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MODULES WITH FI-EXTENDING HULLS

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Abstract. It is shown that every finitely generated projective module P_R over a semiprime ring R has the smallest FI-extending essential module extension $H_{\mathfrak{FI}}(P_R)$ (called the absolute FI-extending hull of P_R) in a fixed injective hull of P_R. This module hull is explicitly described. It is proved that $\widehat{Q}_{\mathfrak{FI}}(\mathrm{End}(P_R)) \cong \mathrm{End}(H_{\mathfrak{FI}}(P_R))$, where $\widehat{Q}_{\mathfrak{FI}}(\mathrm{End}(P_R))$ is the smallest right FI-extending right ring of quotients of $\mathrm{End}(P_R)$ (in a fixed maximal right ring of quotients of $\mathrm{End}(P_R)$). Moreover, we show that a finitely generated projective module P_R over a semiprime ring R is FI-extending if and only if it is a quasi-Baer module and if and only if $\mathrm{End}(P_R)$ is a quasi-Baer ring. An application of this result to C^*-algebras is considered. Various examples which illustrate and delimit the results of this paper are provided.

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An important technique used to study an algebraic object is to search for an overobject that has the following properties: (1) it belongs to a class $\mathfrak{K}$ with some desirable properties; (2) it is explicitly computable; (3) information can be transferred between the base object and its overobject (thus one tries to find an overobject from $\mathfrak{K}$ which is 'close to' the base object).

In module theory the class of injective modules and, its generalization, the class of extending modules have the property that every submodule of a member is essential in a direct summand of the member. This property, originated by Chatters and Hajarnavis in [**21**], ensures a rich structure theory for these classes. Although every module has an injective hull, it is usually hard to compute. For many modules a minimal essential extension which belongs to the class of extending modules may not exist (e.g. $\oplus_{n=1}^{\infty}\mathbb{Z}_{\mathbb{Z}}$, see comment above Proposition 8). Moreover the class of extending modules lacks some important closure properties (e.g. it is not closed under direct sums).

Throughout this paper all rings are associative with identity and R denotes such a ring. All modules are unitary. Recall from [**12**] that a right R-module M_R is *FI-extending* if every fully invariant submodule of M_R is essential in a direct summand of M_R. A ring R is *right FI-extending* if R_R is FI-extending. Note that the set of fully invariant submodules of a module M_R includes the socle, Jacobson radical, torsion submodule for a torsion theory (e.g., $Z(M_R)$, the singular submodule), and MI for all right ideals I of R, etc. Hence, the FI-extending condition provides an 'economical use' of the extending condition by targeting only the fully invariant submodules, and thus some of the most important submodules of M_R for an essential splitting of M_R. Natural examples of FI-extending rings and modules abound: direct sums of uniform modules, more specifically all finitely generated Abelian groups, semisimple modules, prime rings, serial rings, semiprime right finitely pseudo-Frobenius (FPF) rings. Also it is shown in [**13**, Corollary 1.9] that semiprime right Noetherian group algebras over a field are right FI-extending. Note that in this case if the group is Abelian, then the group algebra is extending.

In [**19**] we showed that every semiprime ring R has the smallest right FI-extending right ring of quotients $\widehat{Q}_{\mathfrak{FI}}(R)$. In this paper, we further develop the FI-extending concept by showing that over a semiprime ring R, every finitely generated projective module P_R has the smallest FI-extending essential extension $H_{\mathfrak{FI}}(P_R)$ (called the absolute FI-extending hull of P_R) in a fixed injective hull of P_R. Moreover, $H_{\mathfrak{FI}}(P_R)$ is easily computable (see Theorem 6 and Proposition 8), it is from a class for which direct sums and direct summands are FI-extending, and since $H_{\mathfrak{FI}}(P_R)$ is finitely generated and projective over $\widehat{Q}_{\mathfrak{FI}}(R)$, we are assured of a reasonable transfer of information between P_R and $H_{\mathfrak{FI}}(P_R)$ (e.g. see Theorem 12 and Corollary 13).

Since many well-known types of Banach algebras are semiprime (e.g. C^*-algebras), our results are applicable. Finitely generated modules over a Banach algebra are considered in [**26**]. Kaplansky [**28**] defined AW^*-modules over a C^*-algebra and used them to answer several questions concerning automorphisms and derivations on certain types of C^*-algebras. Furthermore work using these modules appeared in [**6**]. Moreover, from [**20**, p. 352], every algebraically finitely generated C^*-module M is projective, hence $H_{\mathfrak{FI}}(M)$ exists. Since every C^*-algebra A is both semiprime and non-singular, $\widehat{Q}_{\mathfrak{FI}}(A)$ always exists by [**19**]. Also in [**19**], we characterized all C^*-algebras with only finitely many minimal prime ideals and showed that for such A, $\widehat{Q}_{\mathfrak{FI}}(A)$ is also a C^*-algebra. Thus our results should yield fruitful applications to projective modules over C^*-algebras, as well as many other algebras in Functional Analysis.

According to [**15**] a module M_R is called *strongly FI-extending* if every fully invariant submodule of M_R is essential in a fully invariant direct summand of M_R (e.g. a non-singular FI-extending module). Note that the class of strongly FI-extending modules is closed under direct summands and is contained in the class of FI-extending modules. A ring R is said to be *right strongly FI-extending* if R_R is strongly FI-extending. We use $\mathfrak{FI}$ (resp., $\mathfrak{E}$, $\mathfrak{SFI}$) to denote the class of FI-extending (resp., extending, strongly FI-extending) right modules or the class of right FI-extending (resp., right extending, right strongly FI-extending) rings according to the context (see [**7**–**10**, **12**, **13** and **33**] for details and examples of the (strongly) FI-extending property of modules and rings, also see [**21**] and [**23**] for the extending property of modules and rings).

