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ON SECOND DERIVATIVE ESTIMATES FOR EQUATIONS OF MONGE-AMPÈRE TYPE

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We derive interior second derivative estimates for solutions of equations of Monge-Ampère type which extend those of Pogorelov for the case of affine boundary values. A key ingredient in our method is the existence of a strong solution of the homogeneous Monge-Ampère equation.

1. Introduction

Interior second derivative estimates for convex solutions of the Monge-Ampère equation

(1.1) $$\det D^2 u = g(x)$$

were derived by Pogorelov [8],[9], under the restriction that the solution u have affine boundary values. Here g is a positive function in $C^{1,1}(\Omega)$ and Ω a convex domain in Euclidean n space, $\mathbb{R}^n$. Pogorelov's method was subsequently extended to encompass Monge-Ampère type equations of the form,

(1.2) $$\det D^2 u = g(x,u,Du),$$

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where g is a positive function in $C^{1,1}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, in Lions [6,7] and Gilbarg and Trudinger [2]. A somewhat different approach, embracing less smooth functions g, was given by Ivochkina [3]. In this paper we establish interior estimates for solutions subject to $C^{1,1}$ boundary data. In particular we prove the following.

THEOREM 1. *Let* Ω *be a* $C^{1,1}$ *, uniformly convex domain in* $\mathbb{R}^n$ *,* φ *a function in* $C^{1,1}(\bar{\Omega})$ *and* g *a positive function in* $C^{1,1}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ *. Then if* u *is a convex classical solution of the Dirichlet problem*

$$(1.3)\qquad \det D^2u = g(x,u,Du) \quad in \quad \Omega\ , \quad u=\varphi \quad in \quad \partial\Omega\ ,$$

we have for any $\Omega' \subset\subset \Omega$

$$(1.4)\qquad \sup_{\Omega'} |D^2u| \leqslant C$$

where C *is a constant depending only on* $n,\Omega\ ,\Omega'\ ,|\varphi|_{1,1;\Omega}, g\ , |u|_{0;\Omega}$ *and the modulus of continuity of* u *on* $\partial\Omega$*.*

Our derivation of Theorem 1 rests on the following existence theorem for the *homogeneous* Monge-Ampère equation.

THEOREM 2. *Let* Ω *be a* $C^{1,1}$*, uniformly convex domain in* $\mathbb{R}^n$ *and* $\varphi \in C^{1,1}(\bar{\Omega})$*. Then there exists a unique, convex solution* $u \in C^{1,1}(\Omega) \cap C^{0,1}(\bar{\Omega})$ *of the Dirichlet problem.*

$$(1.5)\qquad \det D^2u = 0 \quad \text{in} \quad \Omega\ , \quad u=\varphi \quad \text{in} \quad \partial\Omega\ .$$

Theorem 2 improves earlier work, in particular that of Rauch and Taylor [10], concerning the existence of generalized solutions of (1.5). As in [10], the solution u is characterized as the lower boundary of the convex hull in $\mathbb{R}^{n+1}$ of the boundary manifold $(\partial\Omega,\varphi)$. We have also been informed that a result similar to Theorem 2 has been proved by Bedford and Taylor. The passage from Theorem 2 to Theorem 1 will be accomplished with the aid of the Pogorelov method.

Theorems 1 and 2 are proved in Sections 2 and 3 of this paper. Notation, unless otherwise indicated, will follow that of the book [2] .

2. The Homogeneous Equation

In this section we will prove Theorem 2. The existence of a unique convex generalized solution of (1.5) was proved in [10] under the weaker hypotheses that Ω is bounded and strictly convex and $\varphi \in C^0(\bar{\Omega})$, so we need only prove the regularity assertion. Geometrically, the graph of u is the lower boundary of the convex hull of graph $(\varphi|_{\partial\Omega})$.

To prove that $u \in C^{0,1}(\bar{\Omega})$ we can assume without loss of generality that φ is convex, so that φ is a lower barrier for u. Also, using the convexity of u, we have

$$\frac{u(x)-u(y)}{|x-y|} \leqslant |D\varphi|_{0;\Omega}$$

for all $y \in \partial\Omega$ and $x \in \Omega$. We thus obtain a global gradient bound for u.

