only one irreducible representation of W. If $\{\alpha_1, \ldots, \alpha_m\} = \{\beta_1, \cdots, \beta_m\}$, the n is necessarily even and Λ corresponds to a direct sum of two irreducible representations of W. An irreducible representation of W is *special* if the corresponding symbol satisfies the conditions

$$\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \beta_2 \leqslant \cdots \leqslant \alpha_m \leqslant \beta_m$$

or

$$\beta_1 \leqslant \alpha_1 \leqslant \beta_2 \leqslant \alpha_2 \leqslant \cdots \leqslant \beta_m \leqslant \alpha_m .$$

Here, we understand that if $\{\alpha_1, \ldots, \alpha_n\} = \{\beta_1, \ldots, \beta_n\}$, then each of the irreducible components of the representation of W corresonding to Λ is special.

In all cases discussed above, irreducible representations which are not special are called *nonspecial*. Symbols corresponding to special representations will be called special symbols. Lusztig has partitioned $\widehat{W}$ into certain *families* (cf. [9]). Each family contains a unique special irreducible representation of W. The families can be described using symbols as follows. Two irreducible characters of W belong to the same family if and only if they possess symbols for which the unordered sets $\{\alpha_1, \ldots, \alpha_{m+1}, \beta_1, \ldots, \beta_m\}$ (resp. $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}$) are the same if $\mathbf{G}$ is of type $\mathbf{B}$ or $\mathbf{C}$ (resp. $\mathbf{D}$). When $\mathbf{G}$ is of type $\mathbf{D}$ and $\{\alpha_1, \ldots, \alpha_m\} = \{\beta_1, \ldots, \beta_m\}$, then each irreducible component of the character corresponding to the given symbol constitutes one family. The Springer correspondence gives rise to a map $O \mapsto E_{u,\mathbf{1}}$, where O is a unipotent orbit, $u \in O$, and $\mathbf{1} \in \widehat{A(u)}$ is the trivial character. This map allows us to transfer the notions of special, nonspecial and families to unipotent orbits.

DEFINITIONS 2.1.1. (i) O is said to be *special* (resp. *nonspecial*) if $E_{u,\mathbf{1}}$ is special (resp. nonspecial).

(ii) O_1 and O_2 are said to belong to the same family if $E_{u_1,\mathbf{1}}$ and $E_{u_2,\mathbf{1}}$ belong to the same family of irreducible characters. Here $u_i \in O_i$, $i = 1, 2$.

Next, we give a description of the map $O \mapsto E_{u,\mathbf{1}}$ in terms of partitions and symbols. We shall employ the following notation: If $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)$ is a partition, we set

$$[\tfrac{1}{2}\boldsymbol{\mu}] := ([\mu_p/2], \dots, [\mu_1/2]) ,$$
$$[\tfrac{1}{2}\boldsymbol{\mu}] \pm 1 := ([\mu_p/2] \pm 1, \dots, [\mu_1/2] \pm 1) ,$$

(note the change in order).

Now, let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbf{P}(\mathbf{T}_n)$, $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$, and define $\boldsymbol{\lambda}_+ := (\lambda_1 + r - 1, \lambda_2 + r - 2, \dots, \lambda_{r-1} + 1, \lambda_r) =: \boldsymbol{\lambda}_+^0 \cup \boldsymbol{\lambda}_+^e$. If $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, then the orbit corresponding to $\boldsymbol{\lambda}$ gets mapped to the character with symbols $\binom{[\frac{1}{2}\boldsymbol{\lambda}_+^0]}{[\frac{1}{2}\boldsymbol{\lambda}_+^e]}$. If $\mathbf{T} = \mathbf{C}$, then the orbit corresponding to $\boldsymbol{\lambda}$ gets mapped to the character with symbol $\binom{[\frac{1}{2}\boldsymbol{\lambda}_+^e]}{[\frac{1}{2}\boldsymbol{\lambda}_+^0]}$ if $\ell(\boldsymbol{\lambda})$ is odd, and gets mapped to the character with symbol $\binom{0, [\frac{1}{2}\boldsymbol{\lambda}_+^0]+1}{[\frac{1}{2}\boldsymbol{\lambda}_+^e]}$ if $\ell(\boldsymbol{\lambda})$ is even. This follows easily from results in Sections 11.4 and 13.3 of [5].

NOTATION. Given $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$, we shall denote by $\mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n)$ the set of partitions corresponding to the special orbits in groups of type $\mathbf{T}_n$. The following description of $\mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n)$ is well known (cf. 13.4 in [5], supplemented by 3.9 and 3.11 in Ch. III of [13]).

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$.

(i) $\boldsymbol{\lambda} \in \mathbf{P}_{\mathrm{sp}}(\mathbf{B}_n) \Leftrightarrow \lambda_{2i}$ and λ_{2i+1} have the same parity for all i with $1 \leqslant i \leqslant [r/2] \Leftrightarrow {}^t\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{B}_n)$.

(ii) $\boldsymbol{\lambda} \in \mathbf{P}_{\mathrm{sp}}(\mathbf{C}_n) \Leftrightarrow \lambda_{2i-1}$ and λ_{2i} have the same parity for all i with $1 \leqslant i \leqslant [r/2] \Leftrightarrow {}^t\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{C}_n)$.

(iii) $\boldsymbol{\lambda} \in \mathbf{P}_{\mathrm{sp}}(\mathbf{D}_n) \Leftrightarrow \lambda_{2i-1}$ and λ_{2i} have the same parity for all i with $1 \leqslant i \leqslant [r/2]$.

Next, we describe the families of orbits alluded to above. Let $\boldsymbol{\lambda} \in \mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n)$, $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$. Write

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 \cup \boldsymbol{\lambda}^e \text{ Set } \boldsymbol{\lambda}^* := \begin{cases} \boldsymbol{\lambda}^0, & \text{if } \mathbf{T} = \mathbf{B} \text{ or } \mathbf{D} \\ \boldsymbol{\lambda}^e, & \text{if } \mathbf{T} = \mathbf{C} \end{cases} .$$

Assume that $\boldsymbol{\lambda}^* =: (\mu_1^{a_1}, \dots, \mu_r^{a_r})$, and define $\boldsymbol{\lambda}_1^*$, $\boldsymbol{\lambda}_2^*$ by

$$\boldsymbol{\lambda}^* =: \boldsymbol{\lambda}_1^* \cup \boldsymbol{\lambda}_2^* \text{ and } \boldsymbol{\lambda}_1^* := (\mu_1^{\varepsilon_1}, \dots, \mu_r^{\varepsilon_r}) =: (\nu_1, \nu_2, \dots, \nu_k),$$

where $\varepsilon_i = a_i - 2$ if $a_i > 2$ and $\varepsilon_i = a_i$ otherwise $(1 \leqslant i \leqslant r)$. If $\mathbf{T} = \mathbf{C}$ and $\ell(\lambda)$ is even we add a zero entry $v_{k+1} = 0$ to λ_1^*. Now define A_λ as follows.

(i) If $\mathbf{T} = \mathbf{B}$, $A_\lambda := \phi$ unless $v_{2i-1} - v_{2i} = 0$ or 2 for all i with $1 \leqslant i \leqslant [k/2]$, and $v_k = 1$. If these conditions are satisfied then we set $A_\lambda := \{1 \leqslant i \leqslant [k/2] : v_{2i-1} - v_{2i} = 2\}$.

(ii) If $\mathbf{T} = \mathbf{C}$, $A_\lambda := \phi$ unless $v_{2i} - v_{2i+1} = 0$ or 2 for all i with $1 \leqslant i \leqslant [k/2]$, and $v_k = 2$. If these conditions are satisfied then we set $A_\lambda := \{1 \leqslant i \leqslant [k/2] : v_{2i} - v_{2i+1} = 2\}$.

(iii) If $\mathbf{T} = \mathbf{D}$, $A_\lambda := \phi$ unless $v_{2i} - v_{2i+1} = 0$ or 2 for all i with $1 \leqslant i \leqslant [k/2]$, and $v_k = 1$. If these conditions are satisfied then we set $A_\lambda := \{1 \leqslant i \leqslant [k/2] : v_{2i} - v_{2i+1} = 2\}$.

For any subset $J \subseteq A_\lambda$, define the partition $\lambda_1^*(J) = (v_1^*, \ldots, v_k^*)$ as follows.

(i) If $\mathbf{T} = \mathbf{B}$, then for all i with $1 \leqslant i \leqslant [k/2]$

$$(v_{2i-1}^*, v_{2i}^*) := \begin{cases} (v_{2i-1}, v_{2i}) , & \text{if } i \notin J, \\ (v_{2i-1} - 1, v_{2i} + 1) , & \text{if } i \in J. \end{cases}$$

(ii) If $\mathbf{T} = \mathbf{C}$ or $\mathbf{D}$, then for all i with $1 \leqslant i \leqslant [k/2]$

$$(v_{2i}^*, v_{2i+1}^*) := \begin{cases} (v_{2i}, v_{2i+1}) , & \text{if } i \notin J, \\ (v_{2i} - 1, v_{2i+1} + 1) , & \text{if } i \in J. \end{cases}$$

Now for $J \subseteq I_\lambda$, set $\lambda(J) := \lambda_1^*(J) \cup \lambda_2^* \cup \lambda^{**}$, where

$$\lambda^{**} = \begin{cases} \lambda^e & \text{if } \mathbf{T} = \mathbf{B} \text{ or } \mathbf{D}, \\ \lambda^0 & \text{if } \mathbf{T} = \mathbf{C}. \end{cases}$$

LEMMA 2.1.2. *The assignment $J \subset A_\lambda \mapsto O_{\lambda(J)}$ establishes a bijection between the power set of A_λ and the family containing the special orbit O_λ.*

Proof. Using the prescription given above for the map $O \mapsto E_{u,\mathbf{1}}$ one checks that the symbols corresponding to $\lambda(J)$, $J \subset A_\lambda$, belong to the same family. One then observes that every partition is of the form $\lambda(J)$ for some $\lambda \in \mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n)$ and some $J \subset A_\lambda$. □

2.2. THE GROUP $\overline{A}(u)$ AND THE PACKETS

Assume that $E_{u,\phi}$ is an irreducible character of the Weyl group of some reductive group, where u is a unipotent element and $\phi \in \widehat{A(u)}$. Lusztig has defined a integer $a_{E_{u,\phi}}$ by requiring that $t^{a_{E_{u,\phi}}}$ is the highest power dividing the *generic degree* of $E_{u,\phi}$ (see [9]).

DEFINITION 2.2.1 ([9]) Let $u \in \mathbf{G}$ be a special unipotent element. Set $\widehat{A(u)}_0 := \{\phi \in \widehat{A(u)} : E_{u,\phi} \neq (0)$, and $a_{E_{u,\phi}} = \dim \mathcal{B}_u\}$. The group $\overline{A}(u)$ is the largest quotient of $A(u)$ through which all $\phi \in \widehat{A(u)}_0$ do factor.

NOTATION 2.2.2. We will occasionally write $\overline{A}(O)$ if $O = O(u)$ or $\overline{A}(\lambda)$ if $O(u) = O_\lambda$ in place of $A(u)$. Similar conventions will be applied to $A(u)$ as well as $C(u) := \mathbf{Z_G}(u)/(\mathbf{Z_G}(u))^0$.

Remark 2.2.3. For orthogonal and symplectic groups, the group $\overline{A}(u)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $k \in \mathbb{N}$ which can be calcluated from the symbol attached to λ as follows (see [5]). If **G** is of type **B** or **C**, then the set of entries occuring only once in the symbol has cardinality $2k+1$. If **G** is of type **D**, then the set of entries of the symbol which appear in just one row has cardinality $2k$. The integer k can also be calculated directly from the partition as follows. Let $\lambda \in \mathbf{P}(\mathbf{T}_n)$, and $\lambda^* =: (\lambda_1^{a_1}, \ldots, \lambda_s^{a_s})$, where $\lambda^* =: \lambda^0$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$ and $\lambda^* = \lambda^e$ if $\mathbf{T} = \mathbf{C}$. Let $t = s - 1$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$ and $t = s$ if $\mathbf{T} = \mathbf{C}$. Let $I_*(\lambda)$ be as defined in 1.3.4.

LEMMA 2.2.4. *Let* $\lambda \in \mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n)$. *Then rank* $\overline{A}(\lambda) = \#(I_*(\lambda))$. □

Remark 2.2.5. (i) If **G** is a symplectic, or a unramified quasi-split orthogonal group, and λ a special partition such that $O_\lambda^{\mathrm{st}} \neq \phi$, then the number of packets partitioning O_λ^{st} is equal to the number of irreducible unipotent characters of $\mathbf{G}(\mathbb{F}_q)$ which are associated with λ (see [5]).

(ii) If **G** is symplectic or quasi-split orthogonal, then for any λ such that $O_\lambda^{\mathrm{st}} \neq \phi$, we may parametrize the set of packets partitioning O_λ^{st} using the set $\mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}[\overline{A}(u), F^\times/(F^\times)^2]$.

2.3. DUALITY AND PACKETS

Let **G** be of type $\mathbf{T}_n$, $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$ and let $\hat{\mathbf{G}}$ denote the dual group which we take to be defined over the field F.

In [13], Spaltenstein defines two duality maps. The first map is an order preserving isomorphism $d_{T_n} = d\colon \mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n) \to \mathbf{P}_{\mathrm{sp}}(\hat{\mathbf{T}}_n)(\hat{\mathbf{T}}$ is the type dual to $\mathbf{T}$). The second map is an order reversing isomorphism

$$D_{T_n} = D : \mathbf{P}_{\mathrm{sp}}(\mathbf{T}_n) \to \mathbf{P}_{\mathrm{sp}}(\hat{\mathbf{T}}_n).$$

When **G** is odd orthogonal or symplectic then the map D is related to the map d by: $D(\lambda) = {}^t(d(\lambda))$ for every special λ. When **G** is even orthogonal, then $d(\lambda) = \lambda$ for every special λ.

NOTATION 2.3.1. We shall often write $\hat{\lambda}$ instead of $d(\lambda)$ and ${}^L\lambda$ instead of $D(\lambda)$. The above duality maps have a simple meaning in terms of the Springer correspondence. The map d regarded as a map between special symbols, via the correspondence $O(u) \mapsto E_{u,1}$, is just the identity map. The map D is obtained from d by tensoring the irreducible character corresponding to $d(\lambda)$ by the sign character. It thus follows

https://doi.org/10.1023/A:1001839511409 Published online by Cambridge University Press

that we get isomorphisms

$$\overline{A}(\lambda) \cong \overline{A}(\hat{\lambda}) \cong \overline{A}({}^L\lambda), \quad \lambda \text{ special.}$$

As a consequence, the sets $I_*(\lambda)$, $I_*(\hat{\lambda})$, $I_*({}^L\lambda)$ (see 1.3.4.) do all have the same cardinality. These sets, as subsets of $\mathbb{N}$, are equipped with a natural order⋆ which allows us to define two order-preserving bijections $\iota_{\hat{\lambda}} : I_*(\lambda) \to I_*(\hat{\lambda})$ and $\iota_{L_\lambda} : I_*(\lambda) \to I_*({}^L\lambda)$, where λ is special. These bijections induce the following correspondences of packets.

DEFINITION 2.3.2. Let $\prod(\lambda, \psi)$ be the packet associated to the special partition λ and ψ: $I_*(\lambda) \longrightarrow F^\times/(F^\times)^2$. Define $\widehat{\prod}(\lambda, \psi) := \prod(\hat{\lambda}, \psi \circ (\iota_{\hat{\lambda}})^{-1})$ and ${}^L\prod(\lambda, \psi) := \prod({}^L\lambda, \psi \circ \iota_{L_\lambda})^{-1})$.

For later purposes we shall need to explicitly describe the duality maps d and D.

(i) **G** is of type $\mathbf{B}_n$.
Let $\lambda \in \mathbf{P}_{\mathrm{sp}}(\mathbf{B}_n)$ with $\lambda = \lambda^0 \cup \lambda^e$. λ^0 has an odd number of parts, so we may (and do) write $\lambda^0 := (\mu_1, \dots, \mu_{2r+1})$. Now define $\underline{\lambda}^0 := (\mu_1^*, \dots, \mu_{2r}^*, \mu_{2r+1} - 1)$, where for $1 \leqslant i \leqslant r$

$$(\mu_{2i-1}^*, \mu_{2i}^*) = \begin{cases} (\mu_{2i-1}, \mu_{2i}), & \text{if } \mu_{2i-1} = \mu_{2i}, \\ (\mu_{2i-1} - 1, \mu_{2i} + 1), & \text{if } \mu_{2i-1} > \mu_{2i}. \end{cases}$$

Set $\underline{\lambda} := \underline{\lambda}^0 \cup \lambda^e$. Note that $\lambda \in \mathbf{P}_{\mathrm{sp}}(\mathbf{C}_n)$.

(ii) **G** is of type $\mathbf{C}_n$.
Let $\lambda \in \mathbf{P}_{\mathrm{sp}}(\mathbf{C}_n)$ with $\lambda = \lambda^0 \cup \lambda^e$. Assume first that $\ell(\lambda^e)$ is even. In this case assume that $\lambda^e =: (\mu_1, \mu_2, \dots, \mu_{2r})$. Set $\underline{\lambda}^e := (\mu_1 + 1, \mu_2^*, \dots, \mu_{2r-2}^*, \mu_{2r-1}^*, \mu_{2r} - 1, 1)$, where for $1 \leqslant i \leqslant r-1$

$$(\mu_{2i}^*, \mu_{2i+1}^*) := \begin{cases} (\mu_{2i}, \mu_{2i+1}), & \text{if } \mu_{2i} = \mu_{2i+1}, \\ (\mu_{2i} - 1, \mu_{2i+1} + 1), & \text{if } \mu_{2i} > \mu_{2i+1}. \end{cases}$$

Set $\underline{\lambda} := \lambda^0 \cup \underline{\lambda}^e$.
Next, assume that $\ell(\lambda^e)$ is odd and that $\lambda^e =: (\mu_1, \dots, \mu_{2r+1})$.
Define $\underline{\lambda}^e := (\mu_1 + 1, \mu_2^*, \dots \mu_{2r+1}^*)$, where for $1 \leqslant i \leqslant r$

$$(\mu_{2i}^*, \mu_{2i+1}^*) := \begin{cases} (\mu_{2i}, \mu_{2i+1}), & \text{if } \mu_{2i} = \mu_{2i+1}, \\ (\mu_{2i} - 1, \mu_{2i+1} + 1), & \text{if } \mu_{2i} > \mu_{2i+1}. \end{cases}$$

In this case set $\underline{\lambda} := \lambda^0 \cup \underline{\lambda}^e$.
In both cases considered above, we have $\underline{\lambda} \in \mathbf{P}_{\mathrm{sp}}(\mathbf{B}_n)$.

(iii) **G** is of type $\mathbf{D}_n$.

⋆The referee requested a clarification of this natural order. Each of the sets $I_*(\lambda)$, $I_*(\hat{\lambda})$, $I_*({}^L\lambda)$ is a subset of $\mathbb{N}$ and hence inherits a total ordering from the standard total ordering on $\mathbb{N}$. Presumably this is what is meant by the natural order.

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Let $\lambda \in \mathbf{P}_{\mathrm{sp}}(\mathbf{D}_n)$ with $\lambda = \lambda^0 \cup \lambda^e$. Then $\ell(\lambda^0)$ is even and we will write $\lambda^0 =: (\mu_1, \ldots, \mu_{2r})$, and define $\underline{\lambda}^0 := (\mu_1 + 1, \mu_2^*, \ldots, \mu_{2r-1}^*, \mu_{2r} - 1)$, where for $1 \leqslant i \leqslant r-1$,

$$(\mu_{2i}^*, \mu_{2i+1}^*) := \begin{cases} (\mu_{2i}, \mu_{2i+1}), & \text{if } \mu_{2i} = \mu_{2i+1}, \\ (\mu_{2i} - 1, \mu_{2i+1} + 1), & \text{if } \mu_{2i} > \mu_{2i+1}. \end{cases}$$

Set $\underline{\lambda} := \underline{\lambda}^0 \cup \lambda^e$.

LEMMA 2.3.3. (i) *If* $\mathbf{G}$ *is of type* $\mathbf{B}$ *or* $\mathbf{C}$, *then* $d(\lambda) = \underline{\lambda}$ *for any special* λ.

(ii) *If* $\mathbf{G}$ *is of type* $\mathbf{B}$, $\mathbf{C}$ *or* $\mathbf{D}$, *then* $D(\lambda) = {}^t\underline{\lambda}$ *for any special* λ.

Here $\underline{\lambda}$ *is the partition defined in the preceding discussion.* □

2.4. ENDOSCOPIC INDUCTION

DEFINITION 2.4.1. Let $\mathbf{G}$ denote a connected reductive algebraic group $\mathbf{G}$ defined over F, and $\mathbf{H}$ an endoscopic group of $\mathbf{G}$. Let O_H be a unipotent orbit in $\mathbf{H}$. By the Springer correspondence, the pair $(O_H, \mathbf{1})$ is associated to an irreducible representation σ of $W(\mathbf{H})$, the Weyl group of $\mathbf{H}$. The Weyl group $W(\hat{\mathbf{H}})$ of the dual group $\hat{\mathbf{H}}$ of $\mathbf{H}$ can be identified with a reflection subgroup of the Weyl group $W(\hat{\mathbf{G}})$ of $\hat{\mathbf{G}}$. On the other hand $W(\mathbf{H})$ and $W(\mathbf{G})$ can be identified with $W(\hat{\mathbf{H}})$ and $W(\hat{\mathbf{G}})$ respectively⋆ up to inner automorphisms. Using truncated induction (cf. [5]), σ gives rise to an irreducible representation ρ of $W(\mathbf{G})$. If ρ corresponds to a pair $(O_G, \mathbf{1})$ for some unipotent orbit O_G in $\mathbf{G}$, then we declare that O_H is in the domain of *endoscopic induction* and that O_G is its image. We then write $O_G = \mathrm{Ind}_{\mathbf{H}}^{\mathbf{G}} O_H$.

Remark 2.4.2. (i) The domain of endoscopic induction contains all special orbits.

(ii) Endoscopic induction was, basically, first introduced by Lusztig in ([9]), who regarded it as a map from the special orbits in the dual group $\hat{\mathbf{H}}$ to orbits in $\mathbf{G}$. Endoscopic induction is the composition of the map defined by Lusztig and the duality map.

(iii) In [1], we proved that (over $\mathbb{C}$) endoscopic induction is the unique map between the set of special orbits in $\mathbf{H}(\mathbb{C})$ and the set of unipotent orbits in $\mathbf{G}(\mathbb{C})$ which produces matching unipotent orbital integrals.

Next, recall (cf. [5]) that the unipotent orbits in $\mathbf{G}$ can also be parameterized using weighted Dynkin diagrams and that a unipotent orbit is said to be *even* if all the weights on the associated diagram are even.

The next lemma is stated as an observation in ([4], p. 105).

⋆As the referee points out, these identifications are only canonical modulo inner automorphisms.

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LEMMA 2.4.3. *Let* $\mathbf{G}$ *be a connected semisimple algebraic group* $/F$. *Let* O_G *be a special unipotent orbit in* $\mathbf{G}$. *Assume that the dual orbit* ${}^LO_G$ *is even. Let* $\mathbf{H}$ *be an endoscopic group of* $\mathbf{G}$, *and* O_H *a special orbit in* $\mathbf{H}$. *Assume further that* $O_G = \mathrm{Ind}_{\mathbf{H}}^{\mathbf{G}} O_H$. *Then* ${}^LO_G \cap \hat{\mathbf{H}} = {}^LO_H$. □

Let $\mathbf{G}$ be type $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{D}$. A direct description* of endoscopic induction in terms of partitions is given as follows. Recall that the endoscopic groups of $\mathbf{G}$ are of the following types:

$\mathbf{G}$	$\mathbf{H}$	
$\mathbf{B}_n$	$\mathbf{B}_k \times \mathbf{B}_{n-k}$,	$0 \leqslant k \leqslant [n/2]$,
$\mathbf{C}_n$	$\mathbf{C}_k \times \mathbf{D}_{n-k}$,	$0 \leqslant k \leqslant n$,
$\mathbf{D}_n$	$\mathbf{D}_k \times \mathbf{D}_{n-k}$,	$0 \leqslant k \leqslant [n/2]$.

LEMMA 2.4.4. *Let* $\mathbf{G}$ *be as above and* $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$ *be an endoscopic group of* $\mathbf{G}$. *Let* O_i *be a special orbit in* $\mathbf{H}_i$, $i = 1, 2$ *and* O_G *a unipotent orbit in* $\mathbf{G}$ *such that* $O_G = \mathrm{Ind}_{\mathbf{H}_1 \times \mathbf{H}_2}^{\mathbf{G}} O_1 \times O_2$. *Then*

$$\lambda(O_G) = \inf_{\mathbf{P}(\mathbf{G})}(\lambda(O_1) + \lambda(O_2)).$$

Proof. In ([13], p. 219) Spaltenstein introduced a set of axioms describing a system of maps $\{j_{\mathbf{H},\mathbf{G}}\}$ defined on the set of special orbits in endoscopic groups (and generalized versions thereof) $\mathbf{H}$ of $\mathbf{G}$. He proved the existence and uniqueness of such systems, and that the recipe given in the statement of the lemma is such a solution for groups of type $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$. On the other hand it is well known that endoscopic induction, as defined in 2.4.1, satisfies all the Spaltenstein axioms.** □

2.5. TRANSFER OF PACKETS

Our aim here is to define the endoscopic transfer of the Packets (see Definition 1.3.4) contained in a stable special orbit in an elliptic endoscopic group of a symplectic or split special orthogonal group. The definition will be introduced in three steps of increasing levels of generality. The main step is the first; each of the next two steps reduces to the preceding step. To be more precise, let $\mathbf{G}$ denote a symplectic or split orthogonal group, and $\mathbf{H}$ an elliptic endoscopic group of $\mathbf{G}$. Let O_H denote a special orbit in $\mathbf{H}$, and set $O_G := \mathrm{Ind}_H^G O_H$ and let λ denote the partition with $O_\lambda := O_G$.