If N_R is a submodule of M_R, then N_R is *essential* (resp., *dense* also called *rational*) in M_R if for any $0 \neq x \in M$, there exists $r \in R$ such that $0 \neq xr \in N$ (resp., for any $x, y \in M$ with $0 \neq x$, there exists $r \in R$ such that $xr \neq 0$, and $yr \in N$). For R-modules M_R and N_R, we use $N_R \leq M_R$, $N_R \leq^{\text{ess}} M_R$ and $N_R \trianglelefteq M_R$ to denote that N_R is a submodule of M_R, N_R is an essential submodule of M_R and N_R is a fully invariant submodule of M_R, respectively. Let $E(M_R)$ denote an injective hull of a module M_R. Recall that a right ring of quotients T of a ring R is an overring of R such that R_R is dense in T_R. An overring S of a ring R is called a *right essential overring* of R if R_R is essential in S_R. For a ring R, $\mathbf{J}(R)$, $\mathbf{B}(R)$, $\text{Mat}_n(R)$ and $Q(R)$ denote the Jacobson radical of R, the set of central idempotents of R, the $n \times n$ matrix ring over R and the maximal right ring of quotients of R, respectively.

DEFINITION 1. We fix an injective hull $E(M_R)$ of M_R and a maximal right ring of quotients $Q(R)$ of R.

(i) Let $\mathfrak{M}$ be a class of right R-modules and M_R a right R-module. We call, when it exists, a module $H_{\mathfrak{M}}(M_R)$ the *absolute* $\mathfrak{M}$ *hull* of M_R if $H_{\mathfrak{M}}(M_R)$ is the smallest essential extension of M_R in $E(M_R)$ that belongs to $\mathfrak{M}$.

(ii) Let $\mathfrak{K}$ be a class of rings and R a ring. We call, when it exists, a ring $\widehat{Q}_{\mathfrak{K}}(R)$ the $\mathfrak{K}$ *absolute to* $Q(R)$ *right ring hull* of R if $\widehat{Q}_{\mathfrak{K}}(R)$ is the smallest right ring of quotients of R in $Q(R)$ that belongs to $\mathfrak{K}$.

LEMMA 2 ([**17**, Lemma 1.4]). *Let T be a right ring of quotients of a ring R.*

(i) *For right ideals X and Y of T, if $X_T \leq^{\text{ess}} Y_T$, then $X_R \leq^{\text{ess}} Y_R$.*

(ii) *If $X_R \trianglelefteq T_R$, then $X_R \leq^{\text{ess}} TXT_R$.*

We recall from [**22**, **29**, **36**] that a ring R is called *quasi-Baer* (resp., *Baer*) if the right annihilator of every right ideal (resp., non-empty subset) of R is generated, as a right ideal, by an idempotent (see [**11**, **22**, **36**] for more on quasi-Baer rings). The class of quasi-Baer rings is denoted by $\mathfrak{qB}$. Mewborn [**32**] showed the existence of a Baer absolute to $Q(R)$ ring hull for a commutative semiprime ring R (which, in this case, coincides with $\widehat{Q}_{\mathfrak{qB}}(R)$). For more details on $\widehat{Q}_{\mathfrak{qB}}(R)$ and its applications, see [**19**].

LEMMA 3 ([**19**, Theorem 3.3]). *Assume that R is a semiprime ring. Then $\widehat{Q}_{\mathfrak{FI}}(R) = R\mathbf{B}(Q(R)) = \widehat{Q}_{\mathfrak{qB}}(R)$, where $R\mathbf{B}(Q(R))$ is the subring of $Q(R)$ generated by R and $\mathbf{B}(Q(R))$.*

LEMMA 4. *Let T be a right ring of quotients of a ring R. Then* $\text{End}(T_R) = \text{End}(T_T) \cong T$.

Proof. See [**31**, p. 94] for the proof. □

LEMMA 5. *If M_R is a FI-extending module, then $fM \subseteq M$ for any central idempotent f of* $\text{End}(E(M_R))$.

Proof. Let $f \in \mathbf{B}(\text{End}(E(M_R)))$. Then $fE(M_R) \cap M \trianglelefteq M_R$. Since M_R is FI-extending, there is $g = g^2 \in \text{End}(M_R)$ such that $fE(M_R) \cap M_R \leq^{\text{ess}} gM_R \leq^{\text{ess}} \overline{g}E(M_R)$, where $\overline{g}$ is the projection from $E(M_R) = E(gM_R) \oplus E((1-g)M_R)$ to $E(gM_R)$. Note that $fE(M_R) \cap M_R \leq^{\text{ess}} fE(M_R)$. Thus $f = \overline{g}$ since f is a central idempotent in $\text{End}(E(M_R))$. Therefore $fM = \overline{g}M = gM \subseteq M$. □

The converse of Lemma 5 does not hold. For this, see R_R in Example 9. Observe that when R is semiprime $\text{End}(\oplus^n R_R) \cong \text{Mat}_n(R) \subseteq \text{Mat}_n(\widehat{Q}_{\mathfrak{FI}}(R)) \cong \text{End}(\oplus^n \widehat{Q}_{\mathfrak{FI}}(R)_{\widehat{Q}_{\mathfrak{FI}}(R)}) = \text{End}(\oplus^n \widehat{Q}_{\mathfrak{FI}}(R)_R)$ by Lemma 4. Thus, in the sequel, we identify $\text{End}(\oplus^n R_R)$ as a subring of $\text{End}(\oplus^n \widehat{Q}_{\mathfrak{FI}}(R)_{\widehat{Q}_{\mathfrak{FI}}(R)}) = \text{End}(\oplus^n \widehat{Q}_{\mathfrak{FI}}(R)_R)$.