It remains only to prove that $u \in C^{1,1}(\Omega)$. This will be carried out in the following lemmas. Let $M = \text{graph}(u|_{\Omega})$, $\partial M = \bar{M} \cap (\partial\Omega \times \mathbb{R})$, and for $E \subset \mathbb{R}^{n+1}$ let $\text{conv}(E)$ denote the convex hull of E. If $x,y \in \mathbb{R}^{n+1}$, $[x,y]$ denotes the closed line segment joining x and y.

LEMMA 2.1 *Let T be a supporting hyperplane of M at $\xi \in M$. Then*

(2.1) $$T \cap \bar{M} = \text{conv}(T \cap \partial M).$$

Proof. For convenience we assume that $u(\xi) = 0$ and $T = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Then $T \cap \bar{M} = \{x \in \bar{\Omega} : u(x) = 0\}$ and $T \cap \partial M = \{x \in \partial\Omega : u(x) = 0\}$.

Suppose $y \in T \cap \bar{M} - \text{conv}(T \cap \partial M)$. Then there is an $n-1$ dimensional plane $S \subset T$ passing between y and $\text{conv}(T \cap \partial M)$ such that $d(y,S) > 0$ and $d(S, \text{conv}(T \cap \partial M)) > 0$. Let S^+ and S^- denote the half spaces in $\mathbb{R}^n$ associated with S. We may assume that $S = \{x \in \mathbb{R}^n : x_1 = 0\}$ and $S^- = \{x \in \mathbb{R}^n : x_1 < 0\}$. Assume also that $\text{conv}(T \cap \partial M) \subset S^-$. Then for some $\varepsilon > 0$ we have $u > \varepsilon$ on $S^+ \cap \partial\Omega$ and $u \leqslant 0$ on $S^- \cap \partial\Omega$. Hence for $\delta > 0$ sufficiently small, $Q = \{x \in \mathbb{R}^{n+1} : \delta x_1 - x_{n+1} = 0\}$ is a hyperplane containing S,

$y \in \{x \in \mathbb{R}^{n+1} : \delta x_1 - x_{n+1} > 0\}$ and $\text{graph}(\varphi|_{\partial\Omega}) \subset \{x \in \mathbb{R}^{n+1} : \delta x_1 - x_{n+1} < 0\}$. Thus $y \notin \text{conv}(\text{graph}(\varphi|_{\partial\Omega}))$, which is a contradiction.

LEMMA 2.2 *For each* $\xi \in M$ *there are* $\zeta_1 \in \partial M$ *and* $\zeta_2 \in M$ *such that* $\xi \in [\zeta_1, \zeta_2] \subset \bar{M}$ *and*

$$|\xi - \zeta_2| \geq \frac{1}{2n} |\zeta_1 - \zeta_2| . \tag{2.2}$$

Proof. Let T be a supporting hyperplane of M at ξ. Then by Lemma 2.1 we may choose $n+1$ points $\xi_1, \ldots \xi_{n+1}$ in $T \cap \partial M$ such that $\xi \in \text{conv}\{\xi_i\}_{i=1}^{n+1}$. From these points we may choose k points, say $\xi_1, \ldots, \xi_k$ such that $\xi \in \text{int conv}\{\xi_i\}_{i=1}^{k}$, where $\text{int conv}\{\xi_i\}_{i=1}^{k}$ denotes the $k-1$ dimensional interior of the simplex $P = \text{conv}\{\xi_i\}_{i=1}^{k}$.

Let η_i be the unique point in ∂P such that $\xi \in [\xi_i, \eta_i]$. Since $\xi \in \text{int } P$, we have $\eta_i \in \text{int } F_i$ for some face F_i of P, and no two η_i lie in the same face. We will show that for some i $\zeta_1 = \xi_i$ and $\zeta_2 = \eta_i$ satisfy the conclusion of the lemma.

Suppose this is not the case. Then

$$\xi \in G_i = \{x \in P : d(x, F_i) < \frac{1}{2n} d(\xi_i, F_i)\}$$

for all $i = 1, \ldots, k$. For each $j = 1, \ldots, k$, $\bigcap_{i \neq j} G_i$ is a $k-1$ dimensional parallelogram with side lengths $\frac{1}{2n} d(\xi_i, F_i)$ for $i \neq j$, and $\xi_j \in \bigcap_{i \neq j} F_i \subset \bigcap_{i \neq j} G_i$ is a vertex of this parallelogram. Thus

$$|\xi - \xi_j| < \frac{1}{2n} \sum_{i \neq j} d(\xi_i, F_i) ,$$

and hence

$$\begin{aligned} d(\xi_j, F_j) &\leq |\xi - \xi_j| + d(\xi, F_j) \\ &\leq \frac{1}{2n} \sum_{i=1}^{k} d(\xi_i, F_i) . \end{aligned}$$

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Summing over j from 1 to k we obtain a contradiction, so the lemma is proved.