*The description given in Lemma 2.4.4 is incorrect in cases $\mathbf{C}$ and $\mathbf{D}$, as Waldspurger has observed. The error arises from a misunderstanding of Spaltenstein's map $j_{\mathbf{H},\mathbf{G}}$. What follows from Spaltenstein's work is that $\lambda(O_G) = \inf_{\mathbf{P}(\mathbf{G})}(\lambda(O_1) + \underline{\lambda}(O_2))$, with $\underline{\lambda}$ defined as in the discussion preceding Lemma 2.3.3. This is true for $\mathbf{G}$ of all three types ($\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$); note that when $\mathbf{G}$ is of type $\mathbf{C}_n$ the numbering of O_1 and O_2 must be chosen so that O_2 comes from the factor $\mathbf{D}_{n-k}$ of $\mathbf{H}$. It is interesting that when $\mathbf{G}$ is of type $\mathbf{B}_n$ Assem's version, while different from Spaltenstein's, is also correct.

**Section 12.6 in Ch. III of [13] is useful at this point.

Assume that $O_H^{\mathrm{st}} \neq \phi$. (Since $\mathbf{G}$ is split, we then have $O_G^{\mathrm{st}} \neq \phi$). The first step deals with the situation where the set of parts of $\boldsymbol{\lambda}$ is of the special form $\{2, 4, \ldots, 2r\}$ if $\mathbf{G}$ is symplectic, or of the form $\{1, 3, \ldots, 2r+1\}$ if $\mathbf{G}$ is orthogonal. These orbits are all special and enjoy the property that their dual orbits ${}^L O_\lambda$ are even. In the second step we consider the situation where $\boldsymbol{\lambda}$ consists only of even (resp. odd) parts when $\mathbf{G}$ is symplectic (resp. orthogonal). This situation is reduced to the situation handled in the first step via Lemma 2.5.10. In the third step we treat the general case.

Before we proceed we need to introduce a certain set associated to any partition $\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{T}_n)$.

DEFINITION 2.5.1. Let $\boldsymbol{\lambda} \in \mathbf{P}(\mathbf{T}_n)$, $\mathbf{T} \in \{\mathbf{B}, \mathbf{C}, \mathbf{D}\}$ with $\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 \cup \boldsymbol{\lambda}^e$. Set $\boldsymbol{\lambda}^* := \boldsymbol{\lambda}^0$ if $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$ and set $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^e$ if $\mathbf{T} = \mathbf{C}$. The set $S(\boldsymbol{\lambda})$, of *segments associated with* $\boldsymbol{\lambda}$, is defined as follows. Let $\boldsymbol{\lambda}^* =: (\lambda_1^{a_1}, \ldots, \lambda_s^{a_s})$. Then

(i) $S(\boldsymbol{\lambda}) := \{(\lambda_1^{a_1}, \ldots, \lambda_1^{a_k}) : 1 \leqslant k \leqslant s \wedge \sum_{i=1}^k a_i \text{ is odd }\}$ if $\mathbf{T} = \mathbf{B}$,
(ii) $S(\boldsymbol{\lambda}) := \{(\lambda_1^{a_1}, \ldots, \lambda_1^{a_k} : 1 \leqslant k \leqslant s \wedge \sum_{i=1}^k a_i \text{ is even }\}$ if $\mathbf{T} = \mathbf{C}$ or $\mathbf{D}$.

DEFINITION 2.5.2. If $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, then $\#S(\boldsymbol{\lambda}) = \#I_*(\boldsymbol{\lambda}) + 1$, and if $\mathbf{T} = \mathbf{C}$, then $\#S(\boldsymbol{\lambda}) = \#I_*(\boldsymbol{\lambda})$ (recall defn. 1.3.4.). If $\mathbf{T} = \mathbf{B}$ or $\mathbf{D}$, we define $S_*(\boldsymbol{\lambda}) := S(\boldsymbol{\lambda}) - \{(\lambda_1^{a_1})\}$, and if $\mathbf{T} = \mathbf{C}$, we define $S_*(\boldsymbol{\lambda}) := S(\boldsymbol{\lambda})$. Thus $\#S_*(\lambda) = \#I_*(\lambda)$. The sets $S(\boldsymbol{\lambda})$ will be ordered using the natural order on partitions, thus $(\lambda_1^{a_1}, \ldots, \lambda_k^{a_k}) \leqslant (\lambda_1^{a_1}, \ldots, \lambda_\ell^{a_\ell})$ iff $k \leqslant \ell$.

DEFINITION 2.5.3. Let $b_\lambda : S_*(\lambda) \longrightarrow I_*(\lambda)$ denote the unique order reversing bijection, where $I_*(\lambda)$ is equipped with the natural the ordering (as a subset of $\mathbb{N}$).

EXAMPLE 2.5.4. Let $\boldsymbol{\lambda} := (9^1, 8^2, 7^2, 6^4, 5^3, 3^1, 2^2, 1^4)$. Then $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^0 = (9^1, 7^2, 5^3, 3^1, 1^4)$. Note that $\operatorname{rank} Q_1 = 1$, $\operatorname{rank} Q_2 = 3$, $\operatorname{rank} Q_3 = 6$, $\operatorname{rank} Q_4 = 7$. Hence, $I_*(\lambda) = \{1, 2, 4\}$. On other hand, $S_*(\lambda) = \{(9^1, 7^2), (9^1, 7^2, 5^3, 3^1), (9^1, 7^2, 5^3, 3^1, 1^4)\}$.

Now, we proceed to define the transfer of stability packets in the following context:

- $\mathbf{G} =$ symplectic or split special orthogonal group.
- $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$, an elliptic endoscopic group of $\mathbf{G}$.
- $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ two special partitions corresponding to the orbits O_{μ_1}, O_{μ_2} in $\mathbf{H}_1$, $\mathbf{H}_2$ respectively. We set $O_H := O_{\mu_1} \times O_{\mu_2}$, and $O_G := \operatorname{Ind}_H^G O_H$. Let $\boldsymbol{\lambda}$ be the partition with $O_G := O_\lambda$. We further assume that $O_H^{\mathrm{st}} \neq \phi$.

Definition of transfer of packets (Step 1)

Assume that the set of distinct parts of $\boldsymbol{\lambda}$ is of the form $\{1, 3, \ldots, 2r+1\}$ if $\mathbf{G}$ is orthogonal and is of the form $\{2, 4, \ldots, 2r\}$ if $\mathbf{G}$ is symplectic. Then O_λ is necessarily special and its dual ${}^L O_\lambda$ is even, as can be easily checked using Lemma 2.3.3. Since $\mathbf{H}$ is necessarily quasi-split, it splits over some quadratic extension of F. Let $\tau \in \{1, \varepsilon, \pi, \varepsilon\pi\}$ such that $F(\sqrt{\tau})/F$ is the minimal extension over which $\mathbf{H}$ is split. Let $\prod_i = \prod_i(\boldsymbol{\mu}_i, \varphi_i) \subset O_{\mu_i}^{\mathrm{st}}$, denote the packets associated to $\varphi_i : I_*(\boldsymbol{\mu}_i) \longrightarrow$

$F^\times/(F^\times)^2$, $i = 1, 2$ (see Definition 1.3.4.). We shall use duality (see Definition 2.3.2.) to reduce the problem of 'transferring' $\prod_H := \prod_1 \times \prod_2$ into that of transferring the dual packet ${}^L\prod_H := {}^L\prod_1 \times {}^L\prod_2 \subseteq O^{\mathrm{st}}_{{}^L\mu_1} \times O^{\mathrm{st}}_{{}^L\mu_2} \subseteq \hat{\mathbf{H}}_1(F) \times \hat{\mathbf{H}}_2(F)$. At this point, and before we proceed, we have to be clear about the meaning of $\hat{\mathbf{H}}_i(F)$, $i = 1, 2$. We do regard $\hat{\mathbf{H}}_i$ as a group defined over F which splits over the same field as does $\mathbf{H}_i$. Now, if $\overline{A}(\lambda) = \langle 1 \rangle$, then O^{st}_λ contains only one packet, namely O^{st}_λ itself, in which case we *always* define the transfer of $\prod_H$ to be O^{st}_λ. Thus, we shall *assume* that $\overline{A}(\lambda) \neq \langle 1 \rangle$. This assumption is equivalent to the condition $S_*({}^L\lambda) \neq \phi$.

Consider now the following eight (mutually exclusive) conditions that may be satisfied by an element $z \in S_*({}^L\boldsymbol{\lambda})$:

(i) $\exists x \in S_*({}^L\boldsymbol{\mu}_1)$ such that $z = x$,
(ii) $\exists y \in S_*({}^L\boldsymbol{\mu}_2)$ such that $z = y$,
(iii) $\exists (x, y) \in S_*({}^L\boldsymbol{\mu}_1) \times S_*({}^L\boldsymbol{\mu}_2)$ such that $z = x \cup y$,
(iv) $\exists x \in S_*({}^L\boldsymbol{\mu}_1) \wedge \exists y \in S({}^L\boldsymbol{\mu}_2) \backslash S_*({}^L\boldsymbol{\mu}_2)$ such that $[z = x \cup y \wedge$ every part of y is smaller than every of $x]$,
(v) $\exists x \in S_*({}^L\boldsymbol{\mu}_1) \wedge \exists y \in S({}^L\boldsymbol{\mu}_2) \backslash S_*({}^L\mu_2)$ such that $[z = x \cup y \wedge$ some part of y is larger than some part of $x]$,
(vi) $\exists x \in S({}^L\boldsymbol{\mu}_1) \backslash S_*({}^L\boldsymbol{\mu}_2) \wedge \exists y \in S_*({}^L\mu_2)$ such that $[z = x \cup y \wedge$ every part of x is smaller than every part of $y]$,
(vii) $\exists x \in S({}^L\boldsymbol{\mu}_1) \backslash S_*({}^L\boldsymbol{\mu}_1) \wedge \exists y \in S_*({}^L\mu_2)$ such that $[z = x \cup y \wedge$ some part of x is larger than some part of $y]$.
(viii) $\exists (x, y) \in \left(S({}^L\boldsymbol{\mu}_1) \backslash S_*({}^L\boldsymbol{\mu}_1)\right) \times \left(S({}^L\boldsymbol{\mu}_2) \backslash S_*({}^L\boldsymbol{\mu}_2)\right)$ such that $z = x \cup y$.

Remark 2.5.5. If $\mathbf{G}$ is odd orthogonal, then $S({}^L\boldsymbol{\mu}_i) = S_*({}^L\boldsymbol{\mu}_i)$, $i = 1, 2$, hence the last five conditions are vacuous.

Now define

$$S_*({}^L\boldsymbol{\lambda}, {}^L\boldsymbol{\mu}_1, {}^L\boldsymbol{\mu}_2) := \{z \in S_*({}^L\boldsymbol{\mu}) : \ z \text{ satisfies one of the conditions (i)-(viii) }\}.$$

For $i = 1, 2$, denote the θ_i the composition of the following maps

$$S_*({}^L\boldsymbol{\mu}_i) \xrightarrow{b_{{}^L\mu_i}} I_*({}^L\boldsymbol{\mu}_i) \xrightarrow{(\iota_{{}^L\mu_i})^{-1}} I_*(\boldsymbol{\mu}_i) \xrightarrow{\varphi_i} F^\times/(F^\times)^2 ,$$

and define $\theta : S_*({}^L\boldsymbol{\lambda}, {}^L\boldsymbol{\mu}_1, {}^L\boldsymbol{\mu}_2) \longrightarrow F^\times/(F^\times)^2$ by

$$\theta(z) := \begin{cases} \theta_1(x) & , \ \text{if (i)} \\ \theta_2(y) & , \ \text{if (ii)} \\ \theta_1(x)\theta_2(y) & , \ \text{if (iii)} \\ 1 \bmod (F^\times)^2 & , \ \text{if (iv)} \\ \theta_1(x) & , \ \text{if (v)} \\ 1 \bmod (F^\times)^2 & , \ \text{if (vi)} \\ \theta_2(y) & , \ \text{if (vii)} \\ 1 \bmod (F^\times)^2 & , \ \text{if (viii)} \end{cases} .$$

Set $\theta_\tau(z) := \tau \cdot \theta(z)$ (recall that E_τ is the minimal extension of F over which H splits).

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DEFINITION 2.5.6. Under the above given assumption on $\boldsymbol{\lambda}$, we define the transfer of the packet $\prod_1(\boldsymbol{\mu}_1, \varphi_1) \times \prod_2(\boldsymbol{\mu}_2, \varphi_2)$, denoted by $\mathrm{Tran}_H^G \prod_1(\boldsymbol{\mu}_1, \varphi_1) \times \prod_2(\boldsymbol{\mu}_2, \varphi_2)$ to be the union $\coprod_{\overline{\theta}_\tau} \prod(\boldsymbol{\lambda}, \overline{\theta}_\tau \circ b_{L_\lambda}^{-1} \circ \boldsymbol{\iota}_{L_\lambda})$, where the union is taken over all $\overline{\theta}_\tau : S_*({}^L\boldsymbol{\lambda}) \to F^\times/(F^\times)^2$ which extend θ_τ.

Definition of transfer of packets (Step 2)

Now, assume that $\boldsymbol{\lambda} = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r})$ contains only odd parts if $\mathbf{G}$ is orthogonal, or contains only even parts if $\mathbf{G}$ is symplectic. This situation can be reduced to the one discussed in step 1 via descent as will be discussed below. First we recall the definition of induction of rational orbits from a Levi subalgebra.

DEFINITION 2.5.7. Let $\mathfrak{g}$ denote a reductive Lie algebra, and let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a Levi decomposition of some parabolic subalgebra. Let O denote a nilpotent orbit in $\mathfrak{m}(F)$. Define $\mathrm{Ind}_m^{\mathfrak{g}} O$ to bet the set of all nilpotent orbits in $\mathfrak{g}(F)$ which intersect $O + \mathfrak{n}(F)$ in an open set.

Next, let $\boldsymbol{\lambda}$ be as given above. Define the Levi subgroup $\mathbf{M}_\lambda$ by $\mathbf{M}_\lambda := \mathbf{GL}_\lambda \times \mathbf{G}'_\lambda$, where

$$\mathbf{GL}_\lambda := \prod_{i=1}^{r-1} [\mathbf{GL}(a_1 + \cdots + a_i)]^{\lambda_i - \lambda_{i+1} - 2/2} \ ,$$

and $\mathbf{G}'_\lambda :=$ unique group of the same classical type as $\mathbf{G}$ such that $2\,\mathrm{rank}\,\mathbf{G}'_\lambda + \sum_{i=1}^{r-1}(a_1 + \cdots + a_i)(\lambda_i - \lambda_{i+1} - 2) = 2\,\mathrm{rank}\,\mathbf{G}$.

Let $\boldsymbol{\lambda}'$ denote the partition obtained from $\boldsymbol{\lambda}$ by replacing each λ_{r-i} by either $2i+2$ if $\mathbf{G}$ is symplectic or by $2i+1$ if $\mathbf{G}$ is orthogonal, where $0 \leqslant i \leqslant r-1$. Note then $\boldsymbol{\lambda}'$ satisfies the conditions of step1 and corresponds to an orbit $O_{\lambda'}$ in $\mathbf{G}'$. Moreover, we have

LEMMA 2.5.8. *There exists a one-to-one correspondence $O' \to O$ between $O_{\lambda'}^{\mathrm{st}} \subset \mathbf{G}'_\lambda(F)$ and $O_\lambda^{\mathrm{st}} \subset \mathbf{G}(F)$ given by*

$$O = \mathrm{Ind}_{M_\lambda}^G(\mathbf{1}, O')\,,$$

where, here 1 *denotes the trivial orbit in* $\mathbf{GL}_\lambda(F)$.

Proof. This follows easily from comparing the Prehomogeneous spaces associated to O_λ and $O_{\lambda'}$ and then applying the definition of induction. For more details, see the argument in Lemma 1.3.1. in [2]. □

COROLLARY 2.5.9. The correspondence established in Lemma 2.5.8. gives rise to a 1-1 correspondence between the packets within $O_{\lambda'}^{\mathrm{st}}$ and those within O_λ^{st}.

Proof. Clear. □

LEMMA 2.5.10. *For $i = 1, 2$, there exists a Levi subgroup $\mathbf{M}_{\mu_i} \subset \mathbf{H}_i$ of the form $\mathbf{M}_{\mu_i} = \mathbf{GL}_{\mu_i} \times \mathbf{H}'_{\mu_i}$ and partition $\boldsymbol{\mu}'_i$ corresponding to an orbit $O_{\mu'_i}$ in $\mathbf{H}_i$ such that*

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(i) $\mathbf{H}'_{\mu_1} \times \mathbf{H}'_{\mu_2}$ *is an elliptic endoscopic group of* $\mathbf{G}'_\lambda$ *which splits over the same extension* E_τ/F *as does* $\mathbf{H}$.
(ii) $\mathbf{GL}_{\mu_1} \times \mathbf{GL}_{\mu_2} \cong \mathbf{GL}_\lambda$.
(iii) $O_{\lambda'} = \mathrm{Ind}^{G'_\lambda}_{H'_{\mu_1} \times H'_{\mu_2}} (O_{\mu'_1}, O_{\mu'_2})$.
(iv) *The map* $U'_i \mapsto U_i$ *between* $O^{\mathrm{st}}_{\mu'_i}$ *and* $O^{\mathrm{st}}_{\mu_i}$ *given by*

$$U_i := \mathrm{Ind}^{H'_{\mu_i}}_{M_{\mu_i}} (\mathbf{1}, U'_i)$$

is a one-to-one correspondence which preserves packets. Here, again $\mathbf{1}$ *is the appropriate trivial orbit (a convention which we shall adhere to).*

Proof. Since $O_\lambda = \mathrm{Ind}^G_{H_1 \times H_2} O_{\mu_1} \times O_\mu$, we have $\boldsymbol{\lambda} = \inf_{\mathbf{P}(\mathbf{G})}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)$. The proof then consists of writing down the parts of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, then using the above relation and the definition of $\boldsymbol{\lambda}'$. The details are straightforward. □

Thus we have the following situation

$$\begin{tikzcd}
(\mathbf{1}, O_{\lambda'}) \subseteq \mathbf{GL}_\lambda \times \mathbf{G}'_\lambda = \mathbf{M}_\lambda \arrow[r, hook] & O_\lambda \subseteq \mathbf{G} \\
(\mathbf{1}, \mathbf{1}, O_{\mu'_1}, O_{\mu'_2}) \subseteq \mathbf{GL}_{\mu_1} \times \mathbf{GL}_{\mu_2} \times \mathbf{H}'_{\mu_1} \times \mathbf{H}'_{\mu_2} = \mathbf{M}_{\mu_1} \times \mathbf{M}_{\mu_2} \arrow[r, hook] \arrow[u, "\mathrm{Tran}^{M_\lambda}_{M_{\mu_1} \times M_{\mu_2}}"] & (O_{\mu_1}, O_{\mu_2}) \subseteq \mathbf{H}_1 \times \mathbf{H}_2, \arrow[u, "\mathrm{Tran}^G_{H_1 \times H_2}"']
\end{tikzcd}$$

where the study of the right vertical arrow can be reduced to the study of the left vertical arrow, which then reduces to the study of the transfer of the packets in $O^{\mathrm{st}}_{\mu'_1} \times O^{\mathrm{st}}_{\mu'_2}$ to $O^{\mathrm{st}}_{\lambda}$. This observation leads to the following definition:

DEFINITION 2.5.11. Let $\boldsymbol{\lambda}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2$ be as above, and let $\prod_i \subseteq O^{\mathrm{st}}(\boldsymbol{\mu}_i)$, $i = 1, 2$, be two given packets. Let $\prod'_i \subseteq O^{\mathrm{st}}(\boldsymbol{\mu}'_i)$ denote the packets corresponding to μ'_i as assured by Lemma 2.5.10. Define the transfer of $\prod_1 \times \prod_2$ to O^{st}_λ, denoted by $\mathrm{Tran}^G_H \prod_1 \times \prod_2$, to be $\mathrm{Ind}^G_{M_\lambda}\left[\mathrm{Tran}^{M_\lambda}_{M_{\mu_1} \times M_{\mu_2}}\left((\{\mathbf{1}\} \times \prod'_1) \times (\{\mathbf{1}\} \times \prod'_2)\right)\right]$.

Definition of transfer of packets (Step 3)

Let $\boldsymbol{\lambda}$ denote *any* partition in $\mathbf{P}(\mathbf{T}_n)$, and let $\boldsymbol{\lambda}^*$ be as usual (see Notation 1.3.1). Then, of course, $I_*(\boldsymbol{\lambda}) = I_*(\boldsymbol{\lambda}^*)$, hence there exists a natural one-to-one correspondence between the packets within O^{st}_λ and those within $O^{\mathrm{st}}_{\lambda^*}$. Let $\mathbf{T}_{n^*}$ denote the group containing O_{λ^*}. The following lemma is not difficult to prove.

LEMMA 2.5.11. *Let* $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2$, *be the two special partitions with corresponding orbits* $O_{\mu_1} \subseteq \mathbf{H}_1$, $O_{\mu_2} \subseteq \mathbf{H}_2$ *such that* $O_G = \mathrm{Ind}^G_{H_1 \times H_2}(O_{\mu_1}, O_{\mu_2})$. *Then there exists a pair* $({}^*\boldsymbol{\mu}_1, {}^*\boldsymbol{\mu}_2)$ *of special partitions such that*

(i) $\#I_*({}^*\mu_i) = \#I_*(\mu_i)$, for $i = 1, 2$,
(ii) $\inf_{\mathbf{P}(\mathbf{T}_{n^*})}({}^*\mu_1 + {}^*\mu_2) = \lambda^*$. □

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By (i) of Lemma 2.5.11, there exists a natural bijection, $\prod \mapsto^* \prod$, between the packets within $O^{\rm st}_{\mu_i}$ and those within $O^{\rm st}_{\mu_i}$, induced by the order preserving bijection between $I_*(^*\boldsymbol{\mu}_i)$ and $I_*(\boldsymbol{\mu}_i)$, $i = 1, 2$. As noted above, we also have a natural bijection between the packets within $O^{\rm st}_{\lambda}$ and those within $O^{\rm st}_{\lambda^*}$.

DEFINITION 2.5.12. Let $\prod_i \subseteq O^{\rm st}_{\mu_i}$, $i = 1, 2$, be two given packets. Define the transfer of $\prod_1 \times \prod_2$, denoted $\mathrm{Tran}^G_H \prod_1 \times \prod_2$, to be the union of all packets within $O^{\rm st}_{\lambda}$ which correspond (under the natural bijection discussed above) to the packets within $O^{\rm st}_{\lambda^*}$ obtained by transferring $^*\prod_1 \times ^*\prod_2$ to $O^{\rm st}_{\lambda^*}$ according to the definition of transfer in step 2.

3. A Transfer Calculation

3.1. NOTATION AND SOME UNIPOTENT ORBITS

Let $n \geqslant 1$ be an integer. Consider the following partitions $\lambda(n; k)$ of $2n + 1$:

$$\lambda(n; k) := \begin{cases} \left(n - 2k, n - 2k, 1^{4k+1}\right), & \text{if } 0 \leqslant k \leqslant \frac{n-1}{2},\ n \text{ odd}, \\ \left(n - 2k - 1, n - 2k - 1, 1^{4k+3}\right), & \text{if } 0 \leqslant k \leqslant \frac{n-2}{2},\ n \text{ even}\ . \end{cases}$$

The unipotent orbit in $\mathbf{SO}(2n + 1)$ corresponding to $\lambda(n; k)$ will be denoted, in this section, by $O(n; k)$ instead of $O_{\lambda(n;k)}$. Note that $\lambda(n; k)$ corresponds to the trivial orbit when $k = (n - 1)/2$, n odd; or when $k = (n - 2)/2$, n even. Next, we discuss some basic properties of these orbits.

LEMMA 3.1.1. *Assume that* $0 \leqslant k \leqslant (n - 1)/2$ *for n odd, and* $0 \leqslant k < (n - 2)/2$ *for n even,* $n \geqslant 4$. *Then*

(i) $A(\lambda(n, k)) = \mathbb{Z}/2\mathbb{Z}$, $\overline{A}(\lambda(n, k)) = \langle 1 \rangle$

(ii) $O(n; k)$ *is a Richardson orbit, induced from the trivial orbit in* $[\mathbf{GL}(2)]^{(n-2k-1)/2} \times \mathbf{SO}(4k + 3)$, *if n is odd, and is induced from the trivial orbit in* $[\mathbf{GL}(2)]^{(n-2k-2)/2} \times \mathbf{SO}(4k + 5)$ *is n is even.*

Proof. Clear. □

Let $(\mathbf{M}(n; k), \mathfrak{g}_2(n; k))$ denote PVS associated with $\lambda(n; k)$. Then we have

$$\mathbf{M}_{n,k} \cong \begin{cases} [\mathbf{GL}(2)]^{(n-2k-1)/2} \times \mathbf{SO}(4k + 3)\ , & 0 \leqslant k < \frac{n-1}{2}, n \text{ odd}, \\ [\mathbf{GL}(2)]^{(n-2k-2)/2} \times \mathbf{SO}(4k + 5)\ , & 0 \leqslant k < \frac{n-2}{2}, n \text{ even} \end{cases}$$

and

$$\mathfrak{g}_2(n, k) \cong \begin{cases} [\mathbf{Mat}\,(2, 4k + 3)\ , & 0 \leqslant k < \frac{n-1}{2}, n \text{ odd}, \\ [\mathbf{Mat}\,(2, 4k + 5)\ , & 0 \leqslant k < \frac{n-2}{2}, n \text{ even} \end{cases}$$

$\mathbf{M}_{n,k}$ acts on $\mathfrak{g}_2(n, k)$ by $g_1, \ldots, g_\ell, h) \cdot X := g_\ell X^t h$, where $g_i \in \mathbf{GL}(2)$, $1 \leqslant i \leqslant \ell$,

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$h \in \mathbf{SO}(2m+1)$, and $X \in \mathbf{Mat}(2, m)$. Here we used ℓ, m for the appropriate integers given in the descriptions of $\mathbf{M}_{n,k}$ and $\mathfrak{g}_2(n,k)$ above.