We first show the existence of the absolute FI-extending hull for every finitely generated projective module over a semiprime ring. Also this module hull is explicitly described.

THEOREM 6. *Every finitely generated projective module P_R over a semiprime ring R has the absolute FI-extending hull $H_{\mathfrak{FI}}(P_R)$. Explicitly, $H_{\mathfrak{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathfrak{FI}}(R)_R)$ where $P \cong e(\oplus^n R_R)$, for some n and $e = e^2 \in$* $\text{End}(\oplus^n R_R)$.

Proof. Step 1. $\widehat{Q}_{\mathfrak{FI}}(R)_R$ is strongly FI-extending. □

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Proof of Step 1. Let $T = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$. Then $T = R\mathbf{B}(Q(R))$ (by Lemma 3) is semiprime and quasi-Baer. To show that T_R is strongly FI-extending, let $X_R \trianglelefteq T_R$. By Lemma 2(ii), $X_R \leq^{\text{ess}} TXT_R$. From [**12**, Theorem 4.7], there exists $c \in \mathbf{B}(T)$ such that $TXT_T \leq^{\text{ess}} cT_T$. By Lemma 2(i), $TXT_R \leq^{\text{ess}} cT_R$. By Lemma 4, $\text{End}(T_R) = \text{End}(T_T) \cong T$. Hence for any $\lambda \in \text{End}(T_R)$, $\lambda(cT) = c(\lambda T)$. Thus $cT_R \trianglelefteq T_R$, so T_R is strongly FI-extending as $X_R \leq^{\text{ess}} cT_R$.

Step 2. $H_{\mathfrak{F}\mathfrak{I}}(\oplus^n R_R) = \oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$.

Proof of Step 2. Note that $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$ is FI-extending by Step 1, so $\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$ is FI-extending [**12**, Theorem 1.3]. Suppose that N_R is FI-extending with $\oplus^n R_R \leq N_R \leq E(\oplus^n R_R) = \oplus^n E(R_R)$. Note that $\mathbf{B}(Q(R)) = \mathbf{B}(\text{End}(E(R_R))$. Take $f \in \mathbf{B}(Q(R))$. Let fI_n be the n-by-n diagonal matrix with f on the diagonal, where I_n is the identity matrix in $\text{End}(\oplus^n E(R_R)) \cong \text{Mat}_n(\text{End}(E(R_R)))$. Then $fI_n \in \mathbf{B}(\text{End}(\oplus^n E(R_R)))$. Thus by Lemma 5,

$$fI_nN \subseteq N, \quad \text{so} \quad fI_n \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} \subseteq N, \quad \text{where} \quad \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} = \oplus^n R_R.$$

Observe that $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) = R\mathbf{B}(Q(R))$ from Lemma 3. Therefore $\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R \leq N_R$, hence $H_{\mathfrak{F}\mathfrak{I}}(\oplus^n R_R) = \oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$.

Step 3. $H_{\mathfrak{F}\mathfrak{I}}(e(\oplus^n R_R)) = e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$.

Proof of Step 3. We may assume that $P_R = e(\oplus^n R_R)$. Therefore, $\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R = e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) \oplus (1-e)(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$. Since $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$ is strongly FI-extending by Step 1, $\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$ is strongly FI-extending by [**15**, Theorem 3.3]. Hence $e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$ is strongly FI-extending [**15**, Theorem 2.4]. Let N_R be FI-extending such that $e(\oplus^n R_R) \leq N_R \leq E(e(\oplus^n R_R))$. Then $\oplus^n R_R = e(\oplus^n R_R) \oplus (1-e)(\oplus^n R_R) \leq N_R \oplus (1-e)(\oplus^n R_R) \leq N_R \oplus E[(1-e)(\oplus^n R_R)]$. Now since N_R is FI-extending and $E[(1-e)(\oplus^n R_R)]$ is injective, $N_R \oplus E[(1-e)(\oplus^n R_R)]$ is FI-extending [**12**, Theorem 1.3]. Hence, by Step 2, $H_{\mathfrak{F}\mathfrak{I}}(\oplus^n R_R) = \oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R \leq N_R \oplus E[(1-e)(\oplus^n R_R)]$.

To prove that $e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) \leq N_R$, take $e\alpha \in e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$ with $\alpha \in \oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R$. Since $e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) \leq N_R \oplus E[(1-e)(\oplus^n R_R)]$, $e\alpha = n + y$ for some $n \in N$ and $y \in E[(1-e)(\oplus^n R_R)]$. Thus

$$e\alpha - n = y \in [e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) + N] \cap E[(1-e)(\oplus^n R_R)].$$

Note that $E[e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)] = E[e(\oplus^n R_R)]$ because $e(\oplus^n R_R) \leq^{\text{ess}} e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$. So $[e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) + N] \cap E((1-e)(\oplus^n R_R)] \leq E[e(\oplus^n R_R)] \cap E[(1-e)(\oplus^n R_R)] = 0$. Hence $e\alpha - n = y = 0$, so $e\alpha = n \in N$. Therefore $e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) \leq N_R$. Consequently, $H_{\mathfrak{F}\mathfrak{I}}(e(\oplus^n R_R)) = e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$.

Step 4. $H_{\mathfrak{F}\mathfrak{I}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$.