LEMMA 2.3 *Let* $x_0 \in \Omega$ *and* $\xi_0 = (x_0, u(x_0))$. *Then if* $x \in B = \{x \in \Omega : |x-x_0| < \frac{1}{6n} d(x_0, \partial\Omega)\}$, *we have*

$$u(x) \leqslant u(x_0) + Du(x_0) \cdot (x-x_0) + C|x-x_0|^2 , \tag{2.3}$$

where C *depends only on* n , $|\varphi|_{1,1;\Omega}$, $\operatorname{diam}\Omega$, $d(x_0, \partial\Omega)$ *and a positive lower bound* R *on the principal radii of curvature of* $\partial\Omega$.

Proof. Let $\zeta_1 \in \partial M$ and $\zeta_2 \in M$ be the points associated with ξ_0 as in Lemma 2.2. Assume for convenience that $\zeta_2 = (0,0)$. Let $\psi : \bar{\Omega} - \{0\} \to \partial\Omega$ be the radial retraction. Then we clearly have for $x \in \bar{\Omega} - \{0\}$,

$$|x| \leqslant |\psi(x)| , \tag{2.4}$$

and by Lemma 2.2,

$$d(x_0, \partial\Omega) \leqslant |\psi(x_0)| \leqslant 2n|x_0|. \tag{2.5}$$

If $x \in B' = \{x \in \Omega : |x-x_0| < \frac{1}{4n} d(x_0, \partial\Omega)\}$, we have

$$\begin{aligned} d(x,\partial\Omega) &\geqslant d(x_0, \partial\Omega) - |x-x_0| \\ &\leqslant (1 - \frac{1}{4n}) \, d(x_0, \partial\Omega) , \end{aligned} \tag{2.6}$$

and also, from (2.5) ,

$$\begin{aligned} \frac{1}{|x|} &\leqslant \frac{1}{|x_0| - |x-x_0|} \\ &\leqslant \frac{4n}{d(x_0, \partial\Omega)} . \end{aligned} \tag{2.7}$$

Assuming initially that $\partial\Omega \in C^2$ and $\varphi \in C^2(\bar{\Omega})$ we define $w : \bar{\Omega} \to \mathbb{R}$ by

$$w(x) = \begin{cases} \frac{|x|}{|\psi(x)|} \varphi(\psi(x)) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 . \end{cases} \tag{2.8}$$

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Geometrically, the graph of w is the cone with base graph$(\varphi|_{\partial\Omega})$ and vertex(0,0). Clearly, $w \in C^2(\bar{\Omega}-\{0\})$, and $u \leqslant w$ in Ω, by the convexity of u. Consequently the graphs of u and w are tangent at x_0 whence u is differentiable there with $Du(x\ ) = Dw(x_0)$.

Furthermore, differentiating (2.8) twice, and using (2.4) and (2.7), we obtain

$$(2.9) \qquad |D^2w|_{0;B'} \leqslant C(n,|\varphi,\psi|_{2;B'},d(x_0,\partial\Omega)) .$$

We now proceed to obtain a bound for $|\psi|_{2;B'}$. Clearly, we have $|\psi|_{0;B'} \leqslant \operatorname{diam}\Omega$. To obtain derivative bounds, it is convenient to use polar coordinates. We write

$$(2.10) \qquad \psi(x) = \tilde{\psi}(\theta\ ,\ldots,\theta_{n-1}) ,$$

where $\theta_1,\ldots,\theta_{n-1}$ are the angular variables. Then we obtain in $\bar{\Omega} - \{0\}$,

$$(2.11) \qquad D_i\psi = \sum_{k=1}^{n-1} D_{\theta_k}\tilde{\psi}D_i\theta_k$$

and

$$(2.12) \qquad D_{ij}\psi = \sum_{k,\ell=1}^{n-1} D_{\theta_k\theta_\ell}\tilde{\psi}D_i\theta_k D_j\theta_\ell + \sum_{k=1}^{n-1} D_{\theta_k}\tilde{\psi}D_{ij}\theta_k.$$