Assume now that

$$k \neq \begin{cases} \frac{n-1}{2}, & n \text{ odd} \\ \frac{n-2}{2}, & n \text{ even} \end{cases}$$

i.e. $O(n,k)$ is not the trivial orbit. Then $X \in \mathfrak{g}_2(n,k)$ is a generic point iff $\det(XJ_{4k+3+\varepsilon}{}^tX) \neq 0$. Here, J_p is the form used to define $\mathbf{SO}(p)$, and

$$\varepsilon = \begin{cases} 0, & \text{if } n \text{ odd} \\ 1, & \text{if } n \text{ even} \end{cases}$$

Thus to each generic point $\in \mathfrak{g}_2(n;k)(F)$, there is an F-rank 2 quadratic form attached to it, namely, the quadratic form determined by the 2×2 symmetric matrix $XJ_{4k+3+\varepsilon}{}^tX$. The next Lemma is then obvious.

LEMMA 3.1.2. *Fix an orbit $O(n;k)$ as above.*

(i) *If $k = 0$ and n is odd, then $O^{\mathrm{st}}(n,k)$ splits into four $\mathbf{SO}(2n+1, F)$-conjugacy classes. Moreover, if $X_1, X_2 \in \mathfrak{g}_2(n,k)(F)$ are generic, then X_1 is conjugate to X_2 under $\mathbf{SO}(2n+1, F)$ iff $\det(X_1J_{4k+3+\varepsilon}{}^tX_1) \equiv \det(X_2J_{4k+3}{}^tX_2) \bmod (F^\times)^2$.*
(ii) *If otherwise, then $O^{\mathrm{st}}(n,k)$ splits into seven $\mathbf{SO}(2n+1, F)$-conjugacy classes. Moreover, if $X_1, X_2 \in \mathfrak{g}_2(n,k)(F)$ are generic, then X_1 is conjugate to X_2 under $\mathbf{SO}(2n+1, F)$ iff the quadratic forms determined by X_1 and X_2 are equivalent.*

NOTATION 3.1.3. We shall label the F-rational orbits in $O^{\mathrm{st}}(n,k)$ as follows. If $k = 0$, and n odd, then for each $\tau \in \{1, \epsilon, \pi, \epsilon\pi\}$, we let $O_\tau(n)$ denote the rational orbit containing a generic point $X \in \mathfrak{g}_2(n,k)(F)$ satisfying: $\det(XJ_{4k+3}{}^tX) \equiv \tau \bmod (F^\times)^2$. If $k > 0$, $\tau \in \{\epsilon, \pi, \epsilon\pi\}$, and $\eta \in \{\pm 1\}$, then we let $O_{\tau,\eta}(n,k)$ denote the rational orbit containing a generic $X \in \mathfrak{g}_2(n,k)(F)$ such that the quadratic form corresponding to $XJ_{4k+3+\varepsilon}$ has discriminant τ and Hasse-invariant η; the orbit corresponding to $(F^\times)^2$ will be denoted by $O_1(n;k)$.

We shall be interested in the stable orbits $O^{\mathrm{st}}(n;k_0)$, where $k_0 := (n-3)/2$ if n odd $\geqslant 3$, and $k_0 := (n-4)/2$ if n even $\geqslant 4$, in other words, we are dealing with the partition 331^{2n-5}, $n \geqslant 3$.

Next, we review some facts about the sub-regular orbits in $\mathbf{SO}(5, F)$. The PVS associated with the subregular orbit by $\mathbf{SO}(5)$ is given by the pair $(\mathbf{GL}(1) \times \mathbf{SO}(3), \mathbf{Mat}(1,3))$, where the action is given by: $(g,h) \cdot X := gX^tg$, $g \in \mathbf{GL}(1)$, $h \in \mathbf{SO}(3)$, $X \in \mathbf{Mat}(1,3)$. Let $X = [x,y,z] \in \mathbf{Mat}(1,3)(F)$. The sub-regular orbits in $\mathbf{SO}(5,F)$ are then in one-to-one correspondence with the square classes of the relative invariant $\Delta(x,y,z) := 2xy - z^2$. We shall denote the stable

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subregular orbit in $O^{\mathrm{st}}_{\mathrm{sub}}$. The subregular orbit defined by the condition: $\Delta \equiv \tau \bmod (F^{\times})^2$, $\tau \in \{1, \epsilon, \pi, \epsilon\pi\}$, will be denoted by $O_{\mathrm{sub}}(\tau)$.

The next lemma will be needed.

LEMMA 3.1.4.

(i) $\mathrm{Ind}^{\mathrm{SO}(5)}_{\mathrm{GL}(1)\times \mathrm{SO}(3)} \mathbf{1} = O^{\mathrm{st}}_{\mathrm{sub}}$.
(ii) $\mathrm{Ind}^{\mathrm{SO}(5)}_{\mathrm{GL}(2)} \mathbf{1} = O_{\mathrm{sub}}(1)$.
(iii) *Let $O_{\min}$ denote the (unique) F-rational orbit in $\mathbf{SO}(2n-1, F)$ with corresponding partition* $2\,2\,1^{2n-5}$. *Then* $\mathrm{Ind}^{\mathrm{SO}(2n+1)}_{\mathrm{GL}(1)\times \mathrm{SO}(2n-1)}(\mathbf{1}, O_{\min}) = O_1(n, k_0)$. □

Next, we introduce more notation. Let $n \geqslant 1$ be an integer. Let $\mathbf{G}_n = \mathbf{G} := \mathbf{SO}(2n+1)$. The identity connected component of the Langlands dual group is

$$\hat{\mathbf{G}}_n = \hat{\mathbf{G}} = \mathbf{Sp}(2n, \mathbb{C}) = \{g \in \mathbf{SL}(2n, \mathbb{C}) : {}^t g J'_n g = J'_n\},$$

where

$$J'_n := \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}, \quad I_n := n \times n \text{ identity matrix.}$$

Then $\hat{\mathbf{T}}_n := \{\mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) : t_i \in \mathbb{C}^{\times}, 1 \leqslant i \leqslant n\}$ is a maximal torus of $\hat{\mathbf{G}}$.

Let $K_{G_n} = K_G := \mathbf{G}(O_F)$, a hyperspecial maximal compact subgroup of $\mathbf{G}(F)$. Let $\mathcal{H}(G, K_G)$ denote the corresponding spherical Hecke algebra, i.e. the convolution algebra consisting of all complex valued, compactly supported, and K_G-bi-invariant functions on $\mathbf{G}(F)$. Let $W(B_n) = W :=$ Weyl group of $\mathbf{G}$. Thus $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where S_n is the symmetric group on n letters. It is known that the dominant integral weights of $\hat{\mathbf{G}}_n$ can be indexed by the set $P_n^{++} = \{\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n : m_1 \geqslant m_2 \geqslant \cdots \geqslant 0\}$. Let $\mathbf{m} = (m_1, \ldots, m_n) \in P_n^{++}$. Set $f_{\mathbf{m}} :=$ characteristic function of the double coset

$$K_G \mathrm{diag}(1, \pi^{m_1}, \ldots, \pi^{m_n}, \pi^{-m_1}, \ldots, \pi^{-m_n}) K_G .$$

Then $\{f_{\mathbf{m}} : \mathbf{m} \in P_n^{++}\}$ is a $\mathbb{C}$-basis for $\mathcal{H}(G, K_G)$. The Hecke algebra $\mathcal{H}(G, K_G)$ is isomorphic to $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]^W$, via the *Satake transform*. If $f \in \mathcal{H}(G, K_G)$, then $\check{f}$ will denote the Satake transform of f.

3.2. SOME ORBITAL INTEGRAL CALCULATIONS

Next we start by recalling some results of Igusa. Let $m \geqslant r \geqslant 1$ be integers. Let $\mathbf{X} = \mathbf{M}(r, m) := \mathbf{Mat}(r, m)$. For $x \in \mathbf{X}$, we denote by $\pi_{i_1 \ldots i_r}(x)$ the determinant of the $r \times r$ matrix with the i_1-th, …, i_r-th column of x as its 1-st, …, r-th columns. Following Igusa (cf. [7], page 220), we let i_X denote the morphism from $\mathbf{X}$ to

$\mathbb{A}^p$, where $p = \binom{m}{r}$ is defined by

$$i_X(x) := \left(\pi_{i_1,\dots,i_r}(x)\right)_{1 \leqslant i_1 < \cdots < i_r \leqslant m}, \quad x \in \mathbf{X} ,$$

and set

$$\mathbf{X}' := \mathbf{X} - i_X^{-1}(0), \qquad I(\mathbf{X}) := i_X(\mathbf{X}), \qquad I(\mathbf{X})' := i_X(\mathbf{X}')$$

$\mathbf{GL}(m)$ acts naturally on $\mathbf{X}$ and $\mathbb{A}^m$. The latter action defines one on the space $\Lambda^r(\mathbb{A}^m)$ of alternating forms of rank r, which in turn induces an action of $\mathbf{GL}(m)$ on $I(\mathbf{X})$. Note that the actions of $\mathbf{GL}(m)$ on $\mathbf{X}'$ and $I(\mathbf{X})'$ are equivariant relative to i_X. Moreover, the action of $\mathbf{GL}(m)$ on $I(\mathbf{X})'$ is transitive.

LEMMA 3.2.1. *There is a volume form di on $I(\mathbf{X})'$ satisfying $d(g \cdot i) = (\det g)^r \cdot di$, which defines a measure on $I(\mathbf{X})'(F)$, denoted also by di, which can be normalized so that for any continuous function ϕ on $I(\mathbf{X})(O_F)$ and for $U := I(\mathbf{X})(O_F) - \pi I(\mathbf{X})(O_F)$, we have*

$$\int_{\mathbf{X}(O_F)} \phi(i_X(x))\mathrm{d}x = \prod_{2 \leqslant i \leqslant r} (1 - q^{-i}) \cdot \sum_{j_1,\dots,j_r \geqslant 0} \times \left(\prod_{1 \leqslant k \leqslant r} q^{-(m-k+1)j_k} \right) \cdot \int_U \phi(\pi^{j_1+\dots+j_r} \cdot i)\mathrm{d}i \, .$$

Here, the measure $\mathrm{d}x$ *on* $\mathbf{X}(F)$ *is normalized so that* $\mathrm{vol}(\mathbf{X}(O_F), \mathrm{d}x) = 1$.

Proof. This is Lemma 8 in [7]. □

Next, consider the prehomogenous vector space $(\mathbf{GL}(r) \times \mathbf{SO}(m), \mathbf{M}(r, m))$, where the action is given by $(g, h) \cdot x = gx^t h, \quad x \in \mathbf{M}(r, m)$. Recall, from Section 3.1 that the fundamental relative invariant, f, is given by $f(x) = \det(xJ^t x), \quad x \in \mathbf{M}(r, m)$. Here J is the form used to define $\mathbf{SO}(m)$. In this section, we shall be interested only in the case where $r = 2$; $m \geqslant 3$ and odd. The measure $\mathrm{d}x$ on $\mathbf{M}(2, m)(F)$ is normalized as in Lemma 3.2.1.

LEMMA 3.2.2. *For $s \in \mathbb{C}$, $\mathrm{Re}(s) \geqslant 0$, and $t := q^{-s}$, we have*

$$\int_{\mathbf{M}(2,m)(O_F)} |f(x)|^s \mathrm{d}x = \frac{(1 - q^{-1})(1 - q^{-3}t)(1 - q^{-m+1})}{(1 - q^{-1}t)(1 - q^{-3}t^2)(1 - q^{-m+1}t^2)}$$

Proof. This is a special case of the formula given in ([6], page 236). □

Define $\Omega := \{x \in \mathbf{M}(2, m)(O_F) : \min_{1 \leqslant i_1 < i_2 \leqslant m}(\mathrm{val}(\pi_{i_1,i_2}(x))) = 0\}$.

LEMMA 3.2.3. *For* $s \in \mathbb{C}$, $\mathrm{Re}(s) \geqslant 0$, *and* $t := q^{-s}$, *we have*

$$\int_\Omega |f(x)|^s \mathrm{d}x = \frac{(1-q^{-1})(1-q^{-3}t)(1-q^{-m+1})(1-q^{-m}t^2)}{(1-q^{-1}t)(1-q^{-3}t^2)} .$$

Proof. First note that $f(x) = \varphi(i_X(x))$, $\quad x \in \mathbf{M}(2,m)(F)$, where φ is a quadratic homogeneous polynomial in the $\binom{m}{2}$ variables: $\pi_{i_1,i_2}(x)$, $1 \leqslant i_1 < i_2 \leqslant m$. Next, note that $\boldsymbol{\Omega}$ is equal to the $\mathbf{GL}_m(O_F)$-orbit of the matrix $(a_{ij}) \in \mathbf{M}(2,m)(F)$, where $a_{11} = a_{22} = 1$, and $a_{ij} = 0$ otherwise. The arguments given in ([7], p. 225), show then that

$$\int_\Omega |f(x)|^s \mathrm{d}x = (1-q^{-2}) \int_U |\varphi(i)|^s \mathrm{d}i . \tag{$*$}$$

Using the formula given by Lemma 3.2.1., and the homogeneity of φ, we get

$$\begin{aligned}\int_{\mathbf{M}(2,m)(O_F)} |f(x)|^s \mathrm{d}x &= (1-q^{-2}) \sum_{j_1,j_2 \geqslant 0} q^{-mj_2} \cdot q^{(-m+1)j_2} \cdot \int_U |\varphi(\pi^{j_1+j_2} i)|^s \mathrm{d}i \\ &= (1-q^{-2}) \sum_{j_1,j_2 \geqslant 0} (q^{-m}t^2)^{j_1} \cdot (q^{-m+1}t^2)^{j_2} \int_U |\varphi(i)|^s \mathrm{d}i \qquad (**) \\ &= \frac{(1-q^{-2})}{(1-q^{-m}t^2)(1-q^{-m+1}t^2)} \int_U |\varphi(i)|^s \mathrm{d}i\end{aligned}$$

Combining $(*)$ and $(**)$, we get

$$\int_\Omega |f(x)|^s \mathrm{d}x = (1-q^{-m}t^2)(1-q^{-m+1}t^2) \cdot \int_{\mathbf{M}(2,m)(O_F)} |f(x)|^s \mathrm{d}x .$$

The desired result follows now from Lemma 3.2.2. □

LEMMA 3.2.4. *Let* $\chi : F^\times/(F^\times)^2 \to \mathbb{C}^\times$ *denote the character defined by:* $\chi(\tau) := (-1)^{\mathrm{val}(\tau)}$, $\tau \in F^\times/(F^\times)^2$. *Then for* $\mathrm{Re}(s) \geqslant 0$

(i)

$$\int_{\mathbf{M}(2,m)(O_F)} |f(x)|^s \cdot \chi(f(x)) \mathrm{d}x = \frac{(1-q^{-1})(1+q^{-3}t)(1-q^{-m+1})}{(1+q^{-1}t)(1-q^{-3}t^2)(1-q^{-m+1}t^2)}$$

(ii)

$$\int_\Omega |f(x)|^s \cdot \chi(f(x)) \mathrm{d}x = \frac{(1-q^{-1})(1+q^{-3}t)(1-q^{-m+1})(1-q^{-m}t^2)}{(1+q^{-1}t)(1-q^{-3}t^2)}$$

Proof. Let Y denote any nonempty compact open subset of $\mathbf{M}(2,m)(O_F)$. Then

$$\int_Y |f(x)|^s \mathrm{d}x = \sum_{n=0}^{\infty} t^n \cdot \mathrm{vol}(\{x \in Y : |f(x)| = q^{-n}\}, \mathrm{d}x), \tag{$*$}$$

https://doi.org/10.1023/A:1001839511409 Published online by Cambridge University Press

while

$$\int_Y |f(x)|^s \cdot \chi(f(x))\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n t^n \cdot \mathrm{vol}(\{x \in Y : |f(x)| = q^{-n}\}, \mathrm{d}x) \tag{$**$}$$

Thus $(**)$ is obtained from $(*)$ by changing t to $-t$ (which amounts to changing s to $s - (i\pi/\ln q)$. Our result follows now from Lemma 2.2. and 2.3. upon specializing Y to $\mathbf{M}(2, m)(O_F)$ and Ω, respectively. $\square$

We are interested in integrals of certain spherical functions over the rational orbits contained within the stable unipotent orbits $O^{\mathrm{st}}(n; k_0)$. Recall, from Section 3.1, that $O^{\mathrm{st}}(n; k_0)$ is a union of 4 orbits: $O_\tau(n; k_0)$, $\tau \in F^\times/(F^\times)^2$, if $n = 3$; and is a union of 7 orbits: $O_1(n; k_0)$, $O_{\tau,\eta}(n; k_0)$, if $n \geqslant 4$. One easily checks that for $i > 0$, we have $\mathfrak{g}_i \neq (0) \Leftrightarrow i = 2$ or 4. Moreover, $\dim \mathfrak{g}_2 = 4n - 6$, and $\mathfrak{g}_4 = \langle E_{2,n+3} - E_{3,n+2}\rangle$. A general element $X \in \mathfrak{g}_2 \oplus \mathfrak{g}_4$ will be represented in matrix form as following:

$$X = \left[\begin{array}{ccccccccccc}
0 & 0 & 0 & & & & -x & -a & & & \\
x & 0 & 0 & y_1 & & y_{n-2} & 0 & t & z_1 & & z_{n-2} \\
a & 0 & 0 & b_1 & & b_{n-2} & -t & 0 & c_1 & & c_{n-2} \\
 & & & & & & -z_1 & -c_1 & & & \\
 & & & & & & & & & & \\
 & & & & & & -z_{n-2} & -c_{n-2} & & & \\
 & & & & & & 0 & 0 & & & \\
 & & & & & & 0 & 0 & & & \\
 & & & & & & -y_1 & -b_1 & & & \\
 & & & & & & & & & & \\
 & & & & & & -y_{n-2} & -b_{n-2} & & &
\end{array}\right]$$

Note that

$$\begin{aligned}
\exp(X) &= I_{2n+1} + X + \frac{X^2}{2} \\
&= I_{2n+1} - \frac{P}{2}E_{2,n+2} + \left(t - \frac{Q}{2}\right)E_{2,n+3} + \left(-t - \frac{Q}{2}\right)E_{3,n+2} - \frac{R}{2}E_{3,n+3}.
\end{aligned}$$

Here

$$P = P(X) := x^2 + 2\sum_{j=1}^{n-2} y_j z_j \ ,$$

$$Q = Q(X) := a^2 + 2\sum_{j=1}^{n-2} b_j c_j \ ,$$

and

$$R = R(X) := ax + \sum_{j=1}^{n-2} c_j y_j + \sum_{j=1}^{n-2} b_j z_j \ .$$

Set $D(X) := PQ - R^2$, and note that D is a fundamental relative invariant for the Prehomogeneous space $(\mathbf{GL}(2) \times \mathbf{SO}(2n-3), \mathbf{M}(2, 2n-3))$.

Next, consider the two spherical functions $f_{(1,1,0,\ldots,0)}$ and $f_{(2,0,\ldots,0)}$ on $\mathrm{SO}(2n+1, F)$, $n \geqslant 3$.

LEMMA 3.2.5. Let $X \in \mathfrak{g}_2 \oplus \mathfrak{g}_4$. *Then*

(i) $X \in \operatorname{supp}(f_{(1,1,0,\ldots,0)} \circ \exp) \sqcup \operatorname{supp}(f_{(2,0,\ldots,0)} \circ \exp)$
$\Leftrightarrow [X \in O_F^{4n-5} \wedge \mathrm{val}(t) = -1] \vee$
$[\min\{\mathrm{val}(x), \mathrm{val}(a), \mathrm{val}(b_i), \mathrm{val}(c_i), \mathrm{val}(y_i), \mathrm{val}(z_i), 1 \leqslant i \leqslant n-2\} = -1 \wedge \mathrm{val}(t)$
$\geqslant -1 \wedge \mathrm{val}(D) \geqslant -2]$

(ii) $X \in \operatorname{supp}(f_{(2,0,\ldots,0)} \circ \exp) \Leftrightarrow [\min\{\mathrm{val}(x), \mathrm{val}(a), \mathrm{val}(b_i), \mathrm{val}(c_i), \mathrm{val}(y_i), \mathrm{val}(z_i);$
$1 \leqslant i \leqslant n-2\} = -1] \vee [\min\{\mathrm{val}(P), \mathrm{val}(Q), \mathrm{val}(R)\} = -2 \wedge \mathrm{val}(t) \geqslant -1 \wedge$
$\mathrm{val}(D) \geqslant -2]$.

(iii) $\operatorname{supp}(f_{(1,0,\ldots,0)} \circ \exp) \sqcap \mathfrak{g}_2 = \phi$.

(iv) $\operatorname{supp}(f_{(2,1,0,\ldots,0)} \circ \exp) \sqcap \mathfrak{g}_2 = \phi$. Here 'supp' stands for 'support', and $q^{-\mathrm{val}(t)} = |t|$, $t \in F$.

Proof. Given $g \in \mathbf{G}(F)$, and $\mathbf{m} = (m_1, \ldots, m_n) \in P_n^{++}$, we have* $g \in \operatorname{supp}(f_{\mathbf{m}}) \Leftrightarrow -m_1 - \cdots - m_\ell = \min\{$ valuation of all $\ell \times \ell$ subdeterminants of $g\}$, $\forall \ell, 1 \leqslant \ell \leqslant n$.

Apply this to $g = \exp Y$ for $Y \in \mathfrak{g}_2 \oplus \mathfrak{g}_4$, and note that for $\mathbf{m} = (1, 1, 0, \ldots, 0)$ or $(2, 0, \ldots, 0)$, only the relations corresponding to $\ell = 1, 2$ do matter. The others are redundant. A careful and lengthy analysis of these two relations gives the claimed result. We omit the details. □

DEFINITION 3.2.6. Let $n \geqslant 3$, and $O(n; k_0)$ as above. Let $f \in C_c^\infty(\mathbf{G}(F))$. We say that f satisfies condition (C_n) if:

$$\int_{O_\pi(3;0)} f = \int_{O_{\pi\varepsilon}(3;0)} f, \quad \text{if} \quad n = 3 \ ,$$
$$\sum_{\eta \in \{\pm 1\}} \int_{O_{\pi,\eta}(n;k_0)} f = \sum_{\eta \in \{\pm 1\}} \int_{O_{\pi\varepsilon,\eta}(n;k_0)} f, \quad \text{if} \quad n \geqslant 4 \ .$$

LEMMA 3.2.7.

(i) $\int_{O_1(3;0)} f_{(2,0,0)} = \int_{O_\varepsilon(3;0)} f_{(2,0,0)} = q \int_{O_\pi(3;0)} f_{(2,0,0)}$, *if* $n = 3$,
$\int_{O_1(n;k_0)} f_{(2,0,\ldots,0)} = \sum_{\eta \in \{\pm 1\}} \int_{O_{\varepsilon,\eta}(n;k_0)} f_{(2,0,\ldots,0)} = q \cdot \sum_{\eta \in \{\pm 1\}} \int_{O_{\pi,\eta}(n;k_0)} f_{(2,0,\ldots,0)}$,
if $n \geqslant 4$.

(ii) *The spherical functions* $f_{(0,\ldots,0)}$, $f_{(2,0,\ldots,0)}$, *and* $f_{(2,0,\ldots,0)} + f_{(1,1,0,\ldots,0)}$ *satisfy condition* (C_n), $n \geqslant 3$.

*In other words one can tell which K-double coset g is in by looking at the norms of the exterior powers of the matrix g. This is well-known for the general linear group and works essentially the same way for split odd orthogonal groups.

