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Leading terms of Artin L-functions at $s = 0$ and $s = 1$

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Abstract

We formulate an explicit conjecture for the leading term at $s = 1$ of the equivariant Dedekind zeta-function that is associated to a Galois extension of number fields. We show that this conjecture refines well-known conjectures of Stark and Chinburg, and we use the functional equation of the zeta-function to compare it to a natural conjecture for the leading term at $s = 0$.

1. Introduction

In this paper we continue the study of the leading terms at integer points of equivariant Dedekind zeta-functions that was initiated in [Bur01].

To be more specific we fix a Galois extension of number fields L/K and a sufficiently large finite set S of places of K which in particular includes all archimedean places and all places which ramify in L/K. We set $G := \mathrm{Gal}(L/K)$ and write $\zeta_{L/K,S}(s)$ for the S-truncated equivariant zeta-function of L/K which takes values in the centre $\mathrm{Z}(\mathbb{C}[G])$ of the group ring $\mathbb{C}[G]$. The function $\zeta_{L/K,S}(s)$ can be considered as the vector consisting of the S-truncated Artin L-functions for all irreducible characters of G. For each rational integer m the element $\zeta^*_{L/K,S}(m)$ which is given by the leading non-zero coefficients in the Taylor expansions of these Artin L-functions at $s = m$ lies in the unit group of $\mathrm{Z}(\mathbb{R}[G])$.

In this paper we shall formulate an explicit conjectural formula for the image of $\zeta^*_{L/K,S}(1)$ under the canonical homomorphism $\hat{\partial}$ from the unit group of $\mathrm{Z}(\mathbb{R}[G])$ to the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{R})$. Our formula involves the Euler characteristic (in the sense of [BrB05]) of a natural perfect complex of $\mathbb{Z}[G]$-modules that is constructed by using methods that are both explicit and comparatively elementary. The explicit nature of this formula allows us to prove rather easily that it simultaneously refines both the 'main conjecture' of Stark at $s = 1$ (as described by Tate in [Tat84]) and the 'Ω_1-conjecture' formulated by Chinburg in [Chi85].

In a subsequent paper we will show that our formula is equivalent, under certain hypotheses, to the 'equivariant Tamagawa number conjecture' of [BF01], as applied to the pair $(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$. The latter comparison result is interesting for several reasons: it will allow us to deduce the validity of our conjectural formula for $\hat{\partial}(\zeta^*_{L/K,S}(1))$ in the case that L is an abelian extension of $\mathbb{Q}$, it answers a question raised by Flach and the second named author in [BF96], and it also establishes the link between our explicit conjecture and the very general (and rather abstract) 'main conjecture of non-commutative Iwasawa theory' that was recently formulated by Fukaya and Kato in [FK05]. Indeed, this comparison result combines with the philosophy described by Huber and

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Kings in [HK02, § 3.3] (and by Fukaya and Kato in [FK05, § 2.3.5]) to suggest that the validity of our conjectural formula for $\hat{\partial}(\zeta^*_{L/K,S}(1))$ for all Galois extensions L/K could itself provide a pivotal step in proving the conjecture of Fukaya and Kato in full generality.

The main result in the present paper concerns the compatibility of our conjecture with the explicit conjectural formula for $\hat{\partial}(\zeta^*_{L/K,S}(0))$ that is studied in [Bur01]. Using the functional equation of the equivariant zeta-function, we prove that the mutual compatibility of these conjectures is equivalent to the validity of the 'epsilon constant conjecture' formulated in [BlB03]. This result allows us to interpret results in [BlB03, Bre04a] and [Bre04b] as evidence for our conjecture. For example, by these means we deduce that the conjectural formulas for $\hat{\partial}(\zeta^*_{L/K,S}(0))$ and $\hat{\partial}(\zeta^*_{L/K,S}(1))$ are mutually compatible whenever L/K is tamely ramified or L is either an abelian extension of $\mathbb{Q}$ with odd conductor or a non-abelian extension of $\mathbb{Q}$ of degree 6.