Let ν be the outer unit normal to $\partial\Omega$ at $\psi(x)$, and T the tangent $n-1$ plane to $\partial\Omega$ at $\psi(x)$. Then

$$(2.13) \qquad \frac{\psi(x).\nu}{|\psi(x)|} = \frac{d(0,T)}{|\psi(x)|} \geqslant \frac{d(x,\partial\Omega)}{\operatorname{diam}\ \Omega} .$$

Using (2.6), (2.7) and (2.13) we obtain from (2.11) the estimate

$$(2.14) \qquad |D\psi|_{0;B'} \leqslant C(n,\operatorname{diam}\ \Omega,\ d(x_0,\partial\Omega)) ,$$

and from (2.13), also using

$$(2.15) \qquad |D_{\theta_k\theta_\ell}\tilde{\psi}|_{0;B'} \leqslant C(n,R,d(x_0,\partial\Omega)) ,$$

we obtain

(2.16) $$|D^2\psi|_{0;B'} \leqslant C(n,R,\text{diam}\,\Omega\,,d(x_0\,,\partial\Omega))\ .$$

Thus we have a bound for $|\psi|_{2;B'}$ and hence

(2.17) $$|D^2w|_{0;B'} \leqslant C\ ,$$

where C depends on $n,R,\text{diam}\,\Omega\,,d(x_0\,,\partial\Omega)$ and $|\varphi|_{2;\Omega}$.

Now let $\{\Omega_m\}$ be an increasing sequence of C^2 uniformly convex subdomains of Ω, $\cup\,\Omega_m = \Omega$, and $\{\varphi_m\} \subset C^2(\bar\Omega)$ a sequence of functions converging in $C^{1,\alpha}(\bar\Omega)$ to φ, $\alpha < 1$, and satisfying $|\varphi_m|_{2;\Omega} \leqslant 2|\varphi|_{1,1;\Omega}$. Let $w_m : \bar\Omega \to \mathbb{R}$ be the function defined by

$$w_m(x) = \begin{cases} \dfrac{|x|}{|\psi_m(x)|}\ \varphi_m(\psi_m(x)) & \text{if } x \neq 0 \\ \\ 0 & \text{if } x = 0\ , \end{cases}$$

where $\psi_m : \bar\Omega - \{0\} \to \partial\Omega_m$ is the radial retraction. For m sufficiently large, we then have uniform bounds for $|D^2w_m|_{0;B}$ where $B = \{x \in \Omega : |x-x_0| < \frac{1}{6n}\,d(x_0\,,\partial\Omega)\}$, and therefore since w_m converges to w in $C^{1,\alpha}(\bar\Omega-\{0\})$, $\alpha < 1$, we obtain $w \in C^{1,1}(\bar\Omega-\{0\})$ and

(2.19) $$[Dw]_{1:B} \leqslant C\ ,$$

where C depends only on $n,R,\text{diam}\,\Omega,d(x_0,\partial\Omega)$ and $|\varphi|_{1,1;\Omega}$.

We can now obtain the conclusion of the lemma by using Taylor's theorem and the fact that $Dw(x_0) = Du(x_0)$.

We are now ready to complete the proof of Theorem 2. Let γ be a unit vector in $\mathbb{R}^n$, and form the second order difference quotient of u with respect to γ

(2.20) $$\Delta^h_{\gamma\gamma}u(x) = \frac{1}{h^2}\,\{u(x+h\gamma) + u(x-h\gamma) - 2u(x)\}\ .$$

Then for each $\Omega' \subset\subset \Omega$, we have, from Lemma 2.3, for all $h > 0$ sufficiently small,

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(2.21)
$$|\Delta^h_{\gamma\gamma} u|_{L^\infty(\Omega')} \leq C .$$

Hence we can extract a subsequence $\{h_m\}$ converging to zero such that $\Delta^{h_m}_{\gamma\gamma} u$ converges in the weak* topology on $L^\infty(\Omega')$ to a function $w_\gamma \in L^\infty(\Omega')$. Thus it follows that the distributional derivative $D_{\gamma\gamma}u$ is representable by a function in $L^\infty_{\text{loc}}(\Omega)$. Since γ is an arbitrary direction, we conclude that $u \in C^{1,1}(\Omega)$, and for each $\Omega' \subset\subset \Omega$,

(2.22)
$$[Du]_{1;\Omega'} \leq C,$$

where C depends only on $n, R, \text{diam}\,\Omega, d(\Omega', \partial\Omega)$ and $|\varphi|_{1,1;\Omega}$.