Proof. If $n = 3$, let $V_\tau(3,0)$, $\tau \in F^\times/(F^\times)^2$, denote the $(\mathbf{GL}(2) \times \mathbf{SO}(3))$-open orbit in $\mathbf{M}(2,3)(F)$, corresponding to $O_\tau(3;0)$. If $n \geqslant 4$, let $V_1(n;k_0)$, $V_\tau(n;k_0)$, $\tau \in \{\varepsilon, \pi, \varepsilon\pi\}$, $\eta \in \{\pm 1\}$, denote the $(\mathbf{GL}(2) \times \mathbf{SO}(2n-3))(F)$-open orbit in $\mathbf{M}(2, 2n-3)(F)$, corresponding to $O_1(n;k_0)$ and $O_{\tau,\eta}(n;k_0)$, we use the Ranga Rao integral formula to get

$$\int_{O_?(n;k_0)} f = \mathrm{vol}(\{X \in V_?(n;k_0) + \mathfrak{g}(4) : \exp X \in \mathrm{supp}\,(f)\}, \mathrm{d}X), \quad f \in \mathcal{H}(G_n, K_n).$$

Here, the question marks are reserved for the subscripts indicated above. We now consider the case $n = 3$, and $f = f_{(2,0,0)}$. The arguments for $n \geqslant 4$ and $f = f_{(2,0,\ldots,0)}$ are similar, and will not be given. We shall write an element $X \in \mathfrak{g}_2(3,0) \oplus \mathfrak{g}_4(3,0)$ as following: $X = (x, y, z, a, b, c, t) \in F^7$. Then for $\tau \in F^\times/(F^\times)^2$, we have (using Lemma 3.2.5.):

$$\begin{aligned}\mathrm{vol}(\{X \in V_\tau(3;0) : \exp X \in \mathrm{supp}\,(f_{(2,0,0)})\}, \mathrm{d}x) = q^7 \mathrm{vol}(\{&X \in O_F^6 - \pi O_F^6 \\ &: (P(X), Q(X), R(X)) \in O_F^3 - \pi O_F^3 \wedge \mathrm{val}(D) \geqslant 2 \wedge D(X) \\ &\equiv \tau \mathrm{mod}\,(F^\times)^2\}\ .\end{aligned}$$

To proceed, we need to recall the following general fact. Let $n \geqslant k \geqslant 1$ be integers, and

$$f = (f_1, \ldots, f_k) \colon F^n \Longrightarrow F^k\ ,$$

$f_j \in F[x_1, \ldots, x_n]$, $1 \leqslant j \leqslant k$. The critical set C_f of f is, by definition, the set

$$\{x \in F^n : \mathrm{rank}\left(\frac{\partial f_j}{\partial x_i}\right)_{\substack{1 \leqslant j \leqslant k \\ 1 \leqslant i \leqslant n}} < k.\}$$

Let $t \in F^k - C_f$, and let $\left|dx/df\right|_t$ denote the measure on the fiber $f^{-1}(t)$ constructed in the standard way. Next, let Φ denote a Bruhat–Schwartz function on F^n, whose support is disjoint from C_j. Then the fiber integral: $t \longmapsto \int_{f^{-1}(t)} \Phi\left|dx/df\right|_t$ is locally constant, and

$$\int_{F^n} \Phi(x)\mathrm{d}x = \int_{F^k} \left[\int_{f^{-1}(t)} \Phi\left|\frac{\mathrm{d}x}{\mathrm{d}f}\right|_t\right] \mathrm{d}t\ ,$$

where $\mathrm{d}x$, $\mathrm{d}t$ are the normalized Lebesgue measures on F^n and F^k, respectively. Apply now the above discussed generality to the following situation:

$$\begin{aligned}&n = 6, \quad k = 3, \quad f = (f_1, f_2, f_3) := (P, Q, R), \quad \text{and} \\ &\Phi_\tau : = 1_Y 1_{Z_\tau}, \quad \tau \in F^\times/(F^\times)^2\end{aligned}$$

where

$$Y := \{(x, y, z, a, b, c) \in O_F^6 - \pi O_F^6 : (P, Q, R) \in O_F^3 - \pi O_F^3\}\ ,$$

and

$$\begin{aligned} Z_\tau := \{&(x, y, z, a, b, c) \in O_F^6 - \pi O_F^6 : D = PQ - R^2 \\ &\equiv \tau \bmod (F^\times)^2, \quad \text{and } \operatorname{val}(PQ - R^2) \geqslant 2\}\,. \end{aligned}$$

Here 1_A denotes the characteristic function of A. Thus

$$\begin{aligned} \int_{O_\tau(3;0)} f_{(2,0,0)} &= q^7 \int_{F^6} 1_Y 1_{Z_t} \mathrm{d}X \\ &= q^7 \int_{F^3} \int_{f^{-1}(t)} 1_Y 1_{Z_\tau} \left|\frac{\mathrm{d}X}{\mathrm{d}f}\right|_t \mathrm{d}t \\ &= q^7 \int_{F^3} 1_{Z_\tau} \int_{f^{-1}(t)} 1_Y \left|\frac{\mathrm{d}X}{\mathrm{d}f}\right|_t \mathrm{d}t \end{aligned}$$

The last identity follows from the observation that 1_{Z_τ} is constant on each fiber $f^{-1}(t)$. Now, the fiber integral: $\psi(t) := \int_{f^{-1}(t)} 1_Y \left|\mathrm{d}X/\mathrm{d}f\right|_t \mathrm{d}t$ is a locally constant function supported on $O_F^3 - \pi O_F^3$. Write $\psi(t) = \sum_{\lambda\in\Lambda} a_\lambda 1_{U_\lambda}$, where the sum is taken over a countable set Λ, and $(U_\lambda)_{\lambda\in\Lambda}$ is a mutually disjoint family of compact open subsets of $O_F^3 - \pi O_F^3$, and $a_\lambda \geqslant 0$, $\forall \lambda \in \Lambda$. Thus, for $\tau \in F^\times/(F^\times)^2$, we have

$$\begin{aligned} \int_{O_\tau(3;0)} f_{(2,0,0)} &= q^7 \sum_{\lambda\in\Lambda} a_\lambda \int_{F^3} 1_{Z_\tau} \cdot 1_{U_\lambda} \mathrm{d}t \\ &= q^7 \sum_{\lambda\in\Lambda} a_\lambda \cdot \operatorname{vol}(U_\lambda \cap D^{-1}(\tau(F^\times)^2 \cap P_F^2), \mathrm{d}t) \end{aligned}$$

Now, using ([1], Proposition 2.2.), $\forall \lambda \in \Lambda$, $\forall \tau \in F^\times/(F^\times)^2$, we have

$$\operatorname{vol}(U_\lambda \cap D^{-1}(\tau(F^\times)^2 \cap P_F^2), \mathrm{d}t) = \int_{\tau(F^\times)^2 \cap P^2} [\lim_{e\to\infty} q^{-2e} N_{(e,U_\lambda)}(i)]\mathrm{d}i\,,$$

where, for any $i \in O_F$, and any $e \in \mathbb{N}$, $e \geqslant 1$, $N_{(e,U_\lambda)}(i)$ is defined to be the order of the set $\{(z_1, z_2, z_3) \in \overline{U}_\lambda : \overline{D}(z_1, z_2, z_3) = \overline{i}\}$, where the overbars indicate reduction modulo P_F^e. For $\lambda \in \Lambda$, let $N_0(\lambda) :=$ order of the set $\{(z_1, z_2, z_3) \in \overline{U}_\lambda := \overline{D}(z_1, z_2, z_3) = 0\}$, where, this time, the overbars indicate reduction modulo P_F. Since $U_\lambda \subseteq O_F^3 - \pi O_F^3$, it follows, as can be easily checked that for $i \in P_F^2$, $e \in \mathbb{N}$, $e \geqslant 1$, we have

$$N_{(e,U_\lambda)}(i) = N_0(\lambda) \cdot q^{-2(e-1)}\,.$$

In other words, $N_{(e,U_\lambda)}(i)$ is independent of $i \in P_F^2$. The claimed result follow now from the above discussions, and the fact that for each $n \in \mathbb{N}$,

$$\begin{aligned} &\operatorname{vol}((F^\times)^2 \cap P_F^{2n} - P_F^{2n+1}) \\ &\quad = \operatorname{vol}(\varepsilon(F^\times)^2 \cap (P_F^{2n} - P_F^{2n+1})) = \frac{1-q^{-1}}{2} q^{2n}\,, \end{aligned}$$

and

$$\begin{aligned}&\mathrm{vol}(\pi(F^\times)^2 \cap P_F^{2n+1} - P_F^{2n+2})\\ &\quad = \mathrm{vol}(\varepsilon\pi(F^\times)^2 \cap (P_F^{2n+1} - P_F^{2n+1})) = \frac{1-q^{-1}}{2} q^{-(2n+1)}\ .\end{aligned}$$

This concludes the proof of part (i) of the Lemma. Part (ii) can be proven using similar arguments and we omit the details. □

The next Lemma reviews some results, needed later, about some subregular orbital integrals in $\mathbf{SO}(5, F)$.

LEMMA 3.2.8.

(i) *The dimension of the complex vector space of linear forms on $\mathcal{H}(\mathbf{SO}(5, F), \mathbf{SO}(5, O_F))$ spanned by integration over the four subregular orbits $O_{\mathrm{sub}}(\tau)$, $\tau \in F^\times/(F^\times)^2$ is three dimensional.*

(ii) $\int_{O_{\mathrm{sub}}(\pi)} f = \int_{O_{\mathrm{sub}}(\varepsilon\pi)} f\ ,\quad f \in \mathcal{H}(\mathbf{SO}(5, F), \mathbf{SO}(5, O_F))$

(iii) *Let $\mathbf{m} = (m_1, m_2) \in P_2^{++}$. Then*

(a) $\int_{O_{\mathrm{sub}}(1)} f_{(0,0)} = \frac{1}{2}$,

$$\int_{O_{\mathrm{sub}}(\varepsilon)} f_{(0,0)} = \frac{1}{2}\frac{(1-q^{-1})(1+q^{-3})}{(1+q^{-1})(1-q^{-3})},$$

$$\int_{O_{\mathrm{sub}}(\pi)} f_{(0,0)} = \frac{1}{2}\frac{q^{-1}(1-q^{-1})}{1-q^{-3}}.$$

(b) $\int_{O_{\mathrm{sub}}(1)} f_{(m,m)} = \int_{O_{\mathrm{sub}}(\varepsilon)} f_{(m,m)} = \frac{1}{2} q^{2m} q^{-1}(1-q^{-1})$

$\int_{O_{\mathrm{sub}}(\pi)} f_{(m,m)} = \frac{1}{2} q^{2m} q^{-1}(1-q^{-1})$, if m is odd.

(c) $\int_{O_{\mathrm{sub}}(1)} f_{(m,0)} = \frac{1}{2} q^{\frac{3m}{2}}(1+q^{-1})$,

$\int_{O_{\mathrm{sub}}(\varepsilon)} f_{(m,0)} = \frac{1}{2} q^{\frac{3m}{2}} 1 - q^{-1})$,

$\int_{O_{\mathrm{sub}}(\pi)} f_{(m,0)} = 0$, if $m > 0$, *and even.*

(d) $\int_{O_{\mathrm{sub}}(\tau)} f_{\mathbf{m}} = 0$, $\tau \in F^\times/(F^\times)^2$, *if m_1 and m_2 have different parity.*

Proof. See Section 2 in [1]. □

3.3. A DESCENT LEMMA

LEMMA 3.3.1. *Consider the following two arrows between connected unramified groups defined over F.*

$$\begin{array}{ccc} M_G & \longrightarrow & G \\ M_H & \longrightarrow & H \end{array}$$

where the source groups are Levi subgroups of the target groups, and the lower source (resp. target) group is an endoscopic group of the upper source (resp. target) group.

https://doi.org/10.1023/A:1001839511409 Published online by Cambridge University Press

Assume that D_G, D_H, D_{M_G}, D_{M_H} are all tempered invariant distributions on $\mathbf{G}(F)$, $\mathbf{H}(F)$, $\mathbf{M}_G(F)$, $\mathbf{M}_\mathbf{H}(F)$, *respectively. Assume further that*

(i) $D_G = \mathrm{Ind}_{M_G}^G D_{M_G}$, $D_H = \mathrm{Ind}_{M_H}^H D_{M_H}$,
(ii) $D_{M_G}(g) = D_{M_H}(g^{M_H})$, $g \in \mathcal{H}(M_G, K_{M_G})$. *Here '*Ind*' indicates parabolic induction of invariant distributions, and $g^{M_H} \in \mathcal{H}(M_H, K_{M_H})$ is the 'transfer of g'. Then $D_G(f) = D_H(f^H)$, $f \in \mathcal{H}(G, K_G)$, where f^H is the transfer of f.*

Proof. From (i) and (ii), we have, for $f \in \mathcal{H}(G, K_G)$

$$D_G(f) = D_{M_G}(f^{M_G}) = D_{M_H}\left((f^{M_G})^{M_H}\right) ,$$

and

$$D_H(f^H) = D_{M_H}\left((f^H)^{M_H}\right) .$$

Now, for any unramified group $\mathbf{L}$, and tempered invariant distribution D_L on $\mathbf{L}(F)$, there exists a measure* μ_{D_L} on the space $\hat{L}_{\mathrm{unr.}}$, of tempered unramified principal series, such that, for $f \in \mathcal{H}(L, K_L)$

$$D(f) = \int_{\hat{L}_{\mathrm{unr.}}} \check{f}(z) \mathrm{d}\mu_{D_L}(z) \qquad (*) ,$$

where $f \mapsto \check{f}$ denotes the Satake transform of f. Now, set $\mathbf{L} := \mathbf{M}_H$, and note that the Satake transforms of $(f^{M_G})^{M_G}$ and $(f^H)^{M_H}$ are the same for all $f \in \mathcal{H}(G, K)$. Applying $(*)$ to D_{M_H} and using the above stated identities, we obtain the claimed result. □

COROLLARY 3.3.2. Let $\mathbf{G} := \mathbf{SO}(2n+1)$, $\mathbf{H} := \mathbf{SO}(5) \times \mathbf{SO}(2n-3)$, $n \geqslant 3$. *For all $f \in \mathcal{H}(G, K_G)$ we have*

(i) $\int_{(O^{\mathrm{st}}_{\mathrm{sub}}, \mathbf{1})} f^H = \alpha_0 \int_{O_1(n;k_0)} f$,
(ii) $\int_{(O_{\mathrm{sub}}(1), \mathbf{1})} f^H = \frac{1}{2} \int_{O^{\mathrm{st}}(n;k_0)} f$.

Here $\mathbf{1}$ denotes the trivial orbit in $\mathrm{SO}(2n-3, F)$, and α_0 is a non-zero constant (which will be computed in Section 3.6).

Proof. In case (i), apply Lemma 3.3.1 to the following data: $\mathbf{M}_\mathbf{H} := \mathbf{GL}(1) \times \mathbf{SO}(3) \times \mathbf{SO}(2n-3)$, $\mathbf{M}_\mathbf{G} := \mathbf{GL}(1) \times \mathbf{SO}(2n-1)$, $D_G := \int_{O_1(n;k_0)} \bullet$, $D_H := \int_{(O^{\mathrm{st}}_{\mathrm{sub}}, \mathbf{1})} \bullet$ $D_{M_G} := \int_{O_{\min}} \bullet$, where $O_{\min}$ denotes the (unique) F-rational orbit in $\mathbf{M}_\mathbf{G}(F)$ with corresponding partition $2\ 2\ 1^{2n-5}$, and $D_{M_H} :=$ Dirac delta measure at the identity in $\mathbf{M}_\mathbf{H}(F)$. Thanks to Lemma 3.1.4, and ([1], Theorem 3.2), the hypothesis of Lemma 3.3.1 are satisfied, up to a nonzero constant α_0. Hence the result.

*The referee points out that, in general, one needs distributions not just measures.

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(ii) In this case, we apply Lemma 3.3.1 to the following data: $\mathbf{M}_H = \mathbf{M}_G = \mathbf{GL}(2) \times \mathbf{SO}(2n-3)$,

$$D_{M_H} = D_{M_G} := \text{ Dirac delta measure at the identity },$$
$$D_G := \int_{O^{\mathrm{st}}(n;k_0)} \bullet\ , \text{ and } D_H := \int_{(O_{\mathrm{sub}}(1),\mathbf{1})} \bullet\ .$$

The hypothesis of Lemma 3.3.1. are satisfied, up to a nonzero constant, by virtue of Lemma 3.1.4. The constant can be calculated by evaluating D_G and D_H at the identity elements of the Hecke algebras, using Lemma 3.1.4. (iiia). □

3.3. A FORMULA FOR f^H

Fix an integer $n \geqslant 3$. Let $\mathbf{G} := \mathbf{SO}(2n+1)$ and $\mathbf{H}_1 := \mathbf{SO}(5) \times \mathbf{SO}(2n-3)$. $\mathbf{H}_2 = \mathbf{SO}(2n-1) \times \mathbf{SO}(3)$. $\mathbf{H}_1(F)$ and $\mathbf{H}_2(F)$ are both elliptic endoscopic groups of $\mathbf{G}(F)$. We are interested in calculating the endoscopic transfer map: $f \mapsto f^{H_i}$, $i = 1, 2$, for certain functions $f \in \mathcal{H}(G, K_G)$.

First we recall some definitions and facts. For $k \in \mathbb{N}$, $k \geqslant 1$, the Harish-Chandra spherical $\mathbf{c}$-function $\mathbf{c}_{B_k}$ is defined by

$$\mathbf{c}_{B_k}(z_1, \ldots, z_k) := \prod_{1 \leqslant i<j \leqslant k} \frac{1 - q^{-1} z_i^{-1} z_j}{1 - z_i^{-1} z_j} \cdot \prod_{1 \leqslant i<j \leqslant k} \frac{1 - q^{-1} z_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}} \cdot \prod_{1 \leqslant i \leqslant n} \frac{1 - q^{-1} z_i^{-2}}{1 - z_i^{-2}}$$

(the empty products equal 1 in the case $k = 1$). Following the notation of 3.1, let $\mathbf{G}_k := \mathrm{SO}(2k+1)$, and $f \in \mathcal{H}(G_k, K_k)$. The *Satake transform* $\check{f}$ of f, is explicitly given by Macdonald's formula (cf.[10]) as follows. If $\mathbf{m} = (m_1, \ldots, m_k) \in P_k^{++}$, then

$$\check{f}_{\mathbf{m}}(z_1, \ldots, z_k) = \frac{q^{\frac{1}{2}[(2k-1)m_1 + (2k-3)m_2 + \cdots + m_k]}}{Q_{\mathbf{m}}(q^{-1})} \sum_{\sigma \in W(B_k)} [\mathbf{c}_{B_k}(z_1, \ldots, z_k) z_1^{m_1} \cdots z_k^{m_k}]^{\sigma},$$

where $Q_{\mathbf{m}}(q^{-1})$ denotes the *Poincaré polynomial* of the stabilizer of $\mathbf{m}$ in the Weyl group $W(B_k) \cong S_k \ltimes (\mathbb{Z}/2\mathbb{Z})^k$. Next, we recall a suitable version of the *Plancherel Theorem* for $\mathbf{G}_k(F)$.

PROPOSITION 3.3.1. Let $\mathbf{m}, \mathbf{m}' \in P_k^{++}$. Then

$$\frac{Q_k(q^{-1})}{|W(B_k)|} \left(\frac{1}{2\pi i}\right)^k \int_{\hat{\mathbf{T}}_{k,0}} \check{f}_{\mathbf{m}}(\mathbf{z}) \overline{\check{f}_{\mathbf{m}'}(\mathbf{z})} \mathrm{d}\mu_k(\mathbf{z}) = \begin{cases} q^{[(2k-1)m_1 + (2k-3)m_2 + \cdots + m_k]} \frac{Q_k(q^{-1})}{Q_{\mathbf{m}}(q^{-1})}, & \textit{if } \mathbf{m} = \mathbf{m}', \\ 0, & \textit{otherwise} \end{cases}$$

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Here

$$\hat{\mathbf{T}}_{k,0} := \{\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k : |z_1| = \cdots = |z_k| = 1\} ,$$

$$\mathrm{d}\mu_k(\mathbf{z}) := |\mathbf{c}_{B_k}(\mathbf{z})|^{-2} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_k}{z_k} , \quad \mathbf{z} = (z_1, \ldots, z_k) \in \hat{\mathbf{T}}_{k,0} ,$$

$$Q_k(q^{-1}) := \quad \textit{Poincare polynomial of } W(G_k) ,$$

and the overbar denotes complex conjugation.

Proof. See [10]. □

Now, as before, we fix $n \in \mathbb{N}$, $n \geqslant 3$, and set $\mathbf{G} := \mathbf{SO}(2n+1)$. Fix $k, \ell \in \mathbb{N}$ such that $k + \ell = n$. Set $\mathbf{H} := \mathbf{SO}(2k+1) \times \mathbf{SO}(2\ell+1)$. Let $f \in \mathcal{H}(G, K_G)$. Recall that $f^H \in \mathcal{H}(H, K_H)$ is defined by $\check{f}^H := \check{f}$,where $\check{f}^H$ is the Satake transform of f^H defined on $\mathcal{H}(H, K_H)$, and $\check{f}$ is the Satake transform of f defined on $\mathcal{H}(G, K_G)$. Write

$$f^H = \sum_{\substack{\mathbf{m} \in P_k^{++} \\ \mathbf{n} \in P_\ell^{++}}} a_{\mathbf{m},\mathbf{n}} g_{\mathbf{m}} \otimes h_{\mathbf{n}}, \quad a_{\mathbf{m},\mathbf{n}} \in \mathbb{Q} .$$

Here $g_{\mathbf{m}} \in \mathcal{H}(\mathbf{SO}(2k+1, F), \mathbf{SO}(2k+1, O_F))$, and $h_{\mathbf{n}} \in \mathcal{H}(\mathbf{SO}(2\ell+1, F), \mathbf{SO}(2\ell+1, O_F))$ are the basic spherical functions corresponding to $\mathbf{m}$ and $\mathbf{n}$ respectively (see 3.1). The following Lemma provides a formula for calculating the coefficients $a_{\mathbf{m},\mathbf{n}}$.

LEMMA 3.3.2. *The coefficient $a_{\mathbf{m},\mathbf{n}}$ is given by*

$$\begin{aligned} a_{\mathbf{m},\mathbf{n}} = {} & q^{-\frac{1}{2}[((2k-1)m_1+(2k-3)m_2+\cdots+m_k)+((2\ell-1)n_1+(2\ell-3)n_2+\cdots+n_\ell)]} \cdot \\ & \cdot \left(\frac{1}{2\pi i}\right)^n \int_{\hat{\mathbf{T}}_{n,0}} \mathbf{c}_{B_k}(z_1^{-1}, \ldots, z_k^{-1}) \cdot \mathbf{c}_{B_\ell}(z_{k+1}^{-1}, \ldots, z_n^{-1}) \check{f}(z_1, \ldots, z_n) \cdot \\ & \cdot z_1^{m_1} \cdots z_k^{m_k} z_{k+1}^{n_1} \cdots z_n^{n_\ell} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} . \end{aligned}$$

Proof. Using the Plancherel Theorem (Proposition 3.3.1) for $\mathbf{H}(F)$, and the explicit formulae for $\check{g}_{\mathbf{m}}$ and $\check{h}_{\mathbf{n}}$, we get the identity

$$\begin{aligned} a_{\mathbf{m},\mathbf{n}} = {} & q^{\frac{1}{2}[(2k-1)m_1+(2k-3)m_2+\cdots+m_k]} \cdot q^{\frac{1}{2}[(2\ell-1)n_1+(2\ell-3)n_2+\cdots+n_\ell]} \cdot \frac{1}{|W(B_k)||W(B_\ell)|} \cdot \\ & \cdot \left(\frac{1}{2\pi i}\right)^n \int_{\hat{\mathbf{T}}_{n,0}} \sum_{\sigma \in W(B_k)} \sum_{\sigma \in W(B_\ell)} [\mathbf{c}_{B_k}(z_1, \ldots, z_k) z_1^{m_1} \cdots z_k^{m_k}]^\sigma \\ & \cdot [\mathbf{c}_{B_\ell}(z_{k+1}, \ldots, z_n) z_{k+1}^{n_1} \cdots z_n^{n_\ell}]^\tau \cdot \\ & \cdot |\mathbf{c}_{B_k}(z_1, \ldots, z_k)|^{-2} |\mathbf{c}_{B_\ell}(z_{k+1}, \ldots, z_n)|^{-2} \overline{\check{f}(z_1, \ldots, z_n)} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} . \end{aligned}$$

Now, note that $\check{f}$ is invariant under the Weyl group $W(B_k) \times W(B_\ell)$, and that for $\mathbf{z} \in \hat{\mathbf{T}}_{n,0}$, $\overline{\check{f}(z_1, \ldots, z_n)} = \check{f}(z_1^{-1}, \ldots, z_n^{-1}) = \check{f}(z_1, \ldots, z_n)$. An appropriate change of

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variables applied to each term in the above double sum will give the integrand in the formula stated in the lemma. The rest is clear. □

3.4. SOME AUXILIARY SPHERICAL FUNCTIONS AND THEIR TRANSFERS

Let **G**, $\mathbf{H}_1$ and $\mathbf{H}_2$ be as in Section 3.3. Our aim is to explicitly calculate the functions $f^{H_i}_{(1,1,0,\dots,0)}, f^{H_i}_{(2,0,\dots,0)}$, $i = 1, 2$. In principle this can be accomplished using Lemma 3.3.2. In practice, however, it is easier to work first with certain auxiliary functions in $\mathcal{H}(G, K_G)$ which we now introduce. Let $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{H}(G, K_G)$ be defined as follows.

$$\check{\varphi}_1(z_1, \dots, z_n) := \frac{1}{2^{n-1}(n-1)!} \sum_{\sigma \in W(B_n)} z_1^\sigma = z_1 + \cdots + z_n + z_1^{-1} + \cdots + z_n^{-1} ,$$

$$\check{\varphi}_2(z_1, \dots, z_n) := \frac{1}{2^{n-1}(n-2)!} \sum_{\sigma \in W(B_n)} (z_1 z_2)^\sigma = \sum_{e_i, e_j \in \{\pm 1\}} \sum_{1 \leqslant i < j \leqslant n} z_i^{e_i} z_j^{e_j} ,$$

$$\check{\varphi}_3(z_1, \dots, z_n) := \frac{1}{2^{n-1}(n-1)!} \sum_{\sigma \in W(B_n)} (z_i^2)^\sigma = z_1^2 + \cdots + z_n^2 + z_1^{-2} + \cdots + z_n^{-2} .$$

Next, for any positive integer r, define the following subsets of P_r^{++}.

$$A^r(1) := \{(0, \dots, 0), (1, 0, \dots, 0)\} ,$$
$$A^r(2) := \{(0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0)\} ,$$
$$A^r(3) := \{(0, \dots, 0), (1, 0, \dots, 0), (1, 1, 0, \dots, 0), (2, 0, \dots, 0)\} .$$

LEMMA 3.4.1. For $i = 1, 2, 3$, we have

[a] $\varphi_i = \sum_{\mathbf{m} \in A^n(i)} a^i_{\mathbf{m}} f_{\mathbf{m}}$, where $a^i_{\mathbf{m}} \in \mathbb{Q}$.

[b] $a^1_{(1,0,\dots,0)} = q^{-\frac{(2n-1)}{2}}$, $a^2_{(1,1,0,\dots,0)} = q^{-(2n-2)}$,
$a^3_{(2,0,\dots,0)} = q^{-(2n-1)}$, $a^3_{(1,1,0,\dots,0)} = -(1 - q^{-1}) q^{-(2n-1)}$.