In this regard, we also prove that the validity of the epsilon constant conjecture of [BlB03] would imply that epsilon constants of symplectic characters are uniquely characterized by a natural algebraic invariant of L/K. This result shows that the epsilon constant conjecture implies an affirmative answer to a question that has been open ever since Cassou-Noguès and Taylor proved in [CT83a] and [CT83b] that symplectic epsilon constants of *tamely ramified* Galois extensions of number fields are characterized by algebraic invariants.

When taken together with the results of [Bur01, BlB03] and [Bur05] the present paper demonstrates that the use of equivariant zeta-functions and of the Euler characteristic formalism of [BrB05] provides a universal framework of leading term conjectures which incorporates as consequences a wide variety of seemingly unrelated theorems and explicit conjectures ranging from Hilbert's Theorem 132 to the 'Ω-conjectures' of Chinburg, the explicit Galois structure results on units and ideal class groups proved by Fröhlich in [Frö89] and [Frö92], the refinement of Stark's conjecture formulated by Rubin and the 'refined class number formulas' conjectured by both Gross and Tate. In turn, such a universal approach gives new insight into various long-standing questions and conjectures. For example, in the setting of the present paper, the results we prove in §§ 3 and 5 show that Chinburg's 'Ω_1-conjecture' and 'Ω_3-conjecture' are consequences of leading term conjectures at $s = 1$ and $s = 0$ respectively, whereas his 'Ω_2-conjecture' is most naturally interpreted as a consequence of the compatibility of these leading term conjectures with respect to the relevant functional equation. From a philosophical perspective, this qualitative distinction neatly accounts for the fact that the Ω_2-conjecture is much easier to study than either of the Ω_1-conjecture or Ω_3-conjecture and also provides a satisfactory answer to a problem emphasized by Fröhlich in both [Frö89] and [CCFT91, § 3]. Indeed, in [Frö89, Introduction] Fröhlich writes of the 'amazing analogy' between the Galois structure theories of, on the one hand, unit groups and ideal class groups and, on the other hand, rings of algebraic integers, and he stresses that providing a natural explanation of this analogy is 'an outstanding problem – possibly connected with a new interpretation of the functional equation'. The results we prove in §§ 3 and 5 now provide just such an explanation of this analogy (see Remark 5.5 for further details in this regard).

In a little more detail, the basic contents of this paper are as follows. In § 2 we review some basic algebraic K-theory and give a new (and more conceptual) description of the 'extended boundary homomorphism' $\hat{\partial}$ that was introduced in [BF01]. We also review relevant facts concerning homological algebra and the Euler characteristic construction of [BrB05] and recall the definition and basic properties of equivariant Dedekind zeta-functions. In § 3 we formulate our conjectural description of $\hat{\partial}(\zeta^*_{L/K,S}(1))$, prove some of its basis properties and describe its relation to the conjectures of Stark and Chinburg. In § 4 we review the conjectural description of $\hat{\partial}(\zeta^*_{L/K,S}(0))$ that is formulated in [Bur01]. In § 5 we use the functional equation of the equivariant zeta-function and the 'additivity criterion' proved in [BrB05] to investigate the compatibility of the conjectures of §§ 3 and 4 with the

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conjecture of [BlB03]. We also prove that, if the central conjecture of [BlB03] is valid, then epsilon constants of symplectic characters are uniquely characterized by natural algebraic invariants.

The present paper incorporates updated versions of the unpublished manuscripts [Bur98] and [BrB03].

2. Preliminaries

In this section we summarize the necessary background from algebraic K-theory and homological algebra. Furthermore we recall the definition of the equivariant zeta-function.

2.1 Algebraic K-theory

We recall the definition of the relative K_0-group and describe the extended boundary homomorphism which takes values in such a group.

2.1.1 *Relative K_0-groups.* For any integral domain R of characteristic 0, any extension E of the field of fractions of R and any finite group G let $K_0(R[G], E)$ denote the relative algebraic K-group associated to the ring homomorphism $R[G] \to E[G]$; a description of $K_0(R[G], E)$ in terms of generators and relations is given in [Swa68, p. 215]. Writing $K_0(R[G])$ for the Grothendieck group of the category of finitely generated projective $R[G]$-modules and $K_1(R[G])$ for the Whitehead group there is a long exact sequence of relative K-theory

$$K_1(R[G]) \longrightarrow K_1(E[G]) \xrightarrow{\partial^1_{R[G],E}} K_0(R[G], E) \xrightarrow{\partial^0_{R[G],E}} K_0(R[G]) \xrightarrow{\partial_{R[G],E}} K_0(E[G]) \tag{1}$$