3. Second Derivative Estimates

In this section we will prove Theorem 1. Writing the equation (1.2) in the form

(3.1)
$$F(D^2u) = \log\det D^2u = f(x,u,Du) ,$$

where $f = \log g$, we have

(3.2)
$$F_{ij} = u^{ij}$$

$$F_{ij,k\ell} = -u^{ik}u^{j\ell} = -F_{ik}F_{j\ell} ,$$

where $[u^{ij}]$ denotes the inverse of D^2u .

Next, we note that any pure second derivative $D_{\gamma\gamma}u$ of a solution $u \in C^4(\Omega)$ of (3.1) satisfies the equation

(3.3)
$$F_{ij}D_{ij\gamma\gamma}u = F_{ik}F_{j\ell}D_{ij\gamma}uD_{k\ell\gamma}u + D_{\gamma\gamma}f .$$

Since u is convex, we have $D_{\gamma\gamma}u \geq 0$, so we need only estimate $D_{\gamma\gamma}u$ from above.

We now fix $\Omega' \subset\subset \Omega$ and set $\delta = d(\Omega', \partial\Omega)$, $\Omega'' = \{x \in \Omega : d(x,\partial\Omega) > \delta/2\}$ and $\Omega''' = \{x \in \Omega : d(x,\partial\Omega) > \delta/4\}$. We

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first prove a lower bound for $\inf\limits_{\Omega''} (v-u)$, where v is the convex solution of the Dirichlet problem (1.5). Let $x_0 \in \Omega''$ and for $\sigma > 0$ set

$$\psi(x) = \psi_\sigma(x) = -\sigma((\delta/4)^2 - |x-x_0|^2) .$$

We have $D_{ij}\psi = 2\sigma\delta_{ij}$, so

$$\det(D^2v+D^2\psi) \leqslant C_1(n) \sum_{k=1}^{n} \sigma^k M^{n-k} ,$$

where
$$M = \sup_{\Omega'''} |D^2v| .$$

We also have

$$\sup_{\Omega'''} |Du| \leqslant 8\delta^{-1}|u|_{0;\Omega} ,$$

and hence

$$\inf_{\Omega'''} g(x,u,Du) \geqslant \lambda > 0 ,$$

where λ is a constant depending only on $|u|_{0;\Omega}$, δ and g.

Choosing $\sigma > 0$ so small that

$$C_1(n) \sum_{k=1}^{n} \sigma^k M^{n\ k} \leqslant \lambda ,$$

and using the comparison principle, we obtain $v - u \geqslant -\psi$ in $B_{\delta/4}(x_0)$, and hence

$$\inf_{\Omega''}(v-u) \geqslant \sigma\left(\frac{\delta}{4}\right)^2 . \tag{3.4}$$

We now consider the function

$$\eta = \eta_\varepsilon = (v-u-\varepsilon)^+$$

in the set $\Omega_\varepsilon = \{x \in \Omega : \eta(x) > 0\}$. Setting $\varepsilon = \frac{1}{2}\sigma(\delta/4)^2$, we have $\Omega'' \subset\subset \Omega_\varepsilon$, and we can estimate $d(\Omega_\varepsilon, \partial\Omega)$ from below in terms of ε, $|Dv|_{0;\Omega}$ and a modulus of continuity for u. We then have

$$\sup_{\Omega_\varepsilon} |Du| \leqslant C_2 , \tag{3.5}$$

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and hence also

$$\inf_{\Omega_\varepsilon} g(x,u,Du) \geqslant C_3 > 0 \tag{3.6}$$

and

$$\sup_{\Omega_\varepsilon}(|Df(x,u,Du)| + |D^2 f(x,u,Du)|) \leqslant C_4 , \tag{3.7}$$

where C_2, C_3 and C_4 depend on n, $|u|_{0;\Omega}$, $|\varphi|_{1,1;\Omega}$, Ω , $d(\Omega',\partial\Omega)$, g and a modulus of continuity for u . Here we have used the estimates of the previous section to remove the dependence on v .