Proof. Write $\varphi_i = \sum_{\mathbf{m} \in P_n^{++}} b^i_{\mathbf{m}} f_{\mathbf{m}}$, $b^i_{\mathbf{m}} \in \mathbb{C}$, $1 \leqslant i \leqslant 3$. Applying Lemma 3.3.2. we get

$$b^i_{\mathbf{m}} = * \left(\frac{1}{2\pi i}\right)^n \int_{\hat{\mathbf{T}}_{n,0}} \prod_{1 \leqslant j \leqslant k \leqslant n} \frac{1 - z_j^{-1} z_k}{1 - q^{-1} z_j^{-1} z_k} \cdot \prod_{j=1}^n \frac{1 - z_j^2}{1 - q^{-1} z_j^2} \cdot \varphi_i(z_1, \dots, z_n) \cdot$$
$$\cdot \prod_{j=1}^n z_j^{m_j - 1} \cdot \mathrm{d}z_1 \dots \mathrm{d}z_n ,$$

where $*$ is some nonzero constant which does not concern us at the moment. Note that the integrand can have a z_1-pole only at $z_1 = 0$, and only when $0 \leqslant m_1 \leqslant i$. Next, we first consider the case where $i = 1$, and $m_1 = 1$. Then $0 \leqslant m_2 \leqslant 1$. Assume that

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$m_2 = 1$. Then

$$b^1_{\mathbf{m}} = *\left(\frac{1}{2\pi i}\right)^n \sum_{e_k \in \{\pm 1\}} \sum_{1 \leqslant k \leqslant n} \int_{\hat{\mathbf{T}}_{n,0}} \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j^{-1}}{1 - q^{-1} z_i z_j^{-1}} \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j}{1 - q^{-1} z_i z_j} \cdot \\ \cdot \prod_{i=1}^n \frac{1 - z_i^2}{1 - q^{-1} z_i^2} \cdot \\ \cdot z_k^{m_k + e_k - 1} z_1^{m_1 - 1} z_2^{m_2 - 1} z_3^{m_3 - 1} \cdots \hat{z}_k \cdots z_n^{m_n - 1} \quad dz_1 \cdots dz_n \ .$$

Note that all the integrals vanish except when $k = 1$ and $e = -1$ (otherwise there is no z_1-pole). Thus for $\mathbf{m} = (1, 1, m_3, \dots m_n)$, we have

$$b^1_{\mathbf{m}} = *\left(\frac{1}{2\pi i}\right)^n \int_{\hat{\mathbf{T}}_1} \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j^{-1}}{1 - q^{-1} z_i z_j^{-1}} \cdot \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j}{1 - q^{-1} z_i z_j} \cdot \prod_{i=1}^n \frac{1 - z_i^2}{1 - q^{-1} z_i^2} \cdot \\ \cdot z_1^{-1} z_3^{m_3 - 1} \cdots z_n^{m_n - 1} \quad \mathrm{d}z_1 \cdots \mathrm{d}z_n \ .$$

Taking the residue at the only z_1-pole, namely $z_1 = 0$, we get a contour integral of a function which has no z_2-poles. Thus $b^1_{\mathbf{m}} = 0$ in this case. The same reasoning shows that if $\mathbf{m} = (1, 0, \dots, 0)$, then

$$b^1_{\mathbf{m}} = q^{-\frac{(2n-1)}{2}} \left(\frac{1}{2\pi i}\right)^n \int_{\hat{\mathbf{T}}_1} \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j^{-1}}{1 - q^{-1} z_i z_j^{-1}} \cdot \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j}{1 - q^{-1} z_i z_j} \cdot \\ \cdot \prod_{i=1}^n \frac{1 - z_i^2}{1 - q^{-1} z_i^2} \cdot \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_n}{z_n} \ .$$

Successively, taking the residues at $z_1 = 0, \dots, z_n = 0$, we get $b^1_{\mathbf{m}} = q^{-\frac{(2n-1)}{2}}$ as desired. The proof of the remaining part of statement [a], as well as the identities $a^2_{(1,1,0,\dots,0)} = q^{-(2n-2)}$, and $a^3_{(2,0,\dots,0)} = q^{-(2n-1)}$ is similar and we omit the details. So, it remains only to check the identity $a^3_{(1,1,0,\dots,0)} = -(1 - q^{-1}) q^{-(2n-2)}$. Set $\mathbf{m} = (m_1, m_2, \dots, m_n) = (1, 1, 0, \dots, 0)$. Then, using Lemma 3.3.2., we get

$$a^3_{\mathbf{m}} = q^{-(2n-2)} \sum_{e_k \in \{\pm 1\}} \sum_{1 \leqslant k \leqslant n} \left(\frac{1}{2\pi i}\right)^n \int_{\hat{\mathbf{T}}_{n,0}} \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j^{-1}}{1 - q^{-1} z_i z_j^{-1}} \cdot \\ \cdot \prod_{1 \leqslant i<j \leqslant n} \frac{1 - z_i z_j}{1 - q^{-1} z_i z_j} \\ \cdot \prod_{i=1}^n \frac{1 - z_i^2}{1 - q^{-1} z_i^2} \cdot z_k^{m_k - 1 + 2e_k} z_1^{m_1 - 1} \dots \hat{z}_k \cdots z_n^{m_n - 1} \quad \mathrm{d}z_1 \cdots \mathrm{d}z_n \ .$$

Note that the integrand has no z_1-pole unless $k = 1$ and $e_k = -1$. In this case, $z_1 = 0$ is a z_1-pole of order 2, and there are no other z_1-poles. Thus, taking the residue at

$z_1 = 0$ (using logarithmic differentiation), we get:

$$a^3_{(1,1,0,\ldots,0)} = q^{-(2n-2)}\left(\frac{1}{2\pi i}\right)^{n-1}\int_{|z_2|=\cdots=|z_n|=1}\prod_{2\leqslant i<j\leqslant n}\frac{1-z_iz_j^{-1}}{1-q^{-1}z_iz_j^{-1}}$$

$$\cdot\prod_{2\leqslant i<j\leqslant n}\frac{1-z_iz_j}{1-q^{-1}z_iz_j}\cdot$$

$$\cdot\prod_{i=2}^{n}\frac{1-z_i^2}{1-q^{-1}z_i^2}\cdot\sum_{2\leqslant j\leqslant n}-(1-q^{-1})(z_j+z_j^{-1})\mathrm{d}z_2\quad\frac{\mathrm{d}z_3}{z_3}\cdots\frac{\mathrm{d}z_n}{z_n}\ .$$

Taking the residue at $z_2 = 0$ gives

$$a^3_{(1,1,0,\ldots,0)} = -(1-q^{-1})q^{-(2n-2)}\left(\frac{1}{2\pi i}\right)^{n-2}\int_{|z_3|=\cdots=|z_n|=1}\prod_{3\leqslant i<j\leqslant n}\frac{1-z_iz_j^{-1}}{1-q^{-1}z_iz_j^{-1}}\cdot$$

$$\cdot\prod_{3\leqslant i<j\leqslant n}\frac{1-z_iz_j}{1-q^{-1}z_iz_j}\cdot\prod_{j=3}^{n}\frac{1-z_j^2}{1-q^{-1}z_j^2}\frac{\mathrm{d}z_3}{z_3}\cdots\frac{\mathrm{d}z_n}{z_n}\ .$$

Now, successively taking the residues at $z_3 = 0, \ldots, z_n = 0$, we get the desired identity. □

The next two lemmas will provide formulae (sufficiently explicit for our purposes) for the function $\varphi_j^{H_i}$, $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant 3$.

LEMMA 3.4.2. *For* $j = 1, 2, 3$, *we have*

(a) $\varphi^{H_1} = \sum_{(\mathbf{k},\ell)\in A^2(j)\times A^{n-2}(j)} a^j_{\mathbf{k},\ell}g_{\mathbf{k}}\otimes h_\ell$, *where* $a^j_{\mathbf{k},\ell}\in\mathbb{Q}$.

(b) $a^2_{(1,1),(0,\ldots,0)} = q^{-2}$, $a^3_{(1,1),(0,\ldots,0)} = q^2(1-q^{-1})$, $a^3_{(2,0),(0,\ldots,0)} = q^{-3}$.

Here, $g_{\mathbf{k}}$ and h_ℓ denote basic spherical functions associated to $\mathbf{k}\in P_2^{++}$ and $\ell\in P_{n-2}^{++}$, respectively.

Proof. The verifications are similar to those of the preceeding Lemma,and are omitted. □

LEMMA 3.4.3. *For* $j = 1, 2, 3$, *we have*

(a) $\varphi_j^{H_2} = \sum_{(\mathbf{k},\ell)\in A^{n-1}(j)\times A^1(j)} a^j_{\mathbf{k},\ell}g_{\mathbf{k}}\otimes h_\ell$, $a^j_{\mathbf{k},\ell}\in\mathbb{Q}$.

(b) $b^2_{(1,1,0,\ldots,0)} = q^{-(2n-4)}$, $b^3_{(1,1,0,\ldots,0)} = -(1-q^{-1})q^{(2n-4)}$, $b^3_{(2,0,\ldots,0)} = q^{-(2n-3)}$.

Here, $g_{\mathbf{k}}$ and h_ℓ denote the basic spherical functions associated to $\mathbf{k}\in P_{n-1}^{++}$ and $\ell\in P_1^{++}$, respectively.

Proof. Omitted. □

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3.5. THE FUNCTIONS $f^{H_i}_{(1,1,0,\dots,0)}$ AND $f^{H_i}_{(2,0,\dots,0)}$, $i = 1, 2$.

The purpose of the next two lemmas is to compute, using Lemmas 3.4.2. and Lemma 3.4.3., sufficiently explicit expressions for the transferred maps $f^{H_i}_{(1,1,0,\dots,0)}$ and $f^{H_i}_{(2,0,\dots,0)}$, $i = 1, 2$.

LEMMA 3.5.1

(i) *There exist constants* α, $a_{\mathbf{m}}$, $b_{\mathbf{m}}$ ($\mathbf{m} \in A^2(2)$), *such that*

$$\begin{aligned} f^{H_1}_{(1,1,0,\dots,0)} &= (1+q^{-2})(1-q^{-(2n-4)})q^{2n-2}g_{(0,0)} \otimes h_{(0,\dots,0)} + \\ &+ q^{2n-4}g_{(1,1)} \otimes h_{(0,\dots,0)} + \alpha g_{(1,0)} \otimes h_{(0,\dots,0)} + \\ &+ \sum_{\mathbf{m}\in A^2(2)} g_{\mathbf{m}} \otimes (a_{\mathbf{m}} h_{(1,0,\dots,0)} + b_{\mathbf{m}} h_{(1,1,0,\dots,0)}. \end{aligned}$$

(ii) *There exist constants* $c_{\mathbf{m}}$, $d_{\mathbf{m}}$, $e_{\mathbf{m}}$ ($\mathbf{m} \in A^2(3)$), *such that*

$$\begin{aligned} f^{H_1}_{(2,0,\dots,0)} &= q^{2n-4}g_{(2,0)} \otimes h_{(0,\dots,0)} + \\ &+ \sum_{\mathbf{m}\in A^2(3)} g_{\mathbf{m}} \otimes (c_{\mathbf{m}} h_{(1,0,\dots,0)} + d_{\mathbf{m}} h_{(1,1,0,\dots,0)} + e_{\mathbf{m}} h_{(2,0,\dots,0)}) \end{aligned}$$

Proof. (i) By Lemma 3.4.1. [a], there exist $\lambda, \mu \in \mathbb{Q}$ such that

$$\varphi_2 = \lambda f_{(0,\dots,0)} + q^{-(2n-2)} f_{(1,1,0,\dots,0)} + \mu f_{(1,0,\dots,0)}\ .$$

Thus

$$\varphi_2^{H_1} = \lambda f^{H_1}_{(0,\dots,0)} + q^{-(2n-2)} f^{H_1}_{(1,1,0,\dots,0)} + \mu f^{H_1}_{(1,0,\dots,0)}\ . \tag{$*$}$$

On the other hand, using Lemma 3.4.2, there exist constants v, $\alpha_{\mathbf{m}}$, $\beta_{\mathbf{m}} \in \mathbb{Q}$, $\mathbf{m} \in A^2(2)$, such that

$$\begin{aligned} \varphi_2^{H_1} &= q^{-2}g_{(1,1)} \otimes h_{(0,\dots,0)} + vg_{(0,0)} \otimes h_{(0,\dots,0)} \\ &+ \sum_{\mathbf{m}\in A^2(2)} g_{\mathbf{m}} \otimes (\alpha_{\mathbf{m}} h_{(1,0,\dots,0)} + \beta_{\mathbf{m}} h_{(1,1,0,\dots,0)}) \end{aligned} \tag{$**$}$$

Now, by Lemma 3.4.1.[a], φ_1 is in the linear span of $f_{(0,\dots,0)}$ and $f_{(1,0,\dots,0)}$, and by Lemma 3.4.2.[a], $\varphi_1^{H_1}$ is in the linear span of the function $g_{(0,0)} \otimes h_{(0,\dots,0)}$, $g_{(0,0)} \otimes h_{(0,\dots,0)}$, $g_{(1,0)} \otimes h_{(0,\dots,0)}$, and $g_{(1,0)} \otimes h_{(1,0,\dots,0)}$. Since $f^{H_1}_{(0,\dots,0)} = g_{(0,0)} \otimes h_{(0,\dots,0)}$, we deduce that $f^{H_1}_{(1,0,\dots,0)}$ is in the linear span of the four functions mentioned above. Now substituting into $(*)$ and comparing the result with $(**)$, we see that $f^{H_1}_{(1,1,0,\dots,0)}$ is now in the linear span of the eight basic functions appearing in $(**)$. Let γ denote the coefficient of $g_{(1,1)} \otimes h_{(0,\dots,0)}$ in $f^{H_1}_{(1,1,0,\dots,0)}$. Note that, from the discussion above, the coefficient of $g_{(1,1)} \otimes h_{(0,\dots,0)}$ in $f^{H_1}_{(1,0,\dots,0)}$ (and obviously in $f^{H_1}_{(0,\dots,0)}$) is zero. Thus substituting into $(*)$ and comparing the coefficient of $g_{(1,1)} \otimes h_{(0,\dots,0)}$ with that in $(**)$, we get: $\gamma q^{-(2n-2)} = q^{-2}$. Thus $\gamma = q^{2n-4}$. The coefficient δ, say, of $g_{(0,0)} \otimes h_{(0,\dots,0)}$ in $f^{H_1}_{(1,1,0,\dots,0)}$ is obtained from the results of ([3], Proposition 1.3.5.) In fact δ is equal to $c^{-1}\int_O f_{(1,1,0,\dots,0)}$, where the integral is over the unipotent orbit O in $\mathbf{SO}(2n+1,F)$ parametrized by the partition $2^4 1^{2n-7}$ (the stable orbit contains only one F-rational orbit), and c= value of the Igusa zeta function associated to the prehomogeneous vector space $(\mathbf{GL}(4), \mathbf{Alt}(4))$ at $s = 2n-7$. Note that the measures used in calculating the above orbital integral and the Igusa zeta function are the same, so δ does not depend on the normalization of measure.

(ii) We argue as in (i). First, note that the coefficient of $g_{(0,0)} \otimes h_{(0,\dots,0)}$ in $f^{H_1}_{(2,0,\dots,0)}$ is equal to zero. This follows from the fact that the orbital integral of $f_{(2,0,\dots,0)}$ over the orbit O, indicated in (i) above, is equal to zero (see [3], Proposition 1.3.5.) Next, using Lemma 3.4.1., there exists constants λ, μ such that

$$\varphi_3 = \lambda f_{(0,\dots,0)} + \mu f_{(1,0,\dots,0)} - (1-q^{-1})q^{-(2n-2)} f_{(1,1,0,\dots,0)} + q^{-(2n-1)} f_{(2,0,\dots,0)}\ .$$

Thus

$$\begin{aligned}\varphi_3^{H_1} = \lambda g_{(0,\dots,0)} \otimes h_{(0,\dots,0)} + \mu f^{H_1}_{(1,0,\dots,0)} - (1-q^{-1})^{-(2n-2)} f^{H_1}_{(1,1,0,\dots,0)} + \\ + q^{-(2n-1)} f^{H_1}_{(2,0,\dots,0)}\ . \end{aligned} \qquad (*)$$

On the other hand, using Lemma 3.4.2., there exists constants μ, $\alpha_{\mathbf{m}}$, $\beta_{\mathbf{m}}$, $\gamma_{\mathbf{m}}$, $\mathbf{m} \in A^2(3)$ such that

$$\begin{aligned}\varphi_3^{H_1} = \mu g_{(0,0)} \otimes h_{(0,\dots,0)} - q^{-2}(1-q^{-2}) g_{(1,1)} \otimes h_{(0,\dots,0)} + q^{-3} g_{(2,0)} \otimes h_{(0,\dots,0)} + \\ + \sum_{\mathbf{m}\in A^2(3)} g_{\mathbf{m}} \otimes [\alpha_{\mathbf{m}} h_{(1,0,\dots,0)} + \beta_{\mathbf{m}} h_{(1,1,0,\dots,0)} + \gamma_{\mathbf{m}} h_{(2,0,\dots,0)}] \end{aligned} \qquad (**)$$

As in (i), one then argues that $f^{H_1}_{(2,0,\dots,0)}$ is in the linear span of the basic functions appearing in the right hand side of $(**)$. Suppose that the coefficient of $g_{(1,1)} \otimes h_{(0,\dots,0)}$ (resp. $g_{(2,0)} \otimes h_{(0,\dots,0)}$) in $f^{H_1}_{(2,0,\dots,0)}$ is γ (resp. δ). Substituting into $(*)$, and using the formula for $f^{H_1}_{(1,1,0,\dots,0)}$ established in (i), we then compare the coefficients of $g_{(1,1)} \otimes h_{(0,0,\dots,0)}$ and $g_{(2,0)} \otimes h_{(0,\dots,0)}$ appearing in $(*)$ and $(**)$, and find

$$-q^{-2}(1-q^{-2}) = -(1-q^{-2})q^{-(2n-2)} \cdot q^{(2n-4)} + \gamma q^{-(2n-1)}, \quad \text{and} \quad q^{-3} = \delta q^{-(2n-1)}$$

Thus $\gamma = 0$, and $\delta = q^{2n-4}$. □

LEMMA 3.5.2.

(i) *There exist constants* $\alpha, a_{\mathbf{m}}$, $\mathbf{m} \in A^{n-1}(2)$, *such that*

$$\begin{aligned} f^{H_2}_{(1,1,0,\dots,0)} = (1-q^{-(2n-2)})q^{2n-2} g_{(0,\dots,0)} \otimes h_{(0)} + \\ + q^2 g_{(1,1,0,\dots,0)} \otimes h_{(0)} + \alpha g_{(1,0,\dots,0)} \otimes h_{(0)} + \sum_{\mathbf{m}\in A^{n-1}(2)} g_{\mathbf{m}} \otimes a_{\mathbf{m}} h_{(1)}\ . \end{aligned}$$

(ii) *There exist constants* $b_{\mathbf{m}}$, $c_{\mathbf{m}}$, $\mathbf{m} \in A^{n-1}(3)$, *such that*

$$f^{H_2}_{(2,0,\dots,0)} = q^2 g_{(2,0,\dots,0)} \otimes h_{(0)} + \sum_{\mathbf{m}\in A^{n-1}(3)} g_{\mathbf{m}} \otimes (b_{\mathbf{m}} h_{(1)} + c_{\mathbf{m}} h_{(2)})\ .$$

Proof. The proof is entirely similar to that of Lemma 3.5.1., and we omit it. □

3.6. THE TRANSFER FACTORS

Fix an integer $n \geqslant 3$. Set $\mathbf{G} = \mathbf{SO}(2n+1)$, and $\mathbf{H} = \mathbf{SO}(5) \times \mathbf{SO}(2n-3)$. Our first goal is study the transfer of the integrals over $(O_{\mathrm{sub}}(\tau), \mathbf{1})$, $\tau \in F^\times/(F^\times)^2$, from $\mathbf{H}(F)$ to $\mathbf{G}(F)$.

We begin by evaluating the constant α_0 appearing in Corollary 3.3.2 (i).

LEMMA 3.6.1. $\alpha_0 = 2$.

Proof. We shall evaluate both sides of Corollary 3.3.2(i) at the function $f_{(2,0,\ldots,0)}$. We compute the integral $\int_{O_1(n;k_0)} f_{(2,0,\ldots,0)}$ as follows. By Lemma 3.2.7(i), we have $\int_{O^{\mathrm{st}}(n;k_0)} f_{(2,0,\ldots,0)} = 2(1+q^{-1}) \int_{O_1(n;k_0)} f_{(2,0,\ldots,0)}$. On the other hand, by virtue of Lemma 3.5.1(ii) and Lemma 3.2.8 (d), $f^H_{(2,0,\ldots,0)}$ is the sum of $q^{2n-4} g_{(2,0)} \otimes h_{(0,\ldots,0)}$ and other functions whose integrals over $(O^{\mathrm{st}}_{\mathrm{sub}}, \mathbf{1})$ vanish. Thus, using Corollary 3.3.2(ii) and Lemma 3.2.8(iii) (c) (with $m = 2$), we get

$$\begin{aligned}\frac{1}{2} q^{2n-1}(1+q^{-1}) &= \int_{(O_{\mathrm{sub}}(1),\mathbf{1})} f^H_{(2,0,\ldots,0)} = (1+q^{-1}) \int_{(O_{\mathrm{sub}}(1),\mathbf{1})} f^H_{(2,0,\ldots,0)} \\ &= (1+q^{-1}) \int_{O_1(n;k_0)} f_{(2,0,\ldots,0)} \,.\end{aligned}$$

Thus $\int_{O_1(n;k_0)} f_{(2,0,\ldots,0)} = \frac{1}{2} q^{2n-1}$. Now, evaluating both sides of Corollary 3.3.2(ii) at $f_{(2,0,\ldots,0)}$, and using Lemma 3.2.8. (c) (with $m = 2$), we get

$$q^{2n-1} = \int_{(O^{\mathrm{st}}_{\mathrm{sub}},\mathbf{1})} f^H_{(2,0,\ldots,0)} = \alpha_0 \int_{(O_1(n;k_0)} f_{(2,0,\ldots,0)} = \alpha_0 \frac{1}{2} q^{2n-1}.$$

Thus $\alpha_0 = 2$. □

PROPOSITION 3.6.2. *Let $\tau \in F^\times/(F^\times)^2$, and set $E_\tau := F(\sqrt{\tau})$. Let κ_τ denote the character of $F^\times$ associated to E_τ via local class field theory. The following identities are satisfied:*

(i) $\int_{(O_{\mathrm{sub}}(\tau),\mathbf{1})} f^H = \frac{1}{2} \sum_{\sigma \in (F^\times/F^\times)^2} \kappa_\tau(\sigma) \int_{O_\tau(3;0)} f$, *if* $n = 3$,

(ii) $\int_{(O_{\mathrm{sub}}(\tau),\mathbf{1})} f^H = \frac{1}{2} [\int_{O_1(n;k_0)} f + \sum_{\substack{\sigma \in F^\times/(F^\times)^2 \\ \sigma \neq 1 \\ \eta \in \{\pm 1\}}} \kappa_\tau(\sigma) \int_{O_{\sigma,\eta}(n;k_0)} f]$, *if* $n \geqslant 4$,

where $f \in \{f_{(0,\ldots,0)}, f_{(1,1,0,\ldots,0)}, f_{(2,0,\ldots,0)}\}$.

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Proof. For $\tau \in F^\times/(F^\times)^2$, and $g \in C_c^\infty(\mathbf{H}(F))$, we set $a_\tau(g) := \int_{(O_{\mathrm{sub}}(\tau),\mathbf{1})} g$. For $f \in C_c^\infty(\mathbf{G}(F))$, set

$$A_1(f) := \int_{O_1(n;k_0)} f\ , \quad \text{and for } \tau \in F^\times/(F^\times)^2\ ,\ \tau \not\equiv 1\ , \quad \text{we set}$$
$$A_\tau(f) := \int_{O_\tau(n;k_0)} f\ , \ \text{if } n = 3\ , \quad \text{and}$$
$$A_\tau(f) := \sum_{\eta\in\{\pm 1\}} \int_{O_{\tau,n}(n;k_0)} f\ , \quad \text{if } n \geqslant 4\ .$$

Using this notation, the statement of the may be formulated as following:

$$\begin{bmatrix} a_1(f^H) \\ a_\varepsilon(f^H) \\ a_\pi(f^H) \\ a_{\varepsilon\pi}(f^H) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} A_1(f) \\ A_\varepsilon(f) \\ A_\pi(f) \\ A_{\varepsilon\pi}(f) \end{bmatrix},$$

where $f \in \{f_{(0,\ldots,0)}, f_{(1,1,0,\ldots,0)}, f_{(2,0,\ldots,0)}\}$. First, note that it follows from Corollary 3.3.2 and Lemma 3.6.1, that for all $f \in \mathcal{H}(G, K_G)$, we have

$$a_1(f^H) = \frac{1}{2} \sum_{\tau\in F^\times/(F^\times)^2} A_\tau(f), \tag{$*$}$$

$$\sum_{\tau\in F^\times/(F^\times)^2} a_\tau(f^H) = 2A_1(f). \tag{$**$}$$

We now treat each of the above three functions separately. Since, in each case, the function f is understood, we shall simplify the notation by dropping f and f^H; thus we shall write a_τ and A_τ instead of $a_\tau(f^H)$ and $A_\tau(f)$, etc.

(1) The case $f = f_{(0,\ldots,0)}$.