(cf. [Swa68, ch. 15] or [Bur04] for more details). The exact sequence (1) is functorial in the pair (R, E). The projective class group $\mathrm{Cl}(R[G])$ of $R[G]$ is defined to be the kernel of $\partial_{R[G],E}$ and is in fact independent of E. In the case $R = \mathbb{Z}$ and $E = \mathbb{R}$ we will often write ∂^i_G for the map $\partial^i_{\mathbb{Z}[G],\mathbb{R}}$.

Let $\mathrm{Z}(E[G])^\times$ denote the multiplicative group of the centre of $E[G]$. The reduced norm map induces a homomorphism $\mathrm{nr} : K_1(E[G]) \to \mathrm{Z}(E[G])^\times$ and we denote its image by $\mathrm{Z}(E[G])^{\times +}$. In this paper E will always be either $\mathbb{R}$ or an algebraically closed field or a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ for some prime number p. In all these cases the homomorphism $\mathrm{nr} : K_1(E[G]) \to \mathrm{Z}(E[G])^\times$ is injective (cf. [CR87, Theorem (45.3)]) and we will always identify $K_1(E[G])$ and $\mathrm{Z}(E[G])^{\times +}$ via nr. In particular we will consider $\partial^1_{R[G],E}$ as a map $\mathrm{Z}(E[G])^{\times +} \to K_0(R[G], E)$. If E is algebraically closed or a finite extension of $\mathbb{Q}_p$ then $\mathrm{Z}(E[G])^{\times +} = \mathrm{Z}(E[G])^\times$.

We recall that in the case $R = \mathbb{Z}$ and $E = \mathbb{Q}$ the canonical maps $K_0(\mathbb{Z}[G], \mathbb{Q}) \to K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ induce an isomorphism

$$K_0(\mathbb{Z}[G], \mathbb{Q}) \cong \bigoplus_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p), \tag{2}$$

where the direct sum is over all prime numbers p.

Our main interest will be the case $R = \mathbb{Z}$ and $E = \mathbb{R}$. For every prime number p and every embedding $j : \mathbb{R} \to \mathbb{C}_p$ (where $\mathbb{C}_p$ denotes the completion of an algebraic closure of $\mathbb{Q}_p$) we obtain induced maps $j_* : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ and $j_* : \mathrm{Z}(\mathbb{R}[G])^\times \to \mathrm{Z}(\mathbb{C}_p[G])^\times$.

LEMMA 2.1. *The map*

$$K_0(\mathbb{Z}[G], \mathbb{R}) \to \prod_{p,j} K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$$

is injective. Here the product runs over all prime numbers p and all embeddings $j : \mathbb{R} \to \mathbb{C}_p$.

Proof. We consider the exact sequence (1) for the pairs $(R, E) = (\mathbb{Z}, \mathbb{Q})$, $(\mathbb{Z}, \mathbb{R})$, $(\mathbb{Z}_p, \mathbb{Q}_p)$ and $(\mathbb{Z}_p, \mathbb{C}_p)$ and the maps between these sequences which are induced by the obvious inclusions and by

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an embedding $j : \mathbb{R} \to \mathbb{C}_p$. An easy diagram chase shows that there is a commutative diagram of short exact sequences as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(\mathbb{Z}[G],\mathbb{Q}) & \longrightarrow & K_0(\mathbb{Z}[G],\mathbb{R}) & \longrightarrow & K_1(\mathbb{R}[G])/K_1(\mathbb{Q}[G]) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_0(\mathbb{Z}_p[G],\mathbb{Q}_p) & \longrightarrow & K_0(\mathbb{Z}_p[G],\mathbb{C}_p) & \longrightarrow & K_1(\mathbb{C}_p[G])/K_1(\mathbb{Q}_p[G]) & \longrightarrow & 0 \end{array}$$

Therefore it suffices to show that the maps

$$K_0(\mathbb{Z}[G],\mathbb{Q}) \to \prod_{p,j} K_0(\mathbb{Z}_p[G],\mathbb{Q}_p) \tag{3}$$

and

$$K_1(\mathbb{R}[G])/K_1(\mathbb{Q}[G]) \to \prod_{p,j} K_1(\mathbb{C}_p[G])/K_1(\mathbb{Q}_p[G]) \tag{4}$$

are injective. The injectivity of (3) follows immediately from (2).