Proof of Step 4. Let $\sigma : P_R \to e(\oplus^n R_R)$ be an isomorphism. Then σ can be extended to an isomorphism $\overline{\sigma} : E(P_R) \to E(e(\oplus^n R_R))$. It is easy to check that $H_{\mathfrak{F}\mathfrak{I}}(P_R) = \overline{\sigma}^{-1}(e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)) \cong e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$. □

REMARK. From the proof of Theorem 6, we see that the absolute strongly FI-extending hull and the absolute FI-extending hull of a finitely generated projective module P_R

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over a semiprime ring R coincide (hence both direct sums and direct summands of $H_{\mathfrak{F}\mathfrak{I}}(P_R)$ are FI-extending). Thus $H_{\mathfrak{S}\mathfrak{F}\mathfrak{I}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$, where $P \cong e(\oplus^n R_R)$, for some n and $e = e^2 \in \mathrm{End}(\oplus^n R_R)$.

If R is not semiprime, the above remark does not hold. For example, let $R = \mathbb{Z}_3[S_3]$, the group algebra of S_3 over the field $\mathbb{Z}_3$ of three elements, where S_3 is the symmetric group on $\{1, 2, 3\}$. As was shown in [**15**, Example 1.1], R_R is not strongly FI-extending. Thus, $H_{\mathfrak{S}\mathfrak{F}\mathfrak{I}}(R_R)$ does not exist because R_R is injective.

According to [**25**], $E(M_R)$ is called Σ-*injective* if $\oplus_\Lambda E(M_R)$ is injective for any non-empty set Λ. Thus $E(M_R)$ is Σ-injective if and only if $E(\oplus_\Lambda M_R) = \oplus_\Lambda E(M_R)$ for any non-empty set Λ.

Corollary 7. *Assume that R is a semiprime right Goldie ring. Then every projective right R-module P_R has the absolute FI-extending hull. Moreover, if $P \cong e(\oplus_\Lambda R_R)$ with $e = e^2 \in \mathrm{End}_R(\oplus_\Lambda R_R)$, then $H_{\mathfrak{F}\mathfrak{I}}(P_R) \cong e(\oplus_\Lambda \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$.*

Proof. By [**25**, Lemma 2 and Corollary 3], if a ring R is semiprime right Goldie, then $E(R_R)$ is Σ-injective. Now the rest of the proof follows from that of Theorem 6. □

The FI-extending hull of a module, in general, is distinct from the injective hull of the module or its extending hull (if it exists). From Corollary 7, $H_{\mathfrak{F}\mathfrak{I}}(\oplus_\Lambda \mathbb{Z}_\mathbb{Z}) = \oplus_\Lambda \mathbb{Z}_\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. However in $E(\oplus_\Lambda \mathbb{Z}_\mathbb{Z}) = \oplus_\Lambda \mathbb{Q}_\mathbb{Z}$, where Λ is infinite and $\mathbb{Q}$ is the field of rational numbers, there is not even a minimal extending essential extension of $\oplus_\Lambda \mathbb{Z}_\mathbb{Z}$. Our next result gives an alternative description of $H_{\mathfrak{F}\mathfrak{I}}(P_R)$ from Theorem 6.

Proposition 8. *Assume that P_R is a finitely generated projective module over a semiprime ring R. Then $H_{\mathfrak{F}\mathfrak{I}}(P_R) \cong P \otimes_R \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ as $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$-modules. Hence $H_{\mathfrak{F}\mathfrak{I}}(P_R)$ is also a finitely generated projective $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$-module.*

Proof. The proof is routine. As in Step 3 of the proof of Theorem 6, we may assume that $P = e(\oplus^n R_R)$ with $e = e^2 \in \mathrm{Mat}_n(R)$. Let $\alpha \in P \otimes_R \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$. Then there are $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in R$ and $q_1, q_2, \ldots, q_k \in \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ such that □

$$\alpha = e\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \otimes q_1 + e\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \otimes q_2 + \cdots + e\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \otimes q_k.$$

Since $H_{\mathfrak{F}\mathfrak{I}}(P_R) = e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$ by Theorem 6, $\sigma : P \otimes_R \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) \to H_{\mathfrak{F}\mathfrak{I}}(P_R)$ can be defined by

$$\sigma(\alpha) = e\begin{pmatrix} a_1q_1 + b_1q_2 + \cdots + c_1q_k \\ a_2q_1 + b_2q_2 + \cdots + c_2q_k \\ \vdots \\ a_nq_1 + b_nq_2 + \cdots + c_nq_k \end{pmatrix} \in H_{\mathfrak{F}\mathfrak{I}}(P_R).$$

Then σ is an onto homomorphism. To show that σ is one-to-one, note that $\tau : (\oplus^n R_R) \otimes_R \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) \cong \oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ by corresponding $\gamma_1 \otimes x_1 + \cdots + \gamma_m \otimes x_m \in (\oplus^n R_R) \otimes_R \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ (with $\gamma_i \in \oplus^n R_R$ and $x_i \in \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ for $i = 1, 2, \ldots, m$) to

$\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_m x_m \in \oplus^n \widehat{Q}_{\mathfrak{FI}}(R)$. Since $P = e(\oplus^n R_R)$, so $P \otimes_R \widehat{Q}_{\mathfrak{FI}}(R)$ is a direct summand of $(\oplus^n R_R) \otimes_R \widehat{Q}_{\mathfrak{FI}}(R)$. Thus $\tau|_{P\otimes_R \widehat{Q}_{\mathfrak{FI}}(R)} = \sigma$, hence σ is one-to-one. So $P \otimes_R \widehat{Q}_{\mathfrak{FI}}(R) \cong H_{\mathfrak{FI}}(P_R)$. Moreover, σ is a $\widehat{Q}_{\mathfrak{FI}}(R)$-module isomorphism. □

The existence of absolute FI-extending hulls is not always guaranteed, even in the presence of non-singularity, as the next example shows.