In this case we get the following system of identities:

$$\begin{aligned}
&A_1 + A_\varepsilon + A_\pi + A_{\varepsilon\pi} = 1, \quad \text{clear},\\
&A_1 + A_\varepsilon - A_\pi - A_{\varepsilon\pi} = \tfrac{(1-q^{-1})(1+q^{-3})}{(1+q^{-1})(1-q^{-3})}, \qquad \text{Lemma 3.2.4 (i)} \quad (\text{set } s = 0),\\
&A_1 = \tfrac{1}{2}, \quad (**),\\
&A_\pi = A_{\varepsilon\pi}, \quad \text{Lemma 3.2.7 (ii)}
\end{aligned}$$

Solving the above system, and using Lemma 2.7, we get

$$a_1 = A_1 = \frac{1}{2}, \quad a_\varepsilon = A_\varepsilon = \frac{1}{2}\frac{(1-q^{-1})(1+q^{-3})}{(1+q^{-1})(1-q^{-3})}, a_\pi = a_{\varepsilon\pi} = A_\pi = A_{\varepsilon\pi}$$
$$= \frac{1}{2}\frac{q^{-1}(1-q^{-1})}{1-q^{-3}}\ .$$

The claimed result follows in this case.

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(2) The case $f = f_{(2,0,\dots,0)}$.

In this case it is enough to use $(*)$ and $(**)$, together with the identities:

$a_\pi = a_{\varepsilon\pi} = 0$ (Lemma 2.7) ,

$A_1 = A_\varepsilon$, and $A_\pi = A_{\varepsilon\pi}$ (Lemma 3.2.7 (ii)),

to obtain the same result in this case.

(3) The case $f = f_{(1,1,0,\dots,0)}$.

By case [2], it is sufficient to prove the claimed identities for $f_0 := f_{(2,0,\dots,0)} + f_{(1,1,0,\dots,0)}$. Set

$$\Lambda_1 := \{X \in (P_F^{-1})^{4n-6} - \pi(P_F^{-1})^{4n-6} : \big(P(X), Q(X), R(X)\big) \in (P_F^{-2})^3 - \\ - \pi(P_F^{-2})^3 \wedge D(X) \in P_F^2\} ,$$

and $\Lambda_2 := O_F^{4n-6}$.

According to Lemma 3.2.5 (i), we have

$$\operatorname{supp}(f_0)\circ\exp = \Lambda_1 \times P_F^{-1} \sqcup \Lambda_2 \times (P_F^{-1} - O_F) . \tag{1}$$

Next, for $\tau \in F^\times/(F^\times)^2$, let $U_\tau :=$ union of all $(\mathbf{GL}(2) \times \mathbf{SO}(2n-3))(F)$ – open orbits in $\mathbf{M}(2, 2n-3)(F)$ which are parametrized by the (equivalence classes of) quadratic forms with discriminant τ.

Set $\Lambda_i(\tau) := \Lambda_i \cap U_\tau$, $i = 1, 2$. Using (1), and the Ranga Rao formula, we get, for $\tau \in F^\times/(F^\times)^2$:

$$\begin{aligned} A_\tau(f_0) &= q\mathrm{vol}(\Lambda_1(\tau)) + (q-1)\mathrm{vol}(\Lambda_2(\tau)) \\ &= q\mathrm{vol}(\Lambda_1(\tau) \sqcup \Lambda_2(\tau)) - \mathrm{vol}(\Lambda_2(\tau)) \\ &= q^{4n-5}\mathrm{vol}(\pi(\Lambda_1(\tau) \sqcup \Lambda_2(\tau)) - \mathrm{vol}(\Lambda_2(\tau)) \end{aligned} \tag{2}$$

Next, recall the set Ω defined before (and used) in Lemma 3.2.3. For $\tau \in F^\times(F^\times)^2$, set $\Omega(\tau) := \Omega \cap U(\tau)$ .It is clear, for $\tau \in F^\times/(F^\times)^2$, we have

$$[O_F^{4n-6} \cap U(\tau)] - \Omega(\tau) = \pi(\Lambda_1(\tau) \sqcup \Lambda_2(\tau)) , \quad \text{i.e.}$$
$$A_2(\tau) - \Omega(\tau) = \pi(\Lambda_1(\tau) \sqcup \Lambda_2(\tau)) .$$

Thus, by (2), we get , for $\tau \in F^\times/(F^\times)^2$:

$$\begin{aligned} A_\tau(f_0) &= q^{4n-5}[\mathrm{vol}(\Lambda_2(\tau)) - \mathrm{vol}(\Omega(\tau))] - \mathrm{vol}(\Lambda_2(\tau)) \\ &= (q^{4n-5} - 1)\mathrm{vol}(\Lambda_2(\tau)) - q^{4n-5}\mathrm{vol}(\Omega(\tau)) \\ &= (q^{4n-5} - 1)A_\tau(f_{(0,\dots,0)}) - q^{4n-5}\mathrm{vol}(\Omega(\tau)) . \end{aligned} \tag{3}$$

On the other hand, by Lemma 3.5.1., we have

$$\begin{aligned} f_0^H = {} & (1 + q^{-2})(1 - q^{-(2n-4)})q^{2n-2} g_{(0,0)} \otimes h_{(0,\dots,0)} + \\ & + q^{2n-4} g_{(1,1)} \otimes h_{(0,\dots,0)} + q^{2n-4} g_{(2,0)} \otimes h_{(0,\dots,0)} + \\ & + \text{ other functions whose supports do not meet } (O^{\mathrm{st}}_{\mathrm{sub}}, \mathbf{1}) . \end{aligned}$$

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Since $f^H_{(0,\dots,0)} = g_{(0,0)} \otimes h_{(0,\dots,0)}$, we get for $\tau \in F^\times/(F^\times)^2$:

$$\begin{aligned} a_\tau(f_0^H) = (1+q^{-2})(1-q^{-(2n-4)})q^{2n-2}a_\tau(f^H_{(0,\dots,0)}+ \\ + q^{2n-4}\int_{O_{\mathrm{sub}}(\tau)} g_{(1,1)} + q^{2n-4}\int_{O_{\mathrm{sub}}(\tau)} g_{(2,0)}\ . \end{aligned} \tag{4}$$

Using case [1], and identities (1), (2), the claimed identities (for f_0) are then equivalent to

$$\begin{aligned} &(q^{4n-5} - q^{2n-2} - q^{2n-4} + q^2)a_\tau(f^H_{(0,\dots,0)}) \\ &\quad = \frac{1}{2}\left[\sum_\sigma \kappa_\tau(\sigma)q^{4n-5}\mathrm{vol}(\Omega(\tau)) + q^{2n-4}\int_{O_{\mathrm{sub}}(\tau)}(g_{(1,1)}+g_{(2,0)})\right] \end{aligned} \tag{5}$$

where $\tau \in F^\times/(F^\times)^2$ and the sum ranges over $\sigma \in F^\times/(F^\times)^2$. Now, by Corollary 3.3.2, Lemma 3.6.1, Lemma 3.2.8(i), and Lemma 3.2.7(ii), we see that it is sufficient to prove (5) only for $\tau = \varepsilon$. But now we get, using Lemma 3.2.4(ii) (setting $s = 0$, and $m = 2n-3$):

$$\sum_{\tau \in F^\times/(F^\times)^2} (-1)^{\mathrm{val}(\tau)}\mathrm{vol}(\Omega(\tau)) = \frac{(1-q^{-1})(1+q^{-3})}{(1+q^{-1})(1-q^{-3})}(1-q^{-(2n-4)})(1-q^{-(2n-3)}). \tag{7}$$

Next, using Lemma 3.2.8., the left-hand side of (5) (with $\tau = \varepsilon$) is equal to

$$\frac{1}{2}(q^{4n-5} - q^{2n-2} - q^{2n-4} + q^2) - \frac{(1-q^{-1})(1+q^{-3})}{(1+q^{-1})(1-q^{-3})}. \tag{8}$$

On the other hand, using (7) and Lemma 2.7, the right-hand side of (5) (with $\tau = \varepsilon$) is equal to

$$\begin{aligned} &\frac{1}{2}\left[q^{4n-5}\frac{(1-q^{-1})(1+q^{-3})}{(1+q^{-1})(1-q^{-3})}(1-q^{-(2n-4)})(1-q^{-(2n-3)})+\right. \\ &\quad \left. + q^{2n-4}\big(q^3(1-q^{-1})(1-q^{-1}+q^{-2})\big)\right] \end{aligned} \tag{9}$$

The equality between the terms in (8) and (9) readily follows. This concludes the proof of Proposition 3.6.2. □

3.7. SOME REMARKS

We predict that the identities obtained in 3.6 will extend to all $f \in C_c^\infty(G(F))$. Note that each given orbit corresponding to some partition of the form $\lambda(n;k)$, $k \geqslant k_0$ is induced from an orbit of the form $(\mathbf{1}, O)$, where $\mathbf{1}$ is the trivial orbit in some general linear group, and O is an orbit of the type we have just treated. Moreover, there is a one-to-one correspondence between the rational orbits within O^{st}, and those within the given stable orbits. This one-to-one correspondence is obtained by induction of F-rational orbits. Thus, our prediction carries over to that larger class.

4. Examples and a Conjecture on Transfer Factors for Unipotent Orbital Integrals

4.1. ELLIPTIC UNIPOTENT ENDOSCOPIC DATUM

In this section we first analyze some examples which suggest various features of the transfer factors for the unipotent orbital integrals. Our analysis is based on using the matching results established in [1–3], and the preceding section. These results deal only with certain spherical functions. However, we take our lead from the principle that any identity between unipotent orbital integrals of spherical functions should have a 'natural' extension to all compactly supported and smooth functions. Moreover, these extended indentities have analogues in the ramified situation, by which we mean that the endoscopic group is nonsplit but splits over a ramified extension of the base field. We do not prove any essentially new identities, but we predict, based on our analysis, what the transfer factors should look like in each discussed example. We then present a rough form of the transfer factors, which we then make precise for several families of orbits.

Next, we introduce the concept of *elliptic unipotent endoscopic datum relative to* O_G in a classical split group $\mathbf{G}$.

DEFINITION 4.1.1. An *elliptic unipotent endoscopic datum* consists of a pair $(\mathbf{H}, O_H)$ where

- O_H is a special unipotent orbit in $\mathbf{H}$, with $O_H^{\mathrm{st}} \neq \phi$.
- O_G is a unipotent orbit in $\mathbf{G}$.

such that the following conditions are satisfied

(i) $O_G = \mathrm{Ind}_H^G O_H$ (see def. 2.4.1);
(ii) $\overline{A}(O_H) \cong C(O_G)$, if $\mathbf{G}$ is of type $\mathbf{B}$, and $\overline{A}(O_H) \times \mathbb{Z}/2\mathbb{Z} \cong C(O_G)$, if $\mathbf{G}$ is of type $\mathbf{C}$ or $\mathbf{D}$ (recall that $C(O_G)$ is the group of connected components of the centralizer of some $u \in O_G$).

Remark 4.1.2. Since $\mathbf{G}$ is assumed to be split, we have $O_G^{\mathrm{st}} \neq \phi$.

4.2. EXAMPLES

EXAMPLE 1. Let $\mathbf{G} = \mathbf{SO}(9)$, $\lambda = (5, 3, 1)$. Then O_λ is a special orbit. The PVS associated to λ is given by

$$\mathbf{M}(\lambda) = \mathbf{GL}(1) \times \mathbf{GL}(2) \times \mathbf{SO}(3),$$
$$\mathfrak{g}_2(\lambda) = \mathbf{Mat}(1, 2) \oplus \mathbf{Mat}(2, 3).$$

Let $X = (X_1, X_2) \in \mathfrak{g}_2(\lambda)(F)$, and define

$$Q_1(X) := X_1 X_2 J_3 {}^t X_2 {}^t X_1, \qquad Q_2(X) := X_2 J_3 {}^t X_2.$$

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X is generic $\Leftrightarrow \det(Q_1) \neq 0 \neq \det(Q_2)$. Note that for X generic, $\operatorname{rank}(Q_1(X)) = 1$, $\operatorname{rank}(Q_2(X)) = 2$, hence $I_0(\boldsymbol{\lambda}) = \{1\}$, $I_e(\boldsymbol{\lambda}) = \{2\}$. Thus, there exist four packets within O_λ^{st} determined by the condition $\det Q_1(X) \equiv \tau \bmod (F^\times)^2$. Using Lemma 1.2.5 and Remark 3 (preceeding Lemma 1.2.3), we find that O_λ^{st} contains 10 rational orbits corresponding to 10 pairs of quadratic forms as follows. For $\sigma \in \{1, \varepsilon, \pi, \varepsilon\pi\}$, let $E_\sigma := F(\sqrt{\sigma})$, and let $N_{E_\sigma/F}$ denote the corresponding norm map. Then the ten pairs are $(\langle\sigma\rangle, \langle\rho, 1\rangle)$, where $\sigma \in \{1, \varepsilon, \pi, \varepsilon\pi\}$, and $\rho \in N_{E_\sigma/F}(E_\sigma^\times) \bmod (F^\times)^2$. Let us denote the rational orbit corresponding to the pair $(\langle\sigma\rangle, \langle\rho, 1\rangle)$ by $O_\lambda(\sigma; \rho)$.

Then the four packets within O_λ^{st} are

$$\prod_\sigma := \{O_\lambda(\sigma; \rho) : \rho \in N_{E_\sigma/F}(E_\sigma^\times) \bmod (F^\times)^2\}, \quad \sigma \in \{1, \varepsilon, \pi, \varepsilon\pi\}.$$

The following lemma shows that each $\prod_\sigma$ gives rise to a stable distribution.

LEMMA 4.2.1. *The distribution*

$$f \mapsto \sum_{O \in \prod_\sigma} \int_O f, \quad \sigma \in \{1, \varepsilon, \pi, \varepsilon\pi\},\ f \in C_c^\infty(\mathbf{G}(F))$$

is stable.

Proof. Let $\mathbf{M} := \mathbf{GL}(2) \times \mathbf{SO}(5)$, and $\tau \in F^\times/(F^\times)^2$. Let $O_M(\tau) := (\mathbf{1}, O_{\mathrm{sub}}(\tau))$, where $\mathbf{1}$ is the trivial orbit in $\mathbf{GL}(2, F)$ and $O_{\mathrm{sub}}(\tau)$ is the subregular orbit in $\mathbf{SO}(5, F)$ corresponding to τ (see 3.1.3). Then by Proposition 5.5.1 in [2] the integral over $O_M(\tau)$ is a stable distribution. One then checks that $\mathrm{Ind}_M^G\, O_M(\tau) = \prod_\tau$, $\tau \in \{1, \varepsilon, \pi, \varepsilon\pi\}$. Now, the parabolic induction of a stable distribution is again a stable distribution, and we are done.

Next, we find that there is only one pair $(\mathbf{H}, O_H)$ consisting of an endoscopic group of $\mathbf{G}$, and a special orbit $O_H \subseteq \mathbf{H}$, satisfying $\overline{A}(O_H) = A(O_\lambda) = C(O_\lambda)$, namely

$$\mathbf{H} = \mathbf{SO}(5) \times \mathbf{SO}(5),\ O_H = (O_{\mathrm{sub}}, O_{\mathrm{sub}}) = (O_{311}, O_{311}).$$

Thus O_H^{st} contains 16 orbits forming 16 packets:

$$\sum\nolimits_{\tau,\sigma} := \{(O_{\mathrm{sub}}(\tau), O_{\mathrm{sub}}(\sigma))\}, \quad \tau, \sigma \in \{1, \varepsilon, \pi, \varepsilon\pi\}.$$

The formalism discussed in Section 2 suggests that these 16 packets will transfer to the four packets $\prod_\tau$ as follows. Note that ${}^L\boldsymbol{\lambda} = \{(2, 2, 2, 2)\}$, $S_*({}^L\boldsymbol{\mu}) = \{(2, 2)\}$. Thus Definition 2.5.6. tells us that the packet $\sum_{\tau,\sigma}$ transfers to $\prod_{\tau\sigma}$. Thus, one expects that the integral of f^H over $\sum_{\tau,\sigma}$ should be equal to a linear combination of integrals of f over the various orbits within $\prod_{\tau\sigma}$, the coefficients being the transfer factors. We wish to get some understanding of the transfer factors involved.

LEMMA 4.2.2. *Let f be an element of the three-dimensional space spanned by the spherical functions $f_{(0,0,0)}$, $f_{(1,1,0)}$, $f_{(2,0,0)}$ (which we considered in Section 3). Then*

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for each $\sigma \in F^\times/(F^\times)^2$, $\exists * \neq 0$, *such that*

$$\sum_{\tau\in F^\times/(F^\times)^2} \int_{(O_{\text{sub}}(\tau), O_{\text{sub}}(\sigma))} f^H = * \sum_{\tau,\rho\in F^\times/(F^\times)^2} \langle \rho, \sigma\rangle \int_{O_\lambda(\tau\sigma;\rho)} f .$$

Here, $\langle\ ,\ \rangle$ denotes the Hilbert pairing on $F^\times/(F^\times)^2$, (not to be confused with rank 2 quadratic forms notation).

Proof. Let $\mathbf{M}_H := \mathbf{GL}(1) \times \mathbf{SO}(3) \times \mathbf{SO}(5)$, $\mathbf{M}_G := \mathbf{GL}(1) \times \mathbf{SO}(7)$. Then the following relations are satisfied:

(i) $\int_{(\mathbf{1}, O_{\text{sub}}(\sigma))} \varphi^{M_H} = * \sum_{\tau\in F^\times/(F^\times)^2} \int_{(O_{\text{sub}}(\tau), O_{\text{sub}}(\sigma))} \varphi$, $\varphi \in C_c^\infty(\mathbf{H}(F))$, $\sigma \in F^\times(F^\times)^2$

(ii) $\int_{(\mathbf{1}, O_{\text{sub}}(\sigma))} f^{M_H} = * \sum_{\tau\in F^\times/(F^\times)^2} \langle\sigma, \tau\rangle \int_{O_{331}(\tau)} f$,

for all f in the three dimensional space indicated in the statement of the lemma, and every $\sigma \in F^\times(F^\times)^2$.

(iii) $\int_{O_{331}(\tau)} \psi^{M_G} = * \sum_{\rho\in F^\times/(F^\times)^2} \int_{O_\lambda(\rho,\tau)} \psi$, $\psi \in C_c^\infty(\mathbf{G}(F))$, $\tau \in F^\times/(F^\times)^2$.

Here, and below, $*$ is used as a 'generic' constant which depends only on the normalization of measure. Identity (ii) follows from the work done in Section 3. Identities (i) and (iii) are consequence of a descent argument.

Remark 4.2.3. Note that since $\mathbf{GL}(2) \times \mathbf{SO}(5)$ may be embedded in both $\mathbf{H}$ and $\mathbf{G}$ as a Levi subgroup, we immediately see from the proof of Lemma 4.2.1 that we have the identity

$$\int_{(O_{\text{sub}}(1), O_{\text{sub}}(\tau))} f^H = * \sum_{\sigma\in F^\times/(F^\times)^2} \int_{O_\lambda(\sigma,\tau)} f ,$$

for all $f \in C_c^\infty(\mathbf{G}(F))$, and all $\tau \in F^\times/(F^\times)^2$. Lemma 4.2.2, and Remark 4.2.3 suggest that the following matching result will hold: $\forall f \in C_c^\infty(\mathbf{G}(F))$, and $\forall \sigma, \tau \in F^\times/(F^\times)^2$, we have

$$\int_{(O_{\text{sub}}(\sigma), O_{\text{sub}}(\tau))} f^{H_1} = * \sum_{\rho\in F^\times/(F^\times)^2} \langle \rho, \tau\rangle \int_{O_\lambda(\sigma\tau,\rho)} f .$$

Note that if $\rho \neq N_{E_{\sigma\tau}/F}(E_{\sigma\tau}^\times)$, then $O_\lambda(\sigma\tau, \rho) = \phi$. This observation can be used to show that the right hand side is in fact symmetric in σ and τ.

EXAMPLE 2. In this example we try to argue that there is another ingredient contributing to the transfer factors which appears when the η-exponent of O, $\eta(O)$, is larger than 0. Note that in Example 1, we had $\eta(O) = 0$.

Let $\mathbf{G} = \mathbf{SO}(9)$, $\boldsymbol{\lambda} = (3, 3, 1, 1, 1)$. O_λ has been studied in Section 3. It splits into seven orbits, corresponding to the seven equivalence classes of quadratic forms of rank 2. By Lemma 3.1.1, the Lusztig quotient group $\overline{A}(O_\lambda)$ is trivial, hence O_λ^{st} is a packet. The seven orbits within O_λ^{st} are denoted by $O_\lambda(\tau; \eta)$, where

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$\tau \in F^{\times}/(F^{\times})^2$, and $\eta \in \{\pm 1\}$. Here, of course, we understand that $O_\lambda(1,-1) = \phi$ if $q \equiv 1 \bmod 4$, and that $O_\lambda(1,1) = \phi$ if $q \equiv 3 \bmod 4$. In Section three we considered the pair $(\mathbf{H}_1, O_{H_1})$, where $\mathbf{H}_1 := \mathbf{SO}(5) \times \mathbf{SO}(5)$, $O_{H_1} := (\mathbf{1}, O_{311}) = (\mathbf{1}, O_{\mathrm{sub}})$. The result obtained there can be phrased as following:

$$\int_{(\mathbf{1}, O_{\mathrm{sub}}(\tau)} f^{H_1} = * \sum_{\substack{\rho \in F^{\times}/(F^{\times})^2 \\ \eta \in \{\pm 1\}}} \langle \tau, \rho \rangle \int_{O_\lambda(\rho;\eta)} f\ ,$$

where f is spherical function belonging to a certain three dimensional space. This identity is expected to hold for all $f \in C_c^\infty(\mathbf{G}(F))$. There is, however, another pair $(\mathbf{H}_2, O_{H_2})$ with $\mathrm{Ind}_{H_2}^G O_{H_2} = O_G$, namely: $\mathbf{H}_2 := \mathbf{SO}(3) \times \mathbf{SO}(7)$, $O_{H_2} := (\mathbf{1}, O_{31111})$. Note that O_{31111}^{st} breaks up into four orbits, denoted $O_{31111}(\sigma)$, $\sigma \in F^{\times}/(F^{\times})^2$, forming four packets. In this situation, we expect the following matching result to hold

$$\int_{(\mathbf{1}, O_{31111}(\tau))} f^{H_2} = * \sum_{\substack{\rho \in F^{\times}/(F^{\times})^2 \\ \eta \in \{\pm 1\}}} \mathrm{sgn}\,(\eta) \langle \tau, \rho \rangle \int_{O_\lambda(\rho,\eta)} f\ ,$$

for all $f \in C_c^\infty(G(F))$, and all $\tau \in F^{\times}/(F^{\times})^2$.

This prediction is consistent with the following considerations:

(i) $\sum_{\tau \in F^{\times}/(F^{\times})^2} f^{H_2} = * \int_{O_\lambda(1,1)} f$, f spherical.

This identity follows from the following facts:

(1) O_{31111}^{st} is induced from the trivial orbit in $\mathbf{L} := \mathbf{SO}(3, F) \times \mathbf{SO}(5, F)$.

(2) The trivial orbit in $\mathbf{SO}(3) \times \mathbf{SO}(5)$ endoscopically induces to the orbit O_{22111}^{st} in $\mathbf{SO}(7, F)$, and moreover, by the results of [3], we have $f^L(\mathbf{1}) = * \int_{O_{22111}^{\mathrm{st}}} f$, for f spherical on $\mathbf{SO}(7, F)$

(3) $\mathrm{Ind}_{\mathrm{GL}(1) \times \mathrm{SO}(7)}^{\mathrm{SO}(9)} O_{2211}^{\mathrm{st}} = O_\lambda(1, 1)$.

(ii) The transfer factors in the two identities suggested above, if true, will allow for the expression of the integral over each rational class within O_λ^{st} to be expressed as a linear combination of stable unipotent orbital integrals over the packets within $O_{H_1}^{\mathrm{st}}$ and $O_{H_2}^{\mathrm{st}}$.

(iii) In [14] Waldspurger poses a question (Question 3.1) regarding the dimension of spaces of unipotent orbital integrals, restricted to the Iwahori–Hecke algebra. He then suggests that the similar question with the spherical–Hecke algebra replacing the latter should have the same answer. An affirmative answer to his question(s) implies the following:

(1) The space spanned by the restrictions to the spherical Hecke algebra of $\mathbf{SO}(9, F)$ of the integrals over the seven rational orbits within O_λ^{st} is four-dimensional, and, moreover, one has the following identities:

(i) $\int_{O_\lambda(\pi,1)} f = \int_{O_\lambda(\varepsilon\pi,1)} f$, $\int_{O_\lambda(\pi,-1)} f = \int_{O_\lambda(\varepsilon\pi,-1)} f$, for any spherical f on $\mathbf{SO}(9, F)$.

(2) It is known from [1] that the space spanned by the restrictions to the spherical Hecke algebra of $\mathbf{SO}(7, F)$ of the integrals over the four rational orbits within O^{st}_{31111} is three-dimensional, and moreover, one has the following identity:

$$\int_{O_{31111}(\pi)} f = \int_{O_{31111}(\varepsilon\pi)} f, \qquad \text{for any spherical } f \text{ on } (\mathbf{SO}(7, F)) .$$

Our third consideration is that the above identities are consistent with the suggested transfer factors.

EXAMPLE 3. In this example, expectedly, only the ingredient related to the Hasse-invariant will make a contribution to the transfer factor.

