of an *aggregate as such*, without the introduction of a new logical fundamental concept (such as that of order).

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Dear Sir:

Through the kindness of Professor Fraenkel, I was enabled to see his note before publication. There are one or two remarks which may clarify some of the issues suggested in my article.

I must confess to a difficulty in understanding the various interpretations that have been given to Cantor's definition of "Menge." Perhaps one accepted English translation (E. W. Hobson, *The Theory of Functions of a Real Variable*, Art. 111) will clear up the matter: "A collection of definite distinct objects which is regarded as a single whole is called an aggregate." It will be noticed that "collection," as an English word, could have two distinct meanings—the uncompleted act of collecting, or the result of a completed act of collection. Likewise the word "regarded" is ambiguous. However, I gave Cantor's exact words, although I have not yet any clear idea of what the German words "Anschauung" and "Denkens" mean; an extreme behaviorist might include both words in the category of meaningless noises. Similarly for "regarded," which is metaphorical. The metaphor is unresolved. The apparent fact that there seems to be still some doubt among experts as to the precise meaning of "class," or of "Menge," indicates that something remains to be done in the way of definition. One possibility is to accept "Menge" intuitively; another is to add further definitions of the terms used in the official definition. I believe the net result will be the same: those to whom the original definition is free of mysticism will be confirmed in their insight, while those who lack insight originally will not be enlightened.

It seems to me that this extremely elementary matter of what a "Menge" is, is the parting of the ways, where those who are capable of belief in what is not humanly constructible go to the right, while others turn to the left. The leftists may agree that the words sound like sense but are reluctant to credit the words with more than sound. Few mathematicians, however, are such extreme leftists as this, and even the most skeptical do not let their lack of beliefs interfere with their technical mathematics—which, according to their philosophical nihilism, may be totally devoid of meaning or consistency.

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Even after Professor Fraenkel's lucid remarks on the possibility of distinguishing the *ordered* pairs (a, b), (a, b), I still believe an appeal to extra-mathematical elements (psychological, say) is made in affirming the possibility. Although Professor Fraenkel disavows any intention of going into the psychological or physiological aspects of the question, it seems to me that his own description contains at least some such elements. Further, if we as mathematicians are later to found precise ideas of simultaneity and spatial sequence on the number system, it would seem that we are likely to run into circularities if we appeal to these very ideas in constructing the number system. Would it not be simpler, as some of the formalist-logisticists do, to postulate that a "sign," like (a, b), for instance, has a definite, recognizable "individuality," and that "distinct" signs, like (a, b) and (b, a) are "recognizable" as "distinct"—if they "are" distinct. Although it might be simpler, it would none the less be a highly sophisticated appeal to elements which are not usually included explicitly in the postulate systems from which we construct or deduce the real numbers. In any case, I believe it should be stated what our starting point is to be—marks on paper and verbal rules (not always set out as part of the official formalistic program) for manipulating them, or an intensional attempt to put some meaning into or behind our marks. I do not recall any attempt to do this seemingly simple thing which has withstood destructive criticism to the extent that what is left of the attempt is commonly accepted by a majority of specialists in the foundations of mathematics.

In this connection, Professor Fraenkel's allusion to Cantor's distinction between an "immanent" and a "transient" foundation, brings out sharply one of the points on which leftists and rightists appear to differ beyond hope of reconciliation. The distinction in question is reminiscent of mediaeval doctrines of the theological infinite, which may be appropriate in the philosophy of one who, like Cantor, was sympathetic toward such doctrines, but which, nevertheless, seem curiously vestigial in a mechanized civilization. A few years ago such doctrines reached one of their recurrent high water marks in English-speaking countries with the enthusiastic reception accorded Evelyn Underhill's *Mystic Way*. I believe it was the war which put the quietus on the widespread revival of mysticism inspired by Miss Underhill's work, by demonstrating in a pragmatic fashion that the scholastic philosophy does not suit the brutally direct modern mind. It may be that a little more of such scientific psychology as we have and a little less mediaeval theology in the beginnings of mathematics may be just what mathematics needs in order to

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bring it down from heaven to earth and enable it to reach a humanly acceptable agreement with itself.