EXAMPLE 9. There exists a right non-singular ring R such that R_R does not have the absolute FI-extending hull. Let F be a field and let

$$R = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{pmatrix} \;\middle|\; a, x, y, c \in F \right\}.$$

Then R is right non-singular with $Q(R) = \text{Mat}_3(F)$. Assume to the contrary that there exists $H_{\mathfrak{FI}}(R_R)$. Let

$$H_1 = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix} \text{ and } H_2 = \left\{ \begin{pmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \;\middle|\; a, b, c, x, y \in F \right\}.$$

Then H_1 and H_2 are right FI-extending rings (see [**17**, Example 3.19]). Since H_1 and H_2 are right rings of quotients of R, it follows that H_1 and H_2 are FI-extending right R-modules by [**17**, Proposition 1.8]. Thus $H_{\mathfrak{FI}}(R_R) \subseteq H_1 \cap H_2 = R$, so $H_{\mathfrak{FI}}(R_R) = R_R$. By [**14**, Corollary 1.6], R_R is not FI-extending. Thus we have a contradiction.

EXAMPLE 10. There exists a prime Noetherian ring R (hence $R = H_{\mathfrak{FI}}(R_R)$) but $H_{\mathfrak{E}}(R_R)$ does not exist. Let $R = \text{Mat}_2(F[x, y])$, where F is a field. Then $E(R_R) = Q(R) = \text{Mat}_2(F(x, y))$, where $F(x, y)$ is the field of fractions of $F[x, y]$. Note that $H_{\mathfrak{FI}}(R_R) = R_R$. Let $S = \text{Mat}_2(F(y)[x])$ and $T = \text{Mat}_2(F(x)[y])$. Now both S_S and T_T are extending. Thus by [**17**, Proposition 1.8] S_R and T_R are extending. Assume to the contrary that $H_{\mathfrak{E}}(R_R)$ exists. Then $H_{\mathfrak{E}}(R_R) \leq S_R \cap T_R$. Since $S_R \cap T_R = R_R$ by [**19**, Example 3.10], $R_R = H_{\mathfrak{E}}(R_R)$. So R_R is extending, which is a contradiction because the domain $F[x, y]$ is not Prüfer.

LEMMA 11. *Assume that R is a semiprime ring. Then $\widehat{Q}_{\mathfrak{FI}}(fRf) = f\widehat{Q}_{\mathfrak{FI}}(R)f$ for any $0 \neq f = f^2 \in R$.*

Proof. The proof appears in [**18**]. □

From Osofsky [**34**], there is a prime ring R with $\mathbf{J}(R) = 0$ such that $E(R_R)$ is a non-rational extension of R_R. So $Q(R)_R$ is not injective, thus $\text{End}(E(R_R)) \ncong Q(R)$ as rings by [**31**, p. 95, Proposition 3]. Hence $Q(\text{End}(R_R)) \ncong \text{End}(E(R_R))$ (see also [**16**, Proposition 2.6]). However, a special case of our next result shows that $\widehat{Q}_{\mathfrak{FI}}(R) \cong \text{End}(H_{\mathfrak{FI}}(R_R))$ for a semiprime ring R.

THEOREM 12. *Assume that R is a semiprime ring and P_R is a finitely generated projective module. Then we have the following:*

(i) $\widehat{Q}_{\mathfrak{FI}}(\text{End}(P_R)) \cong \text{End}(H_{\mathfrak{FI}}(P_R))$ *as rings.*

(ii) $\text{Rad}(H_{\mathfrak{FI}}(P_R)_{\widehat{Q}_{\mathfrak{FI}}(R)}) \cap P = \text{Rad}(P_R)$*, where* $\text{Rad}(-)$ *is the Jacobson radical of a module.*

Proof. (i) Since $P_R \cong e(\oplus^n R_R)$ with $e = e^2 \in \text{Mat}_n(R)$, it follows that $\text{End}(P_R) \cong e\text{Mat}_n(R)e$. Also by Theorem 6, $H_{\mathfrak{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathfrak{FI}}(R))$. Thus $\text{End}(H_{\mathfrak{FI}}(P_R)) \cong$

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$e\mathrm{Mat}_n(\mathrm{End}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)e$. Since $\mathrm{End}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R) \cong \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ by Lemma 4, $\mathrm{End}(H_{\mathfrak{F}\mathfrak{I}}(P_R)) \cong e\mathrm{Mat}_n(\mathrm{End}_R(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R))e \cong e\mathrm{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))e$. Now from Lemma 11, $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(e\mathrm{Mat}_n(R)e) = e\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(\mathrm{Mat}_n(R))e$ because $\mathrm{Mat}_n(R)$ is semiprime and $0 \neq e = e^2 \in \mathrm{Mat}_n(R)$. Also by [**17**, Corollary 5.6] since $\mathrm{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(\mathrm{Mat}_n(R))$, it follows that

$$\mathrm{End}(H_{\mathfrak{F}\mathfrak{I}}(P_R)) \cong e\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(\mathrm{Mat}_n(R))e = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(e\mathrm{Mat}_n(R)e) \cong \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(\mathrm{End}(P_R)).$$

(ii) Let $\kappa : P_R \to e(\oplus^n R_R)$ be an isomorphism, where $e = e^2 \in \mathrm{Mat}_n(R)$. Then there exists an isomorphism $\overline{\kappa} : H_{\mathfrak{F}\mathfrak{I}}(P_R) \to e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R)$ which is an extension of κ. Thus to show that $\mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(P_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap P = \mathrm{Rad}(P_R)$, it is enough to see that $\mathrm{Rad}(e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap e(\oplus^n R_R) = \mathrm{Rad}(e(\oplus^n R_R))$.