Let $\mathbf{G} = \mathbf{SO}(11)$, $\boldsymbol{\lambda} = (3, 3, 3, 1, 1)$. The PVS associated with $\boldsymbol{\lambda}$ is given by $\mathbf{M}(\boldsymbol{\lambda}) = \mathbf{GL}(3) \times \mathbf{SO}(5)$, $\mathfrak{g}_2(\boldsymbol{\lambda}) = \mathbf{Mat}\,(3, 5)$, with the usual action. It is clear then that $A(\boldsymbol{\lambda}) = \overline{A}(\boldsymbol{\lambda}) \cong \mathbb{Z}/2\mathbb{Z}$ (see Remark 2.2.3). Moreover, $O^{\mathrm{st}}_{\lambda}$ splits into seven rational orbits which will be denoted by $O_{\lambda}(\tau, \eta)$, $\tau \in F^{\times}/(F^{\times})^2$ and $\eta \in \{\pm 1\}$. Here we are following the convention that $O_{\lambda}(1, \pm 1) = \phi$ if $q \equiv \mp 1 \bmod (F^{\times})^2$. Let $\mathbf{H} := \mathbf{SO}(3) \times \mathbf{SO}(9)$, $O_H := (\mathbf{1}, O_{32211})$. The next lemma contains information about O_G and O_H which we shall use.

LEMMA 4.2.4.

(i) *O_{λ} is a Richardson orbit with respect to two Levi subgroup, namely,* $\mathbf{M}_1 := \mathbf{GL}(4) \times \mathbf{SO}(3)$, *and* $\mathbf{M}_2 := \mathbf{GL}(3) \times \mathbf{SO}(5)$. *Moreover, we have (assuming* $q \equiv 1 \bmod 4$, *for simplicity).*
  [1] $\mathrm{Ind}^G_{M_1} \mathbf{1} = O_{\lambda}(1, 1)$,
  [2] $\mathrm{Ind}^G_{M_2} \mathbf{1} = O^{\mathrm{st}}_{\lambda}$,

(ii) $\mathrm{Ind}^G_H(\mathbf{1}, O_{32211}) = O_{\lambda}$. □

Note that O^{st}_{32211} splits into four orbits $O_{32211}(\tau)$, $\tau \in F^{\times}/(F^{\times})^2$, forming four packets. On the other hand $O^{\mathrm{st}}_{\lambda}$ splits into four packets $\prod_{\tau} := \{O(\tau, \eta) : \eta = \pm 1\}$, $\tau \in F^{\times}/(F^{\times})^2$. According to the formalism explained in Section 2, each packet $\{(\mathbf{1}, O_{32211}(\tau))\}$ transfers to $\prod_{\tau}$. In fact, this can be proven for $\tau = 1$. Indeed, since $\mathbf{GL}(4) \times \mathbf{SO}(3)$ embeds into both $\mathbf{H}$ and $\mathbf{G}$ as a Levi subgroup, it follows from Lemma 4.2.4(i),(ii), that

$$\int_{\left(\mathbf{1}, O_{32211}(1)\right)} f^H = * \int_{O_{\lambda}(1,1)} f, \quad f \in C_c^{\infty}(\mathbf{G}(F))$$

and we expect the following identities to hold

$$\int_{\left(\mathbf{1}, O_{32211}(\tau)\right)} f^H = * \sum_{\eta \in \{\pm 1\}} \mathrm{sgn}\,(\eta) \int_{O_{\lambda}(\tau,\eta)} f, \quad f \in C_c^{\infty}(\mathbf{G}(F))$$

This prediction is further supported by the following two considerations:

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(i) This prediction allows, us in a natural way, to express the integral over any rational orbit within O_λ^{st}, as a linear combination of stable unipotent orbital integrals over O_G^{st} and O_H^{st}.

(ii) An affirmative answer to Waldspurger's question (see consideration (iii) in Example 2) would imply the following:

 (1) The space spanned by the restrictions to the spherical Hecke algebra of $\mathbf{SO}(11, F)$ of the integrals over the seven rational orbits within O_λ^{st} is four dimensional and, moreover, one has the following identities:

$$\int_{O_\lambda(\pi,1)} f = \int_{O_\lambda(\varepsilon\pi,1)} f, \qquad \int_{O_\lambda(\pi,-1)} f = \int_{O_\lambda(\varepsilon\pi,-1)} f,$$

 for any spherical f on $\mathbf{SO}(11, F)$.

 (2) The space spanned by the restrictions to the spherical Hecke algebra of $\mathbf{SO}(9, F)$ of the integrals over the four rational orbit within O_{32211}^{st} is three-dimensional, and moreover, one gets the following identity: $\int_{O_{31111}(\pi)} = \int_{O_{31111}(\varepsilon\pi)} f$,for any spherical f on $\mathbf{SO}(9, F)$.

Note, then, that the suggested transfer factors are consistent with the above identities.

EXAMPLE 4. Let $\mathbf{G} = \mathbf{SO}(11)$, and $\boldsymbol{\lambda} := (3, 3, 2, 2, 1)$. Then O_λ is a nonspecial orbit, and O_λ^{st} breaks up into four rational orbits: $O_\lambda(\tau)$, $\tau \in F^\times/(F^\times)^2$. Note that O_λ^{st} is one whole packet. We wish to give evidence to the effect that the transfer of any stable unipotent orbital integral to O_λ^{st} will involve a linear combination of integrals over *every* rational class within O_λ^{st}. The only pair $(\mathbf{H}, O_H)$ with $\mathrm{Ind}_H^G O_H = O_\lambda$, and $\overline{A}(O_H) \equiv A(O_\lambda)$ is the following: $\mathbf{H} := \mathbf{SO}(5) \times \mathbf{SO}(7)$, $O_H := (\mathbf{1}, O_{31111})$. Now, O_{31111}^{st} contains four rational orbits forming four packets. The rational classes within O_{31111}^{st} will be denoted by $O_{31111}(\tau)$, $\tau \in F^\times/(F^\times)^2$. We shall make use of the following lemma. (For simplicity, we assume $q \equiv 1 \bmod 4$.)

LEMMA 4.2.5.

(i) *Let O_{22221} denote the unique rational orbit within $O_{22221}^{st} \subseteq \mathbf{SO}(9, F)$. Then* $\mathrm{Ind}_{\mathrm{GL}(1)\times\mathrm{SO}(9)}^{\mathrm{SO}(11)}(\mathbf{1}, O_{22221}) = O_\lambda(1)$.

(ii) *Let $\mathbf{H}' := \mathbf{SO}(5) \times \mathbf{SO}(5)$. Then $\mathbf{H}'$ is an endoscopic group of $\mathbf{G}' := \mathbf{SO}(9)$, and* $\mathrm{Ind}_{H'}^{G'} \mathbf{1} = O_{22221}$. *Moreover, we have $f^{H'}(1) = *\int_{O_{22221}} f$, for any spherical f on $\mathbf{G}'(F)$.*

(iii) $\sum_{\sigma\in F^\times/(F^\times)^2} \int_{O_\lambda(\sigma)} f = *\int_{O_{31111}(1)} f^H$, *for any spherical f on $\mathbf{G}(F)$.*

Proof.

(i) Omitted.

(ii) This is a special case of the main result proven in [3].

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(iii) This follows from (i), (ii), and the fact that O^{st}_{31111} is induced from the trivial orbit in $\mathbf{GL}(1,F)\times\mathbf{SO}(5,F)$. □

We predict the following identities to hold

$$\int_{(\mathbf{1},O_{31111}(\tau))} f^H =* \sum_{\sigma\in F^\times/(F^\times)^2} \langle\sigma,\tau\rangle \int_{O_\lambda(\sigma)} f\ ,$$

for all $f\in C_c^\infty(\mathbf{G}(F))$.

We base our prediction on the following considerations:

(i) The given prediction allows us to express the integral over each rational orbit within O^{st}_λ in terms of stable orbital integrals over O^{st}_H.
(ii) It is consistent with Lemma 4.2.5.
(iii) An affirmative answer to Waldspurger's question would imply that the restrictions of the four integrals over $O_\lambda(\tau)$ to the spherical Hecke algebra do span a three-dimensional space. Moreover, one has $\int_{O_\lambda(\pi)} f=\int_{O_\lambda(\varepsilon\pi)} f$, for all spherical f on $\mathbf{G}(F)$. Similarly, it was shown in [1], that the space spanned by restricting the four integrals over $O_{31111}(\sigma)$ to the spherical Hecke algebra is three- dimensional. Moreover, one has $\int_{O_\lambda(\pi)} f=\int_{O_\lambda(\varepsilon\pi)} f$, for all spherical f on $\mathbf{SO}(7,F)$. Our third consideration is that the above two relations are consistent with the given transfer factors.

EXAMPLE 5. Let $\mathbf{G}:=\mathbf{SO}(13)$, and $\boldsymbol{\lambda}:=(4,4,3,1,1)$. Then O_λ is a nonspecial orbit, and O^{st}_λ splits into four rational orbits, denoted by $O_\lambda(\tau)$, $\tau\in F^\times/(F^\times)^2$, forming four packets. Let $\mathbf{H}:=\mathbf{SO}(3)\times\mathbf{SO}(11)$, and $O_H:=(\mathbf{1},O_{33311})$. Note that O_{33311} is special, and that O^{st}_{33311} splits into four rational orbits: $O_{33311}(\tau)$, $\tau\in F^\times/(F^\times)^2$, forming four packets. The formalism is Section 2 predicts that the packet $\{(\mathbf{1},O_{33311}(\tau))\}$ will transfer to the packet $\{O_\lambda(\tau)\}$, $\forall\tau\in F^\times/(F^\times)^2$.

We expect the following identities to hold

$$\int_{(\mathbf{1},O_{33311}(\tau))} f^H =* \int_{O_{44311}(\tau)} f,\quad f\in C_c^\infty(\mathbf{G}(F))\ .$$

We offer the following consideration as a support for the above prediction:

(i) The first consideration is the following Lemma (we assume $q\equiv 1 \bmod 4$).

LEMMA 4.2.6. *For any spherical function f on $\mathbf{G}(F)$, we have*
(a) $\int_{(\mathbf{1},O_{33311}(1))} f^H =* \int_{O_{44311}(1))} f$,
(b) $\sum_{\tau\in F^\times/(F^\times)^2} \int_{(\mathbf{1},O_{33311}(\tau))} f^H =* \sum_{\tau\in F^\times/(F^\times)^2} \int_{O_{44311}(\tau)} f$

Proof. Identity (a) follows from applying the descent Lemma 3.3 to the data (1)–(3) below

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(1) $\mathrm{Ind}_{\mathrm{GL}(4)\times\mathrm{SO}(3)}^{\mathrm{SO}(13)}(\mathbf{1},\mathbf{1}) = O_{33311}(\mathbf{1})$.

(2) Set $\mathbf{G}' = \mathbf{SO}(5)$, and $\mathbf{H}' = \mathbf{SO}(3)\times\mathbf{SO}(3)$. Then $\int_{O_{221}} f = *f^{H'}(\mathbf{1})$, for all spherical f on $\mathbf{G}'(F)$, (see [1]). Here O_{221} is the unique rational orbit within the stable orbit O_{221}^{st}.

(3) $\mathrm{Ind}_{\mathrm{GL}(4)\times\mathrm{SO}(5)}^{\mathrm{SO}(13)}(\mathbf{1}, O_{221}) = O_{44311}(1)$.

Indentity (b) follows from applying Lemma 3.3 to the following to the data (4)–(6) below

(4) $\mathrm{Ind}_{\mathrm{GL}(3)\times\mathrm{SO}(5)}^{\mathrm{SO}(11)}(\mathbf{1},\mathbf{1}) = O_{33311}^{\mathrm{st}}$.

(5) Set $\mathbf{G}'' := \mathbf{SO}(7)$, and $\mathbf{H}'' := \mathbf{SO}(3)\times\mathbf{SO}(5)$. Then $\int_{O_{22111}} f = *f^{H''}(\mathbf{1})$, for all spherical f on $\mathbf{G}'(F)$, (see [1]). Here, O_{22111} is the unique rational orbit contained in O_{22111}^{st}.

(6) $\mathrm{Ind}_{\mathrm{GL}(3)\times\mathrm{SO}(7)}^{\mathrm{SO}(13)}(\mathbf{1}, O_{22111}) = O_{44311}^{\mathrm{st}}$. □

(ii) As a second piece of evidence, we observe that an affirmative answer to Waldspurger's question would imply the following identities which are consistent with the predicted identity:

$$\int_{O_{33311}(\pi)} f = \int_{O_{33311}(\varepsilon\pi)} f, \text{ for all spherical } f \text{ on } \mathbf{SO}(11,F),$$
$$\int_{O_{44311}(\pi)} f' = \int_{O_{44311}(\pi)} f', \text{ for all spherical } f' \text{ on } \mathbf{SO}(13,F).$$

EXAMPLE 6. Let $\mathbf{G} := \mathbf{Sp}(12)$, and $\boldsymbol{\lambda} := (4,4,2,2)$. Then O_λ is a special orbit. The PVS associated to $\boldsymbol{\lambda}$ is given by

$$\mathbf{M}(\boldsymbol{\lambda}) = \mathbf{GL}(2)\times\mathbf{GL}(4), \qquad \mathfrak{g}_2(\boldsymbol{\lambda}) = \mathbf{Mat}\,(2,4)\oplus\mathbf{sym}(4),$$

$$(g,h)\cdot(X,S) = (gXh^{-1}, hS^t h), \quad (g,h)\in\mathbf{M}(\boldsymbol{\lambda}), \quad (X,S)\in\mathfrak{g}_2(\boldsymbol{\lambda}).$$

For $(X,S)\in\mathfrak{g}_2(\boldsymbol{\lambda})(F)$, define

$$Q_1(X,S) := XS^tX, \qquad Q_2(X,S) := S,$$

and set $\Delta_i(X,S) := \det Q_i(X,S)$, $i=1,2$. The set $\{\Delta_1,\Delta_2\}$ is then a set of fundamental relative invariants for the PVS $(\mathbf{M}(\boldsymbol{\lambda}),\mathfrak{g}_2(\boldsymbol{\lambda}))$. The stable orbit O_λ^{st} splits in 49 rational orbits determined by the equivalence classes of pairs of quadratic forms: $(Q_1(X,S), Q_2(X,S))$, (X,S) a generic point in $\mathfrak{g}_2(\boldsymbol{\lambda})(F)$. The pairs of quadratic forms obtained in this way can be easily found using Lemma 1.2.9. Let q_1, q_2 be two quadratic forms of rank 4 and 2 respectively, and assume that they arise from some generic point in $\mathfrak{g}_2(\lambda)(F)$. Let δ_1,δ_2 denote the discriminant of q_1,q_2, respectively, and let ζ_1,ζ_2 denote the Hasse-invariant of q_1,q_2, respectively. The rational orbit $O\subseteq O_\lambda^{\mathrm{st}}$ corresponding to (q_1,q_2) will be denoted by $O_\lambda(\delta_1,\zeta_1;\delta_2,\zeta_2)$. Note that $\overline{A}(\boldsymbol{\lambda}) = A(\boldsymbol{\lambda}) = C(\boldsymbol{\lambda})\cong(\mathbb{Z}/2\mathbb{Z})^2$. The stable orbit O_λ, therefore, breaks up into 16 packets $\prod_{\sigma,\tau}$ where, $(\tau,\sigma)\in[F^\times/(F^\times)^2]^2$, as follows. $\prod_{\sigma,\tau} := \{O_\lambda(\sigma,\eta;\tau,\zeta)\subseteq O_\lambda^{\mathrm{st}} : \eta,\zeta\in\{\pm1\}\}$.

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Next, it can be shown that there exists four pairs $(\mathbf{H}, O_H)$ such that: (a) $\mathbf{H}$ is an elliptic endoscopic group of $\mathbf{G}$, (b) $\mathrm{Ind}_H^G O_H = O_\lambda$, (c) $\overline{A}(O_H) \times \mathbb{Z}/2\mathbb{Z} \cong C(O_\lambda)$. They are given by the following list:

(1) $\mathbf{H}^1 = \mathbf{G}$, $O_{H^1} := O_\lambda$.
(2) $\mathbf{H}^2 = \mathbf{Sp}(10) \times \mathbf{SO}(2)$, $O_{H^2} := (O_{3322}, \mathbf{1})$.
(3) $\mathbf{H}^3 = \mathbf{Sp}(6) \times \mathbf{SO}(6)$, $O_{H^3} := (O_{2211}, O_{2211})$.
(4) $\mathbf{H}^4 = \mathbf{Sp}(4) \times \mathbf{SO}(8)$, $O_{H^4} := (\mathbf{1}, O_{3311})$.

Of course, when studying the transfer of packets, we need to consider all quasi-split inner forms of $\mathbf{H}^i$, $i = 2, 3, 4$.

Next, we need to study the transfer factor for each of the four cases above. There is no mystery about (1). So, we consider only the last three data. Let us first explicate (in these cases) the packet transfer explained in Section 2.

First we have

$$
{}^L\boldsymbol{\lambda} = (5, 3, 3, 1, 1) = \hat{\boldsymbol{\lambda}}, \qquad S({}^L\boldsymbol{\lambda}) = \{(5), (5, 3, 3), (5, 3, 3, 1, 1)\},
$$
$$
S_*({}^L\boldsymbol{\lambda}) = \{((533), (53311)\}.
$$

Next, write $\mathbf{H}^i = \mathbf{H}_1^i \times H_2^i$ $(i = 2, 3, 4)$, where $\mathbf{H}_1^i$ is the symplectic component of $\mathbf{H}^i$ and $\mathbf{H}_2^i$ is the orthogonal component of $\mathbf{H}^i$. We also write $(\boldsymbol{\mu}_1^i, \boldsymbol{\mu}_2^i)$ to denote the pair of partitions corresponding to the orbit O_{H^i} which endoscopically transfers to O_λ.

Now, we have the following data:

- O_{3322}^{st} splits into four rational orbits, denoted $O_{3322}(\tau)$, $\tau \in F^\times/(F^\times)^2$, forming four packets.

$$
{}^L\boldsymbol{\mu}_1^2 = (5, 3, 3), \qquad S({}^L\boldsymbol{\mu}_1^2) = \{(5), (5, 3, 3)\}, \qquad S_*({}^L\boldsymbol{\mu}_1^2) = \{(5, 3, 3)\},
$$
$$
{}^L\boldsymbol{\mu}_2^2 = (1, 1), \qquad S({}^L\boldsymbol{\mu}_2^2) = \{(1, 1)\}, \qquad S_*({}^L\boldsymbol{\mu}_2^2) = \phi.
$$

- O_{2211} (as an orbit in $\mathbf{Sp}(6)$) splits into four orbits: $O_{2211}(\tau)$, $\tau \in F^\times/(F^\times)^2$, forming four packets.

$$
{}^L\boldsymbol{\mu}_1^3 = (5, 1, 1), \qquad S({}^L\boldsymbol{\mu}_1^3) = \{(5), (5, 1, 1)\}, \qquad S_*({}^L\boldsymbol{\mu}_1^3) = \{(5, 1, 1)\}.
$$
$$
{}^L\boldsymbol{\mu}_2^3 = (3, 3), \qquad S({}^L\boldsymbol{\mu}_2^3) = \{(3, 3)\}, \qquad S_*({}^L\boldsymbol{\mu}_2^3) = \phi.
$$

- O_{3311} splits into four rational orbits: $O_{3311}(\tau)$, $\tau \in F^\times/(F^\times)^2$, forming four packets.

$$
{}^L\boldsymbol{\mu}_1^4 = (5), \qquad S({}^L\boldsymbol{\mu}_1^4) = \{(5)\}, \qquad S_*({}^L\boldsymbol{\mu}_1^4) = \phi. \qquad {}^L\boldsymbol{\mu}_2^4 = (3, 3, 1, 1),
$$
$$
S({}^L\boldsymbol{\mu}_2^4) = \{(33), (3, 3, 1, 1)\}, \qquad S_*({}^L\boldsymbol{\mu}_2^4) = \{(3, 3, 1, 1)\}.
$$

Now, for $i = 2, 3, 4$ and $\sigma \in F^\times/(F^\times)^2$, let $\mathbf{H}^{i,\sigma}$ denote the inner quasi-split form of $\mathbf{H}^i$ which splits over E_σ
(but not over F if $\sigma \not\equiv 1 \bmod (F^\times)^2$).

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Now, using the above data, and the recipe for transfer given in Section 2, we get, for $\sigma \in F^\times/(F^\times)^2$

– If $\mathbf{H} = \mathbf{H}^{2,\sigma}$, then the packet $\{(O_{3322}(\tau), \mathbf{1})\}$ transfers to the packet $\prod_{\sigma,\tau\sigma}$, $\tau \in F^\times/(F^\times)^2$.
– If $\mathbf{H} = \mathbf{H}^{3,\sigma}$, then the packet $\{(O_{2211}(\tau), O_{2211})\}$ transfers to the packet $\prod_{\tau\sigma,\sigma}$, $\tau \in F^\times/(F^\times)^2$.
– If $\mathbf{H} = \mathbf{H}^{4,\sigma}$, then the packet $\{(\mathbf{1}, O_{3311}(\tau))\}$ transfers to the packet $\prod_{\tau\sigma,\sigma}$, $\tau \in F^\times/(F^\times)^2$.

We predict the following identities to hold (with $q \equiv 1 \bmod 4$)

(1) If $\mathbf{H} = \mathbf{H}^{2,\sigma}$ and $\tau \in F^\times/(F^\times)^2$, then

$$\int_{(O_{3322}(\tau),\mathbf{1})} f^H = * \sum_{\zeta,\eta\in\{\pm 1\}} \operatorname{sgn}(\eta) \int_{O_\lambda(\sigma,\eta;\tau\sigma,\zeta)} f\ ,$$

$f \in C_c^\infty(\mathbf{G}(F))$, $\sigma \in F^\times/(F^\times)^2$.

(2) If $\mathbf{H} = \mathbf{H}^{3,\sigma}$ and $\tau \in F^\times/(F^\times)^2$, then

$$\int_{(O_{2211}(\tau),O_{2211})} f^H = * \sum_{\zeta,\eta\in\{\pm 1\}} \operatorname{sgn}(\eta)\operatorname{sgn}(\zeta) \int_{O_\lambda(\tau\sigma,\eta;\sigma,\zeta)} f\ ,$$

$f \in C_c^\infty(\mathbf{G}(F))$, $\sigma \in F^\times/(F^\times)^2$.

(3) If $\mathbf{H} = \mathbf{H}^{4,\sigma}$ and $\tau \in F^\times/(F^\times)^2$, then

$$\int_{(\mathbf{1},O_{3311})} f^H = * \sum_{\zeta,\eta\in\{\pm 1\}} \operatorname{sgn}(\zeta) \int_{O_\lambda(\tau\sigma,\eta;\sigma,\zeta)} f\ ,$$

$f \in C_c^\infty(\mathbf{G}(F))$, $\sigma \in F^\times/(F^\times)^2$.

The above predictions are motivated by two following considerations:

LEMMA 4.2.7. *Let $\sigma \in \{1, \varepsilon\}$ and $\mathbf{H}_\sigma = \mathbf{H}^{2,\sigma}$. Then for any spherical f on $\mathbf{G}(F)$, we have*

(a) $\int_{(O_{3322}(1),\mathbf{1})} f^{H_\sigma} = * \sum_{\zeta,\eta\in\{\pm 1\}} \operatorname{sgn}(\eta) \int_{O_\lambda(\sigma,\eta;\sigma,\zeta)} f\ ,$

(b) $\sum_{\tau\in F^\times/(F^\times)^2} \int_{(O_{3322}(\tau),\mathbf{1})} f^{H_\sigma} = * \sum_{\substack{\zeta,\eta\in\{\pm 1\} \\ \tau\in F^\times/(F^\times)^2}} \operatorname{sgn}(\eta) \int_{O_\lambda(\sigma,\eta;\sigma\tau,\zeta)} f\ .$

Proof. Identity [a] follows from applying the descent lemma 3.3. to the following data:

(i) $\operatorname{Ind}_{GL(3)\times Sp(4)}^{Sp(10)} \mathbf{1} = O_{3322}\ (1)$,

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(ii) Let $\mathbf{G}' := \mathbf{Sp}(6)$, and $\mathbf{H}'_\sigma := \mathbf{Sp}(4) \times \mathbf{U}_{E_\sigma}(1)$, $\sigma = 1$ or 2. Then we form a spherical f on $\mathbf{G}'(F)$, we have ([2])

$$f^{H'_\sigma}(1) = \sum_{\eta \in \{\pm 1\}} \operatorname{sgn}(\eta) \int_{O_{2211}(\sigma,\eta)} f ,$$

(iii) $\operatorname{Ind}_{\mathrm{GL}(3)\times\mathrm{Sp}(6)}^{\mathrm{Sp}(10)} (\mathbf{1}, O_{2211}(\sigma,\eta)) = O_\lambda(\sigma,\eta;\sigma,\zeta)$, $\quad \sigma, \tau \in F^\times/(F^\times)^2$. Identity [b] follows from Lemma 3.3 and the following data:

(iv) $\operatorname{Ind}_{\mathrm{GL}(4)\times\mathrm{SL}(2)}^{\mathrm{Sp}(10)} \mathbf{1} = O_{3322}^{\mathrm{st}}$,

(v) Let $\mathbf{G}'' = \mathbf{Sp}(4)$, $\mathbf{H}''_\sigma := \mathbf{SL}(2) \times \mathbf{U}_{E_\sigma}(1)$, $\sigma = \epsilon, 1$. Then for a spherical f on $\mathbf{G}''(F)$, we have (see [2])

$$f^{H''}(1) = \sum_{\eta \in \{\pm 1\}} \operatorname{sgn}(\eta) \int_{O_{22}(\sigma,\eta)} f$$

(vi) $\operatorname{Ind}_{\mathrm{GL}(4)\times\mathrm{Sp}(4)}^{\mathrm{Sp}(12)} (\mathbf{1}, O_{22}(\sigma,\eta)) = \bigsqcup_{\substack{\zeta,\eta\in\{\pm1\} \\ \tau\in F^\times/(F^\times)^2}} O_\lambda(\sigma,\eta;\sigma,\tau,\zeta)$, $\sigma \in F^\times/(F^\times)^2$. $\square$

LEMMA 4.2.8. *Let* $\sigma \in \{1, \varepsilon\}$, *and* $\mathbf{H} := \mathbf{H}^{4,\sigma}$. *Then for any spherical* f *on* $\mathbf{G}(F)$, *we have*

(a) $\int_{(\mathbf{1}, O_{3311}(1,1))} f^{H_\sigma} = * \sum_{\zeta,\eta\in\{\pm1\}} \operatorname{sgn}(\zeta) \int_{O_\lambda(\sigma,\eta;\sigma,\zeta)} f$,

(b) $\sum_{\tau\in F^\times/(F^\times)^2} \int_{(\mathbf{1}, O_{3311}(\tau))} f^{H_\sigma} = * \sum_{\substack{\zeta,\eta\in\{\pm1\} \\ \tau\in F^\times/(F^\times)^2}} \operatorname{sgn}(\zeta) \int_{O_\lambda(\sigma,\eta;\sigma\tau,\zeta)} f$.