Let $x \in K_1(\mathbb{R}[G])$ be such that $j_*(x) \in K_1(\mathbb{Q}_p[G]) \subseteq K_1(\mathbb{C}_p[G])$ for all p and j. Using the isomorphism $\mathrm{nr} : K_1(\mathbb{R}[G]) \xrightarrow{\cong} \mathrm{Z}(\mathbb{R}[G])^{\times +}$ we have $x = \sum_{g\in G} c_g g \in \mathrm{Z}(\mathbb{R}[G])^{\times +}$ such that

$$j_*(x) = \sum_{g\in G} j(c_g)g \in \mathrm{Z}(\mathbb{Q}_p[G])^{\times}. \tag{5}$$

We claim that $\sum_{g\in G} c_g g \in \mathbb{Q}[G]$. Let $g \in G$ and consider the coefficient c_g. If c_g was transcendental over $\mathbb{Q}$ then there would be an embedding $j : \mathbb{R} \to \mathbb{C}_p$ such that $j(c_g) \notin \mathbb{Q}_p$ contradicting (5). Therefore c_g is algebraic over $\mathbb{Q}$. Now $j(c_g) \in \mathbb{Q}_p$ for all p and embeddings j implies that all primes are completely split in the number field $\mathbb{Q}(c_g)$ and therefore $\mathbb{Q}(c_g) = \mathbb{Q}$. Hence

$$x \in \mathrm{Z}(\mathbb{R}[G])^{\times +} \cap \mathbb{Q}[G] = \mathrm{Z}(\mathbb{Q}[G])^{\times +} \cong K_1(\mathbb{Q}[G]).$$

This shows the injectivity of (4). □

2.1.2 *The extended boundary homomorphism.* We give a conceptual description of the 'extended boundary homomorphism' introduced in [BF01, Lemma 9].

Lemma 2.2. *There is a unique homomorphism* $\hat{\partial}^1_G : \mathrm{Z}(\mathbb{R}[G])^{\times} \to K_0(\mathbb{Z}[G],\mathbb{R})$ *such that for every prime number* p *and every embedding* $j : \mathbb{R} \to \mathbb{C}_p$ *the diagram*

$$\begin{array}{ccc} \mathrm{Z}(\mathbb{R}[G])^{\times} & \xrightarrow{\hat{\partial}^1_G} & K_0(\mathbb{Z}[G],\mathbb{R}) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \mathrm{Z}(\mathbb{C}_p[G])^{\times} & \xrightarrow{\partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}} & K_0(\mathbb{Z}_p[G],\mathbb{C}_p) \end{array}$$

commutes. The restriction of $\hat{\partial}^1_G$ *to* $\mathrm{Z}(\mathbb{R}[G])^{\times +}$ *is the map* $\partial^1_{\mathbb{Z}[G],\mathbb{R}}$.

Proof. To define $\hat{\partial}^1_G(x)$ for $x \in \mathrm{Z}(\mathbb{R}[G])^{\times}$ we choose $\lambda \in \mathrm{Z}(\mathbb{Q}[G])^{\times}$ such that $\lambda x \in \mathrm{Z}(\mathbb{R}[G])^{\times +}$ and set

$$\hat{\partial}^1_G(x) := \partial^1_{\mathbb{Z}[G],\mathbb{R}}(\lambda x) - \sum_p \partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}(\lambda).$$

Here the sum is over all prime numbers p, and $\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}(\lambda)$ means the following: Consider λ as an element of $\mathrm{Z}(\mathbb{Q}_p[G])^{\times}$ via the inclusion $\mathrm{Z}(\mathbb{Q}[G])^{\times} \subseteq \mathrm{Z}(\mathbb{Q}_p[G])^{\times}$ then apply $\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}$ and consider the result as an element of $K_0(\mathbb{Z}[G],\mathbb{R})$ by the inclusion $K_0(\mathbb{Z}_p[G],\mathbb{Q}_p) \subseteq K_0(\mathbb{Z}[G],\mathbb{Q}) \subseteq K_0(\mathbb{Z}[G],\mathbb{R})$. One easily checks that $\hat{\partial}^1_G$ is well defined and a homomorphism. Obviously $\hat{\partial}^1_G(x) = \partial^1_{\mathbb{Z}[G],\mathbb{R}}(x)$