Let $M_R = e(\oplus^n R_R)$. Since M_R is projective, we have that $\mathrm{Rad}(M_R) = M\mathbf{J}(R) = e(\oplus^n R_R)\mathbf{J}(R) = e(\oplus^n \mathbf{J}(R))$. Now $H_{\mathfrak{F}\mathfrak{I}}(M_R) = e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))$ by Theorem 6. Therefore, $H_{\mathfrak{F}\mathfrak{I}}(M_R)$ is a projective right $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$-module, so it follows that $\mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(M_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) = (H_{\mathfrak{F}\mathfrak{I}}(M)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)})\mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) = e(\oplus^n \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))\mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) = e(\oplus^n \mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))$. Say $e(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \in \mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(M_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap M$ with $\alpha_i \in \mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))$. Then $e(\alpha_1 + \alpha_2 + \cdots + \alpha_n) = e(r_1 + r_2 + \cdots + r_n)$ for some $r_i \in R$. Let $e = (a_{ij}) \in \mathrm{Mat}_n(R)$. Then

$$(a_{ij})\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix} = (a_{ij})\begin{pmatrix}r_1\\ r_2\\ \vdots\\ r_n\end{pmatrix},$$

so $a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = a_{i1}r_1 + a_{i2}r_2 + \cdots + a_{in}r_n$ for $i = 1, 2, \ldots, n$. Since $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))$, it follows that $a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n \in \mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))$ for $i = 1, 2, \ldots, n$. Thus $a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = a_{i1}r_1 + a_{i2}r_2 + \cdots + a_{in}r_n \in \mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) \cap R$ for $i = 1, 2, \ldots, n$. By [**19**, Theorem 2.2], $\mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) \cap R = \mathbf{J}(R)$, hence $a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = a_{i1}r_1 + a_{i2}r_2 + \cdots + a_{in}r_n \in \mathbf{J}(R)$. Thus

$$e\begin{pmatrix}r_1\\ r_2\\ \vdots\\ r_n\end{pmatrix} = \begin{pmatrix}a_{11}r_1 + \cdots + a_{1n}r_n\\ a_{21}r_1 + \cdots + a_{2n}r_n\\ \vdots\\ a_{n1}r_1 + \cdots + a_{nn}r_n\end{pmatrix} \in e\begin{pmatrix}\mathbf{J}(R)\\ \mathbf{J}(R)\\ \vdots\\ \mathbf{J}(R)\end{pmatrix} = e(\oplus^n R_R)\mathbf{J}(R) = \mathrm{Rad}(M_R).$$

Therefore, $\mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(M_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap M \subseteq \mathrm{Rad}(M_R)$.

On the other hand, note that $\mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) \cap R = \mathbf{J}(R)$ again by [**19**, Theorem 2.2]. Thus it follows that $\mathrm{Rad}(M_R) = e(\oplus^n \mathbf{J}(R)) \subseteq e(\oplus^n \mathbf{J}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))) = \mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(M_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)})$, hence $\mathrm{Rad}(M_R) \subseteq \mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(M_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap M$. So $\mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(M_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap M = \mathrm{Rad}(M_R)$. Therefore, $\mathrm{Rad}(H_{\mathfrak{F}\mathfrak{I}}(P_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}) \cap P = \mathrm{Rad}(P_R)$. □

When P_R is a progenerator, we have the following:

COROLLARY 13. *Let R be a semiprime ring.*

(i) *If P_R is a progenerator of the category Mod-R, then $H_{\mathfrak{F}\mathfrak{I}}(P_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}$ is a progenerator of the category Mod-$\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$.*

(ii) *If R and S are Morita equivalent, then $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ and $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(S)$ are Morita equivalent.*

Proof. (i) Assume that P_R is a progenerator for Mod-R. Let $P_R \cong e(\oplus^n R_R)$ with $e = e^2 \in \text{Mat}_n(R)$ and let $S = \text{End}(P_R)$. Then R is Morita equivalent to S and $S \cong e\text{Mat}_n(R)e$ with $\text{Mat}_n(R)e\text{Mat}_n(R) = \text{Mat}_n(R)$. Now $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(S) \cong e\text{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))e$ by Lemmas 3 and 11 since S is semiprime. Moreover, we have that $\text{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))e\text{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)) = \text{Mat}_n(R\mathbf{B}(Q(R))e\text{Mat}_n(R\mathbf{B}(Q(R)) = [\text{Mat}_n(R)e\text{Mat}_n(R)]\mathbf{B}(Q(R)) = \text{Mat}_n(R)\mathbf{B}(Q(R)) = \text{Mat}_n(R\mathbf{B}(Q(R))) = \text{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))$ by noting that $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) = R\mathbf{B}(Q(R))$ from Lemma 3. Also by Theorem 12(i), it follows that $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(S) \cong \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(\text{End}(P_R)) \cong \text{End}(H_{\mathfrak{F}\mathfrak{I}}(P_R)_R) = e\text{Mat}_n(\text{End}(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)_R))e \cong e\text{Mat}_n(\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R))e = \text{End}(H_{\mathfrak{F}\mathfrak{I}}(P_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)})$. Thus we have that $H_{\mathfrak{F}\mathfrak{I}}(P_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)}$ is a progenerator of the category Mod-$\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$.

(ii) This part was proved in [**18**]. We include a proof for the sake of completeness. Suppose that R and S are Morita equivalent. Then there exists a progenerator P_R of the category Mod-R such that $S = \text{End}(P_R)$. Thus by Theorem 12(i), $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(S) \cong \text{End}(H_{\mathfrak{F}\mathfrak{I}}(P_R)_R) = \text{End}(H_{\mathfrak{F}\mathfrak{I}}(P_R)_{\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)})$, so $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ and $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(S)$ are Morita equivalent. □

We notice that the converse of Corollary 13(ii) is not true. Let $R = \mathbb{Z}[G]$ the group ring of G over $\mathbb{Z}$, where G is the group of order two. Then R is semiprime by [**31**, p. 162, Proposition 8]. Let $S = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$. From Lemma 3 and the fact that $Q(R) = Q(S)$, it follows that $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(S)$. If R and S are Morita equivalent, then $R \cong S$ since R and S are commutative (see [**30**, p. 494, Corollary 18.42]). So we have a contradiction because R is not quasi-Baer by [**13**, Example 1.11].