Proof. The proof is similar in spirit to the one given to Lemma 4.2.7. We omit it. $\square$

4.3. A CONJECTURE ON TRANSFER FACTORS

In this section we shall present a conjecture which partially describes the transfer factors for the unipotent orbital integrals in classical split groups. First we recall some notation and introduce some conventions which will facilitate our presentation.

Let $\mathbf{G}$ be a symplectic or a split special orthogonal group. Let $\boldsymbol{\lambda}$ be a partition corresponding to a unipotent orbit O_λ (not necessarily special) in $\mathbf{G}$. Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\circ \cup \boldsymbol{\lambda}^e$ be the decomposition of $\boldsymbol{\lambda}$ into odd and even parts, and set $\boldsymbol{\lambda}^* := \boldsymbol{\lambda}^\circ$ if $\mathbf{G}$ is orthogonal, and $\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^e$ if $\mathbf{G}$ is symplectic. Write $\boldsymbol{\lambda}^* =: (\lambda_1^{a_1}, \ldots, \lambda_s^{a_s})$. In Section 1, we associated to the PVS $(\mathbf{M}(\boldsymbol{\lambda}^*), \mathfrak{g}_2(\boldsymbol{\lambda}^*))$, a set of functions $Q_1, \ldots, Q_t$ defined on the set of generic points of $\mathfrak{g}_2(\boldsymbol{\lambda}^*)(F)$. Here $t = s - 1$ if $\mathbf{G}$ is orthogonal, and $t = s$ if $\mathbf{G}$ is symplectic. These functions were used to classify the rational orbits within O_λ^{st} as follows. If $O \subseteq O_\lambda^{\mathrm{st}}$, and v is a generic point in $\mathfrak{g}_2(\boldsymbol{\lambda}^*)(F)$ whose $\mathbf{M}(\boldsymbol{\lambda}^*)(F)$ orbit intersects O non-trivially, then the set $Q_1(v), \ldots, Q_t(v)$ may be regarded as a set of quadratic forms whose equivalence classes do not depend on the choice of v. The equivalence classes of these form determine O. In 1.3.1, we denoted the discriminant of $Q_i(v)$ by Δ_i, and Hasse-invariant of $Q_i(v)$ by η_i, $1 \leqslant i \leqslant t$. The orbit O was then denoted by $O_\lambda(\Delta_1, \eta_1; \ldots; \Delta_t, \eta_t)$.

As we noted in Remark 1.2.4, it is not true that for any choice $\Delta'_i \in F^\times/(F^\times)^2$, $\eta'_i \in \{\pm 1\}$, there exists a rational orbit $O' \subseteq O^{\mathrm{st}}_\lambda$ such that $O' = O'_\lambda(\Delta'_1, \eta'_1; \ldots; \Delta'_t, \eta'_t)$. However, it will be very convenient to use the group structure on the set $[F^\times/(F^\times)^2]^t \times [\mathbb{Z}/2\mathbb{Z}]^t$ when discussing transfer factors. This leads to the notion of *ghosts* (as in Shelstad's work) by which we mean a symbol $O_\lambda(\Delta_1, \eta_1; \ldots; \Delta_t, \eta_t)$ where $(\Delta_1, \ldots, \Delta_t) \in [F^\times/(F^\times)^2]^t$ and $(\eta_1, \ldots, \eta_t) \in \{\pm 1\}^t$, which does not correspond to any rational orbit within O^{st}_λ. We shall treat ghosts as empty 'orbits', and agree that any 'integral', $\int_O$, over a ghost to be zero by convention. Recall also that $I(\boldsymbol{\lambda}) := \{1, \ldots, t\} = I_0(\boldsymbol{\lambda}) \cup I_e(\boldsymbol{\lambda})$, where $I_0(\boldsymbol{\lambda}) = \{i \in I(\boldsymbol{\lambda}) : \text{rank } Q_i \text{ is odd}\}$, and $I_e(\boldsymbol{\lambda}) = \{i \in I(\boldsymbol{\lambda}) : \text{rank } Q_i \text{ is even }\}$, and that $I_*(\boldsymbol{\lambda}) := I(\boldsymbol{\lambda})$ if $\mathbf{G}$ is odd orthogonal and $I_*(\boldsymbol{\lambda}) := I_e(\boldsymbol{\lambda})$ if $\mathbf{G}$ is symplectic or even orthogonal. To each map $\psi : I_*(\boldsymbol{\lambda}) \to F^\times/(F^\times)^2$, we associated a packet $\prod(\boldsymbol{\lambda}, \psi) = \{O_\lambda(\Delta_1, \eta_1; \ldots; \Delta_t, \eta_t) \subseteq O^{\mathrm{st}}_\lambda : \Delta_\alpha \equiv \psi(\alpha) \text{mod } (F^\times), \ \forall \alpha \in I_*(\boldsymbol{\lambda})\}$. We shall allow for all ghosts satisfying the defining condition of a packet to be formally included in that given packet}.

Next, let $(\mathbf{H}, O_H)$ denote an elliptic unipotent endoscopic datum. Thus, if $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$, then O_H is equal to $O_{\mu_1} \times O_{\mu_2}$, where $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are special partitions. Let $\prod_H := \prod(\boldsymbol{\mu}_1, \varphi_1) \times \prod(\boldsymbol{\mu}_2, \varphi_2)$, for some maps $\varphi_i : I_*(\boldsymbol{\mu}_i) \to F^\times/(F^\times)^2$, $i = 1, 2$. It can be checked that the transfer of $\prod_H$ to O^{st}_G, is a single packet denoted by $\prod_G$. Let $\boldsymbol{\psi} : I_*(\boldsymbol{\lambda}) \to F^\times/(F^\times)^2$ denote the map corresponding to $\prod_G$ (see def. 2.5.6), i.e., $\prod_G = \prod(\boldsymbol{\lambda}, \boldsymbol{\psi})$. We need one more piece of notation before we state our conjecture. Set $I_{**}(\boldsymbol{\lambda}) := I(\boldsymbol{\lambda}) \backslash I_*(\boldsymbol{\lambda})$. Let $\langle , \rangle : F^\times/(F^\times)^2 \times F^\times/(F^\times)^2 \to \{\pm 1\}$ denote the Hilbert pairing. In the following conjecture, the measures on the F-rational orbits within a stable orbit are *related* in the sense described after the introduction.

CONJECTURE 4.3.1. *There exist*

(i) *Constants $a_O \in \mathbb{Z}$, one for each $O \in \prod_H$. If $\mathbf{H}$ is split then $a_O := 1$, $O \in \prod_H$,*

(ii) *Two maps (depending on $\prod_H$ and $\prod_G$):*

$\chi_d : I(\boldsymbol{\lambda}) \to F^\times/(F^\times)^2$, *such that* $\chi_d(\alpha) = 1 \text{ mod } (F^\times)^2$, $\forall \alpha \in I_*(\lambda)$

$\chi_h : I(\boldsymbol{\lambda}) \to \{\pm 1\}^\wedge$ (= *the Pontryagin dual of* $\{\pm 1\}$),

(iii) *A nonzero constant $*$ which depends only on O^{st}_G and O^{st}_H, i.e. is independent of the packets $\prod_H$, $\prod_G$*
such that the following is satisfied:

(1) *The distribution*

$$\varphi \mapsto \sum_{O \in \prod_H} a_O \int_O \varphi \quad , \qquad \varphi \in C^\infty_c(\mathbf{H}(F))$$

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is stable.

(2) *For any* $f \in C_c^\infty(\mathbf{G}(F))$, *we have*

$$\sum_{O \in \prod_H} a_O \int_O f^H =* \sum_{\substack{\Delta_\alpha = \psi(\alpha) \\ \forall \alpha \in I_*(\lambda) \\ \Delta_i \in F^\times/(F^\times)^2 \\ \forall i \in I_{**}(\lambda) \\ \eta_k \in \{\pm 1\} \\ \forall k \in I(\lambda)}} b(\Delta_1, \eta_1; \ldots; \Delta_t, \eta_t) \int_{O_\lambda(\Delta_1, \eta_1; \ldots; \Delta_t, \eta_t)} f$$

where $b(\Delta_1, \eta_1; \ldots; \Delta_t, \eta_t) := \langle \chi_d(1), \Delta_1 \rangle \cdots \langle \chi_d(t), \Delta_t \rangle \cdot \chi_h(1)(\eta_1) \cdots \chi_h(t)(\eta_t)$.

Remarks 4.3.2.

(i) Statement (1) in the above conjecture is not new. It is, in fact, part of Conjecture (C) presented in ([2]).

(ii) The main content of statement (2) is the following:

 (a) It asserts that the transfer of the stable distributions associated to the packet $\prod_H$, is a linear combination, with nonzero coefficients, of integrals taken over only the rational orbits within the packet $\prod_G$.

 (b) The transfer factors, i.e., the coefficients appearing in the linear combination alluded to in (a) are values of characters of a group isomorphic to $[F^\times(F^\times)^2]^{|I_{**}(\lambda)|} \times [\mathbb{Z}/2\mathbb{Z}]^{|I(\lambda)|}$, into which every packet is embedded naturally (as a subset).

(iii) The general definition of the maps χ_d and χ_s will not be given. What we have to offer (see below) is a precise definition for these maps for some special, although broad classes, of orbits O_G.

The class of orbits which we wish to discuss consists of those special orbits which correspond to partition λ satisfying the following two properties:

(A) The set of distinct parts of λ^* is a set of the form $\{1, 3, \ldots, 2k+1\}$ if $\mathbf{G}$ is orthogonal, and is a set of the form $\{2, 4, \ldots, 2\ell\}$ if $\mathbf{G}$ is symplectic. (Recall that λ^* consists of all the even parts of λ if G is symplectic, and consists of all odd parts of λ if $\mathbf{G}$ is orthogonal.)

(B) $\overline{A}(\lambda) \cong A(\lambda) \cong C(\lambda)$, if $\mathbf{G}$ is of type $\mathbf{B}$, and $\overline{A}(\lambda) \times \mathbb{Z}/2\mathbb{Z} \cong C(\lambda)$, if $\mathbf{G}$ is of type $\mathbf{G}^\star$ or $\mathbf{D}$.

The next lemma⋆⋆ classifies these orbits.

LEMMA 4.3.2. *A partition* λ *satisfies conditions* (A) *and* (B) *iff the partition* ${}^L\lambda$ *is of the following form:*

- *Type* $\mathbf{B}_n$*: Either*

⋆This is clearly a misprint and presumably should read type $\mathbf{C}$ rather than type $\mathbf{G}$. The condition (B) shows up in the lemma that follows it, but as this lemma seems to be incorrect, it cannot be used to settle the question of how to correct the misprint.

⋆⋆This lemma seems to be incorrect.

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$^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{2r-1}, \lambda_{2r})$, *for some* $r \geqslant 1$, *where* $\boldsymbol{\lambda}_i$ *is even for all* $1 \leqslant i \leqslant 2r$, *and* $\lambda_{2j} \neq \lambda_{2j+1}$ *for all* $1 \leqslant j \leqslant r-1$, *or*
$^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{2r}, \lambda_{2r-1})$, *for some* $r \geqslant 1$, *where* $\boldsymbol{\lambda}_i$ *is even for all* $1 \leqslant i \leqslant 2r+1$, *and* $\lambda_{2j} \neq \lambda_{2j+1}$ *for all* $1 \leqslant j \leqslant r$.

- *Type* $\mathbf{C}_n$:
  $^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{2r+1})$, *for some* $r \geqslant 1$, *where* $\boldsymbol{\lambda}_i$ *is odd for all* $1 \leqslant i \leqslant 2r+1$, *and* $\lambda_{2i-1} \neq \lambda_{2i}$ *for all* $1 \leqslant i \leqslant r$.
- *Type* $\mathbf{D}_n$:
  $^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$, *for some* $r \geqslant 1$, *where* λ_i *is odd for all* $1 \leqslant i \leqslant 2r$, *and* $\lambda_{2i} \neq \lambda_{2i+1}$ *for all* $1 \leqslant j \leqslant r-1$.

Proof. The proof is an exercise in using the formulae for the duality map D given in Section 2.3. □

Remark 4.3.3. Note that $^L\boldsymbol{\lambda}$ is always even. So, we may then use Lemma 2.4.3., when discussing endoscopic induction for $\boldsymbol{\lambda}$.

Fix a partition $\boldsymbol{\lambda}$ satisfying conditions (A) and (B) above, and let O_λ denote the corresponding orbit in the classical split group $\mathbf{G}$. The next lemma will describe all the pairs $(\mathbf{H}, O_H)$ satisfying the following conditions:

(1) $\mathbf{H}$ is an elliptic endoscopic group of G,
(2) $\mathrm{Ind}_H^G\, O_H = O_G$,
(3) $\overline{A}(O_H) \cong C(O_\lambda)$ if $\mathbf{G}$ is an odd special orthogonal group, or $\overline{A}(O_H) \times \mathbb{Z}/2\mathbb{Z} \cong C(O_\lambda) = A(O_\lambda)$ if $\mathbf{G}$ is an even orthogonal group or a symplectic group.

LEMMA 4.3.9. *Let* $\boldsymbol{\lambda}$ *be as above. The pairs* $(\mathbf{H}, O_H)$ *satisfying the conditions (1)–(3) above, are given as follows. (Recall that* $^L\boldsymbol{\lambda}$ *is of the form described by Lemma 4.3.2.)*

- *Type* $\mathbf{B}_n$:
  (a) *Let* $^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{2r})$.
  Let $J_1 \subseteq \{1, 2, \ldots, r\}$, *and* $J_2 := \{1, 2, \ldots, r\} \backslash J_1$. *Define a partition* $^L\boldsymbol{\mu}_{J_i}$, $i = 1, 2$, *corresponding to* J_1, *and* J_2, *respectively, as following:* $^L\boldsymbol{\mu}_{J_i} := \bigvee_{k \in J_i} (\lambda_{2k-1}, \lambda_{2k})$, $i = 1, 2$. *In other words,* $^L\boldsymbol{\mu}_{J_i}$ *is the union of all partitions* $(\lambda_{2k-1}, \lambda_{2k})$ *where* $k \in J_i$. *Set* $\mathbf{H}_{J_1,J_2} := \mathbf{SO}(|^L\boldsymbol{\mu}_{J_1}| + 1) \times \mathbf{SO}(|^L\boldsymbol{\mu}_{J_2}| + 1)$ *and* $O_{J_1,J_2} := (O_{\mu_{J_1}}, O_{\mu_{J_2}})$.
  (b) Let $^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{2r}, \lambda_{2r+1})$.
  Let $J_1 \subseteq \{1, 2, \ldots, r\}$ *and* $J_2 := \{1, 2, \ldots, r\} \backslash J_1$.
  Associate to J_i, $i = 1, 2$, *two partitions* $^L\boldsymbol{\mu}'_{J_i}$, $^L\boldsymbol{\mu}'_{J_i}$ *as following:*

$$^{\mathrm{L}}\boldsymbol{\mu}''_{\mathrm{J_i}} := \bigvee_{\mathrm{k} \in \mathrm{J_i}} (\lambda_{2\mathrm{k}-1}, \lambda_{2\mathrm{k}})\,, \quad ^{\mathrm{L}}\boldsymbol{\mu}''_{\mathrm{J_i}} := \bigvee_{\mathrm{k} \in \mathrm{J_i}} (\lambda_{2\mathrm{k}-1}, \lambda_{2\mathrm{k}}) \cup (\lambda_{2\mathrm{r}+1})\,.$$

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In other words, $^L\mu''_{J_i}$ is obtained from $^L\boldsymbol{\mu}'_{J_i}$ by adding the part λ_{2r+1} at the end. Set

$$\mathbf{H}^1_{J_1,J_2} := \mathbf{SO}(|^L\boldsymbol{\mu}'_{J_1}|+1)\times\mathbf{SO}(|^L\boldsymbol{\mu}''_{J_2}|+1)\ ,$$
$$\mathbf{H}^2_{J_1,J_2} := \mathbf{SO}(|^L\boldsymbol{\mu}''_{J_2}|+1)\times\mathbf{SO}(|^L\boldsymbol{\mu}'_{J_2}|+1)\ ,$$
$$O'_{J_1,J_2} := (O_{\mu'_{J_1}}, O_{\mu''_{J_2}})\ ,$$
$$O''_{J_1,J_2} := (O_{\mu'_{J_1}}, O_{\mu'_{J_2}})\ .$$

- Type $\mathbf{C}_n$:
  Let $^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{2r}, \lambda_{2r+1})$.
  Let $J_1 \subseteq \{1, 2, \dots, r\}$, and $J_2 := \{1, 2, \dots, r\}\backslash J_1$.
  Define $^L\boldsymbol{\mu}_{J_i}$, $i = 1, 2$, as following:
  $^L\boldsymbol{\mu}_{J_1} := (\lambda_1) \cup \bigvee_{k\in J_1}(\lambda_{2k}, \lambda_{2k+1})$,
  $^L\boldsymbol{\mu}_{J_2} := \bigvee_{k\in J_2}(\lambda_{2k}, \lambda_{2k+1})$.
  Set $\mathbf{H}_{J_1,J_2} := \mathbf{Sp}(|^L\boldsymbol{\mu}_{J_1}|-1)\times\mathbf{SO}(|^L\boldsymbol{\mu}_{J_2}|)$, and $O_{J_1,J_2} := (O_{\mu_{J_1}}, O_{\mu_{J_2}})$.
- Type $\mathbf{D}_n$:
  Let $^L\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{2r})$.
  Let $J_1 \subseteq \{1, 2, \dots, r\}$, and $J_2 := \{1, 2, \dots, r\}\backslash J_1$.
  Define $^L\boldsymbol{\mu}_{J_i}$, $i = 1, 2$, as following:
  $^L\boldsymbol{\mu}_{J_i} := \bigvee_{k\in J_i}(\lambda_{2k-1}, \lambda_{2k})$.
  Set $\mathbf{H}_{J_1,J_2} := \mathbf{SO}(|^L\boldsymbol{\mu}_{J_1}|)\times\mathbf{SO}(|^L\boldsymbol{\mu}_{J_2}|)$, and $O_{J_1,J_2} := (O_{\mu_{J_1}}, O_{\mu_{J_2}})$.

Proof. The proof is a combinatorial exercise in applying the formulae for duality given in 2.3, together with Lemma 2.4.3 and Lemma 2.4.4. We omit the (elementary) details. □

Lemma 4.3.4 allows us to count the number of 'distinct' pairs $(\mathbf{H}_1\times\mathbf{H}_2, O_{H_1}\times O_{H_2})$ satisfying conditions (1)–(3). Here, of course, we count pairs up to a switch of factors when $\mathbf{G}$ is orthogonal.

COROLLARY 4.3.5. *Let $\boldsymbol{\lambda}$ be a special partition satisfying conditions (A) and (B). The number of 'distinct' pairs (H, O_H) satisfying conditions (1)–(3) is equal to $2^{\eta(O^{\mathrm{st}})}$.*

Next, note that, for the orbits under consideration, we have $I(\boldsymbol{\lambda}) = I_*(\boldsymbol{\lambda})$. Thus $I_{**}(\boldsymbol{\lambda}) = \phi$, and the formula given by Conjecture 4.3.1. (2), indicates that (aside from the constant) only the ingredient depending on the map χ_h will appear. In order to define χ_h, it will be sufficient to work on the dual group side, and define a map $\hat{\chi}_h\colon S_*(^L\boldsymbol{\lambda}) \to \{\pm1\}^\wedge$. The map χ_h will then be defined to be the composition $\hat{\chi}_h\circ b^{-1}_{\lambda_L}\circ\boldsymbol{\iota}_{\lambda_L}$. We shall define $\hat{\chi}_h$ in a case by case fashion. Fix a pair J_1, J_2 as in Lemma 4.3.4. This pair then determines an elliptic endoscopic group $\mathbf{H}$ and a special orbit O_H. $\hat{\chi}_h$ (and, hence, χ_h) are defined relative to the pair $(\mathbf{H}, O_H)$. In defining $\hat{\chi}_h$, we shall only work with J_2. By an *interval* in J_2 we shall mean a subset of J_2 consisting of consecutive integers and which is maximal (in the sense of set theoretic inclusion) with respect to that property. J_2 is then a disjoint union of intervals. To each interval we associate at most two segments in $S_*(^L\boldsymbol{\lambda})$ as follows. Fix an interval and let $j_{\min}$

and $j_{\max}$ denote the minimum and maximum elements of the fixed interval. Consider now the following cases:

- Let $\mathbf{G}$ be odd orthogonal. If $j_{\min} = 1$, then we associate the segment $(\lambda_1, \ldots, \lambda_{2j_{\max}})$ to the given interval. If $j_{\min} > 1$, then we associate the two segments $(\lambda_1, \ldots, \lambda_{2j_{\max}})$ and $(\lambda_1, \ldots, \lambda_{2j_{\min}-2})$ to the given interval.
- Let $\mathbf{G}$ be symplectic. If $j_{\min} = 1$, we associate the segment $(\lambda_1, \ldots, \lambda_{2j_{\max}+1})$. If $j_{\min} > 1$, we associate the two segments $(\lambda_1, \ldots, \lambda_{2j_{\max}+1})$ and $(\lambda_1, \ldots, \lambda_{2j_{\min}-1})$.
- Let $\mathbf{G}$ be even orthogonal. Then by switching factors if necessary, we may assume that $j_{\max} > 1$. If $j_{\min} = 1$, then we associate the segment $(\lambda_1, \ldots, \lambda_{2j_{\max}})$. if $j_{\max} > 1$, then we associate the two segments $(\lambda_1, \ldots, \lambda_{2j_{\max}})$ and $(\lambda_1, \ldots, \lambda_{2j_{\min}-2})$. Repeating this process for each interval, we get a subset $S \subseteq S_*({}^L\boldsymbol{\lambda})$. Define now $\hat{\chi}_h : S_*({}^L\boldsymbol{\lambda}) \to \{\pm 1\}^\wedge$ by

$$\hat{\chi}_h(z) := \begin{cases} \text{the nontrivial character of } \{\pm 1\} & , \text{if } z \in S \\ \text{the trivial character of } \{\pm 1\} & , \text{if } z \notin S. \end{cases}$$

To illustrate the above construction, we give some examples.

EXAMPLE 4.3.5. Let $\boldsymbol{\lambda} = (7, 5, 5, 3, 3, 1, 1)$. Then ${}^L\boldsymbol{\lambda} = (6, 6, 4, 4, 2, 2)$ and $r = 3$. $S_*({}^L\boldsymbol{\lambda}) = \{(6, 6, 4, 4, 2, 2), (6, 6, 4, 4), (6, 6)\}$. Aside from $(\mathbf{G}, O_\lambda)$, there are three other pairs $(\mathbf{H}, O_H)$ satisfying conditions (1)–(3) above. They are given by the following data:

(a) $\hat{\mathbf{H}} = \mathbf{Sp}(6) \times \mathbf{Sp}(6)$, ${}^L\boldsymbol{\mu}_1 = (4, 4, 2, 2)$, ${}^L\boldsymbol{\mu}_2 = (6, 6)$. Thus $J_2 = \{1\}$.
(b) $\hat{\mathbf{H}} = \mathbf{Sp}(4) \times \mathbf{Sp}(8)$, ${}^L\boldsymbol{\mu}_1 = (4, 4)$, ${}^L\boldsymbol{\mu}_2 = (6, 6, 2, 2)$. Thus $J_2 = \{1, 3\}$.
(c) $\hat{\mathbf{H}} = \mathbf{Sp}(2) \times \mathbf{Sp}(10)$, ${}^L\boldsymbol{\mu}_1 = (2, 2)$, ${}^L\boldsymbol{\mu}_2 = (6, 6, 4, 4)$. Thus $J_2 = \{1, 2\}$.

In case (a) we have only one interval to which the segments (6, 6) is associated. In case (b) we have two intervals: {1} and {3}, to which the segments (6, 6), (6, 6, 4, 4, 2, 2) and (6, 6, 4, 4) are associated. Finally, in case (c) we get the segments (6, 6, 4, 4) and (6, 6). The map $\hat{\chi}_h$ is given as follows: Let sgn denote the nontrivial character of $\{\pm 1\}$, and let denote the trivial character. Then

In case (a): $\hat{\chi}_h(z) := \begin{cases} \text{sgn}, & \text{if } z = (6, 6), \\ \text{id}, & \text{otherwise.} \end{cases}$

In case (b): $\hat{\chi}_h(z) := \begin{cases} \text{sgn}, & \text{if } z = (6, 6), (6, 6, 4, 4, 2, 2), \text{ or } (6, 6, 4, 4), \\ \text{id}, & \text{otherwise.} \end{cases}$

In case (c): $\hat{\chi}_h(z) := \begin{cases} \text{sgn}, & \text{if } z = (6, 6, 4, 4), \text{ or } (6, 6), \\ \text{id}, & \text{otherwise.} \end{cases}$

Acknowledgements

This work started at the University of Saskatchewan and was completed during visits to the University of Chicago and Université de Paris-7, during April–June 1996. I would like to thank all these institutions for their support and hospitality. In particular, I would like to thank Robert Kottwitz for several lively discussions, and Jean-Loup Waldspurger, for inviting me, and for kindly telling me about his results.

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