We now consider, in the set Ω_ε , the function

$$w = \eta h(Du) D_{\gamma\gamma} u ,$$

where $h \in C^2(\mathbb{R}^n)$ is a positive function to be chosen. We then have

$$\frac{D_i w}{w} = \frac{D_i \eta}{\eta} + (\log h)_{p_k} D_{ik} u + \frac{D_{i\gamma\gamma} u}{D_{\gamma\gamma} u} ,$$

$$\frac{D_{ij} w}{w} = \frac{D_i w D_j w}{w^2} + \frac{D_{ij}\eta}{\eta} - \frac{D_i \eta D_j \eta}{\eta^2} + (\log h)_{p_k p_\ell} D_{ik} u D_{j\ell} u + (\log h)_{p_k} D_{ijk} u + \frac{D_{ij\gamma\gamma} u}{D_{\gamma\gamma} u} - \frac{D_{i\gamma\gamma} u D_{j\gamma\gamma} u}{(D_{\gamma\gamma} u)^2} .$$

Using (3.3), we obtain

$$(\eta h)^{-1} F_{ij} D_{ij} w \geqslant D_{\gamma\gamma} u \left\{ \frac{F_{ij} D_{ij}\eta}{\eta} - \frac{F_{ij} D_i \eta D_j \eta}{\eta^2} + (\log h)_{p_k p_\ell} F_{ij} D_{ik} u D_{j\ell} u + (\log h)_{p_k} F_{ij} D_{ijk} u \right\} + F_{ik} F_{j\ell} D_{ij\gamma} u D_{k\ell\gamma} u - \frac{1}{D_{\gamma\gamma} u} F_{ij} D_{i\gamma\gamma} u D_{j\gamma\gamma} u + D_{\gamma\gamma} f . \tag{3.8}$$

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An obvious choice for h is

$$h(p) = \exp(\beta|p|^2/2) \ , \ \beta > 0 \ ,$$

so that

$$(\log h)_{p_k} = \beta p_k \ , \ (\log h)_{p_k p_\ell} = \beta\delta_{k\ell} \ .$$

and hence

$$(\log h)_{p_k p_\ell} F_{ij} D_{ik}u D_{j\ell}u = \beta F_{ij} D_{ik}u D_{jk}u = \beta\Delta u$$

by (3.2).

Next, making use of the estimates (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} D_{\gamma\gamma}u(\log h)_{p_k} F_{ij} D_{ijk}u + D_{\gamma\gamma}f = {} & \beta D_k u D_{\gamma\gamma}u(f_{x_k} + f_z D_k u + f_{p_i} D_{ik}u) + \\ & f_{\gamma\gamma} + 2f_{\gamma z} D_\gamma u + 2f_{\gamma p_i} D_{i\gamma}u + f_{zz}(D_\gamma u)^2 + \\ & 2f_{zp_i} D_\gamma u D_{i\gamma}u + f_{p_i p_j} D_{i\gamma}u D_{j\gamma}u + f_z D_{\gamma\gamma}u + f_{p_i} D_{i\gamma\gamma}u \\ \geqslant {} & g_{p_i} \left(\frac{D_i w}{w} - \frac{D_i \eta}{\eta}\right) D_{\gamma\gamma}u - C_5 \{1 + |D^2u|^2 + \beta(1+|D^2u|)\} \ , \end{aligned}$$

where C_5 depends on the same quantities as C_2, C_3 and C_4 .

In order to handle the other terms in (3.8) we regard $w = w(x,\gamma)$ as a function on $\Omega_\varepsilon \times \partial B_1(0)$ and suppose w takes a maximum value at a point $y \in \Omega_\varepsilon$ and direction γ . The derivative $D_{\gamma\gamma}u(y)$ is then the maximum eigenvalue of the Hessian $D^2u(y)$ and by a rotation of coordinates we can assume that $D^2u(y)$ is in diagonal form with γ a coordinate direction. We now have

$$\begin{aligned} F_{ij} D_{ij}\eta &= F_{ij} D_{ij}v - F_{ij} D_{ij}u \\ &\geqslant -n \ . \end{aligned}$$