Recall from [**37**] that a module M_R is a *quasi-Baer module* if for any $N_R \trianglelefteq M_R$, there exists $h = h^2 \in \Lambda = \text{End}(M_R)$ such that $\ell_\Lambda(N) = \Lambda h$, where $\ell_\Lambda(N) = \{\lambda \in \Lambda \mid \lambda N = 0\}$. It is clear that R_R is a quasi-Baer module if and only if R is a quasi-Baer ring. Also it is shown in [**37**] that M_R is quasi-Baer if and only if for any $I \trianglelefteq \Lambda$ there exists $g = g^2 \in \Lambda$ such that $r_M(I) = gM$, where $r_M(I) = \{m \in M \mid Im = 0\}$. Moreover, if M_R is quasi-Baer, then $\text{End}(M_R)$ is a quasi-Baer ring [**37**, Theorem 4.1]. Close connections between quasi-Baer modules and FI-extending modules are investigated in [**37**].

In the next result, we obtain another close connection between FI-extending modules and quasi-Baer modules which also generalizes some of the equivalences in [**12**, Theorem 4.7].

THEOREM 14. *Assume that P_R is a finitely generated projective module over a semiprime ring R. Then the following are equivalent:*

(i) *P_R is (strongly) FI-extending.*
(ii) *P_R is quasi-Baer.*
(iii) $\text{End}(P_R)$ *is a quasi-Baer ring.*
(iv) $\text{End}(P_R)$ *is a right FI-extending ring.*

Proof. Let $P_R \cong e(\oplus^n R_R)$ with $e = e^2 \in \text{End}(\oplus^n R_R) \cong \text{Mat}_n(R)$ and $n > 0$.

(i)⇒(ii) Assume that P_R is FI-extending. Then $P_R = H_{\mathfrak{F}\mathfrak{I}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)_R)$ by Theorem 6 and Lemma 3. Since $\text{End}(\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)_R) \cong \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ by Lemma 4, we see that $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)_R$ is quasi-Baer. So $\oplus^n \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)_R$ is quasi-Baer [**37**, Proposition 3.19]. Hence $e(\oplus^n \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)_R)$ is quasi-Baer from [**37**, Theorem 3.17]. Therefore P_R is quasi-Baer.

(ii)⇒(iii) It follows from [**37**, Theorem 4.1].

(iii)⇒(i) Suppose that $\text{End}(P_R)$ is a quasi-Baer ring. Then $\text{End}(P_R) \cong e\text{Mat}_n(R)e$ is quasi-Baer. Now $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(\text{Mat}_n(R)) = \text{Mat}_n(R)\mathbf{B}(Q(\text{Mat}_n(R)) = \text{Mat}_n(R)\mathbf{B}(\text{Mat}_n(Q(R))) = \text{Mat}_n(R)\mathbf{B}(Q(R)) = \text{Mat}_n(R\mathbf{B}(Q(R))) = \text{Mat}_n(\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R))$ by Lemma 3. Thus $e\text{Mat}_n(R)e = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(e\text{Mat}_n(R)e) = e\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(\text{Mat}_n(R))e = e\text{Mat}_n$

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$(\widehat{Q}_{\mathfrak{qB}}(R))e$ by Lemma 11. Let $f \in \mathbf{B}(Q(R)) = \mathbf{B}(\mathrm{End}(E(R_R)))$ and let I_n be the identity matrix in $\mathrm{Mat}_n(R)$. Then we have that $fI_n \in \mathbf{B}(\mathrm{Mat}_n(Q(R))) = \mathbf{B}(\mathrm{End}(\oplus^n E(R_R)))$. Thus $e \cdot fI_n \cdot e \in e\mathrm{Mat}_n(\widehat{Q}_{\mathfrak{qB}}(R))e = e\mathrm{Mat}_n(R)e$. Let $e \cdot fI_n \cdot e = (\alpha_{ij}) \in e\mathrm{Mat}_n(R)e \subseteq \mathrm{Mat}_n(R)$. Then we see that

$$e\begin{pmatrix} fR \\ \vdots \\ fR \end{pmatrix} = e \cdot fI_n \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} = e \cdot fI_n \cdot e \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} = e(\alpha_{ij})e \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} \subseteq e \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix}.$$

Hence $e(\oplus^n \widehat{Q}_{\mathfrak{qB}}(R)_R) = e(\oplus^n R_R)$ because $\widehat{Q}_{\mathfrak{qB}}(R) = R\mathbf{B}(Q(R))$ by Lemma 3. Hence from Theorem 6, $H_{\mathfrak{FI}}(e(\oplus^n R_R)) = e(\oplus^n R_R)$, so $e(\oplus^n R_R)$ is (strongly) FI-extending. Therefore P_R is (strongly) FI-extending.

(iii)⇔(iv) Since $\mathrm{End}(P_R)$ is semiprime, this equivalence follows from [**12**, Theorem 4.7]. □

For a ring R, let $\mathrm{Aut}(R)$ denote the *group of ring automorphisms* of R. Let G be a subgroup of $\mathrm{Aut}(R)$. For $r \in R$ and $g \in G$ let r^g denote the image of r under g. We use R^G to denote the *fixed ring* of R under G (i.e. $R^G = \{r \in R \mid r^g = r \text{ for every } g \in G\}$). The *skew group ring*, $R * G$, is defined to be $R * G = \oplus \sum_{g \in G} Rg$ with addition given componentwise and multiplication given as follows: if $a, b \in R$ and $g, h \in G$, then $(ag)(bh) = ab^{g^{-1}}gh \in Rgh$.