In ascribing the *spirit* of the method of ordered pairs to Gauss, I thought I was on fairly safe ground, because it seemed to me that the passages from Gauss quoted presently are as strong a justification for the claim that he had the *spirit* of the method as is some of the evidence put forward by the editors of Gauss' works for other anticipations by the "Princeps mathematicorum."[1]

It seems to me that the man who first clearly saw the two-dimensionality (not necessarily a geometrical intuition) of the complex number $a + bi$ recognized that he was handling an ordered pair of real numbers a, b, the ordering being implicit in the definition of equality for such complex numbers, namely that $a + bi = c + di$ (a, b, c, d, real) when and only when $a = c$ and $b = d$, had the root of the matter. The imaginary unit directs that the reals be correlated in a definite, ordered way, and if we write (a, b) for $a + bi$, and impose then the definition of equality for these particular ordered pairs, $(a, b) = (c, d)$ when and only when $a = c$ and $b = d$, we add only a new notation to what we were doing already, namely manipulating ordered pairs. The first quotation contains no allusion to the definition of equality which (it seems to me) introduces the notion of ordered pairs, nor can I find that Gauss anywhere set down the postulates for equality of complex numbers or the hypercomplex numbers which he discussed. Nevertheless, the germ of the idea does seem to me to be in the first extract, and without the full-blown idea itself I do not see how we are to interpret the equality stated by Gauss in the second extract (on which, in part, his claim to have anticipated Hamilton in the invention of quaternions is based). As the remark "I am more interested in notions than in notations" is sometimes fathered on Gauss, I thought the "notion" in the first quotation, which is independent of the geometrical interpretation of complex numbers, and which blossoms out into the "notation" for hypercomplex numbers in the second, justified ascribing the idea to Gauss. I may say that I was thoroughly familiar with the historical sources cited by Professor Fraenkel, including his own extremely interesting paper on materials for a scientific biography of Gauss, when I wrote my paper.

"Sind aber die Gegenstände von solcher Art, dass sie nicht in Ein, wenn gleich unbegrenzte, Reihe geordnet werden können, sondern sich

[1] See, for instance, as an illuminating essay in reconstructive historical criticism, Gauss' Werke, Bd. X2, pp. 93–105, especially p. 104, and the references there given.

nur in Reihen von Reihen ordnen lassen, oder was dasselbe ist, bilden sie ein Mannigfaltigkeit von zwei dimensionen; verhält es sich dann mit den Relationen einer Reihe zu einer andern oder den Uebergängen aus einer in die andere auf eine ähnliche Weise wie vorhin mit den Uebergängen von einem Gliede einer Reihe zu einem andern Gliede derselben Reihe, so bedarf es offenbar zur Abmessung des Ueberganges von einem Gliede des Systems zu einum andern ausser den vorigen Einheiten $+1$ und -1 noch zweier andern unter sich auch entegegengesetzen $+i$ und $-i$. Offenbar muss aber dabei noch postulirt werden, dass die Einheit i allemal den Uebergang von einem gegebenen Gliede einer Reihe zu einem *bestimmten* Gliede der unmittlebar angrenzenden Reihe bezeichne. Auf dies Weise wird also das System auf ein doppelte Art in Reihen von Reihen geordnet werden können."—Gauss, *Werke*, vol. II, p. 176.

The "notion" appears also in Gauss' *Werke*, vol. VIII pp. 359–60, where the "combination" a, b, c, d is denoted by (a, b, c, d) and "we write"

$$(a, b, c, d)\ (\alpha, \beta, \gamma, \delta) = (A, B, C, D),$$

etc. According to the Editor (Stäckel), internal evidence assigns this fragment to about the year 1819, which is before the date of Hamilton's first published discussion of ordered pairs. That Gauss, in this equality, had broken away from geometrical intuition, seems evident, as he did not discuss 4-dimensional "space."

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A NOTE ON THE VALIDITY OF ARISTOTELIAN LOGIC

Dear Sir:

In a recent issue of this periodical,[1] there appeared an article by Mr. L. Kattsoff entitled "Concerning the Validity of Aristotelian Logic." By use of the formula for the A proposition, $A(ab) = (a < b)\ [(b < a) + (a < b')'\ (b' < a)']$[2] and corresponding formulae for the E, I and O types, Mr. Kattsoff gives the method of Dr. H. B. Smith of establishing the complete generality of Aristotelian logic, its consistency, and the validity of all the classical forms of inference. That Dr. Smith's

[1] *Philosophy of Science,* Vol. 1, No. 2, pp. 149–162, April, 1934.

[2] *Loc. cit.* p. 157. "<" and "′" have their usual meaning in the algebra of classes. "+" indicates a disjunction of propositions.

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