Furthermore, since $Dw(y) = 0$, we have

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$$F_{ij}\frac{D_i\eta D_j\eta}{\eta^2} = \frac{\Sigma F_{ii}|D_i\eta|^2}{\eta^2}$$

$$= \sum_{i\neq\gamma} F_{ii}\left(\beta D_k u D_{ik}u + \frac{D_{i\gamma\gamma}u}{D_{\gamma\gamma}u}\right)^2 + \frac{|D_\gamma\eta|^2}{\eta^2 D_{\gamma\gamma}u}$$

$$\leqslant \frac{|D_\gamma\eta|^2}{\eta^2 D_{\gamma\gamma}u} + \sum_{i\neq\gamma} F_{ii}\left(\frac{D_{i\gamma\gamma}u}{D_{\gamma\gamma}u}\right)^2 - 2\beta\sum_{i\neq\gamma}\frac{D_i\eta D_i u}{\eta}$$

at the point y. Also,

$$\frac{1}{D_{\gamma\gamma}u}\left\{\sum_{i\neq\gamma} F_{ii}(D_{i\gamma\gamma}u)^2 + F_{ij}D_{i\gamma\gamma}uD_{j\gamma\gamma}u\right\}$$

$$= \sum_{i\neq\gamma} F_{\gamma\gamma}F_{ii}(D_{i\gamma\gamma}u)^2 + \sum_{i=1}^{n} F_{\gamma\gamma}F_{ii}(D_{i\gamma\gamma}u)^2$$

$$\leqslant \sum_{i,j=1}^{n} F_{ii}F_{jj}(D_{ij\gamma}u)^2$$

$$= F_{ik}F_{j\ell}D_{ij\gamma}uD_{k\ell\gamma}u$$

at y, by virtue of our choice of coordinates. Taking the above estimates into account in (3.8), and then choosing β sufficiently large, we obtain by virtue of the strong maximum principle, ([2], Theorem 9.6)

$$D_{\gamma\gamma}u(y) \leqslant C_6\left(1 + \frac{1}{\eta(y)}\right),$$

and hence

$$\sup_{\Omega_\varepsilon} w \leqslant C_7$$

where C_6 and C_7 depend on the same quantities as C_2, C_3, C_4 and C_5. Making use of (3.4), we obtain (1.4) as required.

Remarks (i) When φ vanishes on $\partial\Omega$ we can assume that Ω is an arbitrary bounded convex domain in $\mathbb{R}^n$.

(ii) Using Theorem 1, we may infer existence theorems for the Dirichlet problem (1.3) by direct approximation from the globally smooth case treated by Caffarelli, Nirenberg and Spruck [1], Krylov [5] and Ivochkina [4]. In particular we may obtain the results of [6], [7], [11] in this way, without having to invoke regularity considerations for generalized solutions.

(iii) Note that we only need $u \in W^{4,n}_{loc}(\Omega) \cap C^3(\Omega) \cap C^0(\bar{\Omega})$ in the proof of Theorem 1; the assumptions on g automatically ensure such regularity for classical solutions $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, ([2], Lemma 17.16).

References

[1] L. Caffarelli, L. Nirenberg, J. Spruck, "The Dirichlet problem for nonlinear second order elliptic equations, I. Monge-Ampère equation. *Comm. Pure Appl. Math.* 37 (1984), 369-402.

[2] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edition (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983).

[3] N.M. Ivochkina, "Construction of a priori bounds for convex solutions on the Monge-Ampère equation by integral methods", *Ukrain. Math. J.* 30 (1978), 32-38.

[4] N.M. Ivochkina, "Classical solvability of the Dirichlet problem for the Monge-Ampère equation", Zap. Naučn. Sem. Leningrad, Otdel. Mat. Inst. Steklov. (LOMI) 131 (1983) 72-79.

[5] N.V. Krylov, "Boundedly inhomogeneous elliptic and parabolic equations in domains", Izvestia Akad. Nauk. SSSR, 47, (1983), 75-108.

[6] P.L. Lions, "Sur les equations de Monge-Ampère I", *Manuscripta Math.* 41 (1983), 1-43.

[7] P.L. Lions, "Sur les equations de Monge-Ampère II" *Arch. Rat. Mech. Anal.* (to appear).

[8] A.V. Pogorelov, "On the regularity of generalized solutions of the equation $\det(\partial^2 u/\partial x_i \partial x_j) = \varphi(x_1, \ldots, x_n) > 0$", *Dokl. Akad. Nauk SSSR*, 200 (1971), 1436-1440.

[9] A.V. Pogorelov, *The Minkowski multidimensional problem*, (Wiley, New York 1978).

[10] J. Rauch, B.A. Taylor, "The Dirichlet problem for the multi-dimensional Monge-Ampère equation", *Rocky Mountain J. Math.* 7 (1977), 345-364.

[11] N.S. Trudinger, J.I.E. Urbas, "The Dirichlet problem for the equation of prescribed Gauss curvature", *Bull. Austral. Math. Soc.* 28 (1983), 217-231.

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