Let R be a semiprime ring. For $g \in \mathrm{Aut}(R)$, let $\phi_g = \{x \in Q^m(R) \mid xr^g = rx \text{ for each } r \in R\}$, where $Q^m(R)$ is the Martindale right ring of quotients of R. We say that g is *X-outer* if $\phi_g = 0$. A subgroup G of $\mathrm{Aut}(R)$ is called *X-outer* on R if every $1 \neq g \in G$ is X-outer (see [**5**, pp. 139–143]).

Assume that G is a finite group of ring automorphisms of a ring R. Say $G = \{g_1, \ldots, g_n\}$. For $r \in R$ and $\alpha = a_1g_1 + \cdots + a_ng_n \in R * G$ with $a_i \in R$, define $r \cdot \alpha = r^{g_1}a_1^{g_1} + \cdots + r^{g_n}a_n^{g_n}$. Then R is a right $R * G$-module. Moreover, we see that ${}_{R^G}R_{R*G}$ is an $(R^G,\ R * G)$-bimodule. Also $\mathrm{End}\,(R_{R*G}) \cong R^G$.

If A is a C^*-algebra (not necessarily unital), then the set $\mathcal{F}$ of all norm closed essential two-sided ideals forms a filter directed downward by inclusion. The ring $Q_b(A)$ denotes the algebraic inductive limit of $\{M(I)\}_{I \in \mathcal{F}}$, where $M(I)$ is the multiplier C^*-algebra of I. In [**3**], the ring $Q_b(A)$ is called the *symmetric normed algebra of quotients* of A. The norm completion of $Q_b(A)$, i.e. the C^*-algebra inductive limit $M_{\mathrm{loc}}(A)$ of $\{M(I)\}_{I \in \mathcal{F}}$, is called the *local multiplier algebra* of A which was used to solve operator equations on A (see [**24**] and [**35**]). In [**1**–**4**], $Q_b(A)$ and $M_{\mathrm{loc}}(A)$ of a C^*-algebra A have been extensively studied. For more details on local multiplier algebras, see [**5**].

According to [**5**, Definition 3.2.1, p. 73], for a C^*-algebra A, the C^*-subalgebra $\overline{AC_b(A)}$ (i.e., the norm closure of $AC_b(A)$ in $M_{\mathrm{loc}}(A)$) of $M_{\mathrm{loc}}(A)$ is called the *bounded central closure* of A and denoted by ${}^cA$, where $C_b(A)$ is the centre $\mathrm{Cen}(Q_b(A))$ of $Q_b(A)$. If $A = {}^cA$, then A is called *boundedly centrally closed*. It is shown in [**5**, Theorem 3.2.8 and Corollary 3.2.9, pp. 75–76] that the local multiplier algebra and the bounded central closure of a C^*-algebra are boundedly centrally closed. Also, it is shown in [**19**, Lemma 4.12] that a unital C^*-algebra A is boundedly centrally closed if and only if A is quasi-Baer.

By [**5**] a $*$-preserving ring automorphism of a C^*-algebra is called an *$*$-automorphism*. When A is a unital C^*-algebra with a finite group G of $*$-automorphisms of A, it was shown in [**5**, Section 4.4, pp. 139–141] that $A * G$ and A^G are C^*-algebras.

COROLLARY 15. *Let A be a unital C^*-algebra and G a finite group of $*$-automorphisms of A. Then the following conditions (i), (ii) and (iii) are equivalent:*

(i) *A_{A*G} is (strongly) FI-extending.*

(ii) *A_{A*G} is quasi-Baer.*

(iii) *A^G is a boundedly centrally closed C^*-algebra.*
Further, if G is X-outer, then the following conditions (iv) and (v) are equivalent to conditions (i)–(iii):

(iv) *$A * G$ is a boundedly centrally closed C^*-algebra.*

(v) *A is G-quasi-Baer (i.e. the right annihilator of a G-invariant ideal of A is generated by an idempotent).*

Proof. Recall that End $(A_{A*G}) \cong A^G$. Since $|G|$ is invertible, $e = |G|^{-1}(\sum_{g\in G} g) \in A * G$ is an idempotent and $A_{A*G} \cong e(A * G)_{A*G}$ as $A * G$-modules. Thus A_{A*G} is a finitely generated projective module. Thus by [**19**, Lemma 4.12] A^G is quasi-Baer if and only if A^G is boundedly centrally closed. Hence the equivalence of (i)–(iii) follows immediately from Theorem 14.

Further, assume that G is X-outer. Note that the condition (iv) is equivalent to the fact that $A * G$ is quasi-Baer. Hence from [**19**, Lemma 4.12] and [**27**, Theorem 10], (iii), (iv) and (v) are equivalent. □

Note that in [**37**, Example 4.2], there is an example of a module M_R such that End(M_R) is a quasi-Baer ring, but M_R is not quasi-Baer. In this paper we have shown that for $\mathfrak{M} = \mathfrak{F}\mathfrak{I}$, if R is a semiprime ring then $H_{\mathfrak{F}\mathfrak{I}}(R_R) = \widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$. This motivates the following problem:

PROBLEM. For a given class $\mathfrak{M}$ of modules, determine necessary and/or sufficient conditions on R such that $H_{\mathfrak{M}}(R_R) = \widehat{Q}_{\mathfrak{M}}(R)$.

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