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for $x \in \mathrm{Z}(\mathbb{R}[G])^{\times +}$. The diagram commutes for every embedding $j : \mathbb{R} \to \mathbb{C}_p$ because for all prime numbers $q \neq p$ one has $j_*(\partial^1_{\mathbb{Z}_q[G],\mathbb{Q}_q}(\lambda)) = 0$ in $K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$. The uniqueness assertion is a consequence of Lemma 2.1. □

Remark 2.3. The construction of the map $\hat{\partial}^1_G$ in the proof of Lemma 2.2 shows that this map is the same as the homomorphism $\hat{\delta}^1_{\mathbb{Z}[G],\mathbb{R}}$ introduced in [BF01, Lemma 9].

2.1.3 *Change of group.* Let R, E and G be as in § 2.1.1 and let H be a subgroup of G. Since $R[G]$ is free as $R[H]$-module, restriction of scalars is a functor from projective $R[G]$-modules to projective $R[H]$-modules and similarly from $E[G]$-modules to $E[H]$-modules. Therefore one obtains canonical restriction maps res^G_H for all K-groups in the exact sequence (1). Using the identification via the reduced norm we also obtain a restriction map $\mathrm{res}^G_H : \mathrm{Z}(E[G])^{\times +} \to \mathrm{Z}(E[H])^{\times +}$.

The functor $M \mapsto R[G] \otimes_{R[H]} M$ from projective $R[H]$-modules to projective $R[G]$-modules and the corresponding functor from $E[H]$-modules to $E[G]$-modules induce induction maps ind^G_H for all K-groups in the exact sequence (1). Again one also obtains an induction map $\mathrm{ind}^G_H : \mathrm{Z}(E[H])^{\times +} \to \mathrm{Z}(E[G])^{\times +}$.

If H is a normal subgroup of G then the functor $M \mapsto M^H$ from projective $R[G]$-modules to projective $R[G/H]$-modules and the corresponding functor from $E[G]$-modules to $E[G/H]$-modules induce quotient maps $\mathrm{q}^G_{G/H}$ for all K-groups in the exact sequence (1). Again one also obtains a quotient map $\mathrm{q}^G_{G/H} : \mathrm{Z}(E[G])^{\times +} \to \mathrm{Z}(E[G/H])^{\times +}$.

The extended boundary homomorphism is compatible with the restriction, induction and quotient maps. More precisely, the map $\mathrm{res}^G_H : \mathrm{Z}(\mathbb{C}[G])^\times \to \mathrm{Z}(\mathbb{C}[H])^\times$ restricts to a homomorphism $\mathrm{res}^G_H : \mathrm{Z}(\mathbb{R}[G])^\times \to \mathrm{Z}(\mathbb{R}[H])^\times$, and one has $\hat{\partial}^1_H \circ \mathrm{res}^G_H = \mathrm{res}^G_H \circ \hat{\partial}^1_G$ as maps $\mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[H], \mathbb{R})$. Similarly $\hat{\partial}^1_G \circ \mathrm{ind}^G_H = \mathrm{ind}^G_H \circ \hat{\partial}^1_H : \mathrm{Z}(\mathbb{R}[H])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ and $\hat{\partial}^1_{G/H} \circ \mathrm{q}^G_{G/H} = \mathrm{q}^G_{G/H} \circ \hat{\partial}^1_G : \mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G/H], \mathbb{R})$ if H is normal in G.

2.1.4 *Involutions.* We recall the definition of the involution ψ^*_G of $K_0(\mathbb{Z}[G], \mathbb{R})$ that is defined in [Bur01, p. 217]. If P is a projective $\mathbb{Z}[G]$-module then $\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ is a projective $\mathbb{Z}[G]$-module when endowed with the contragredient G-action. Every element in $K_0(\mathbb{Z}[G], \mathbb{R})$ is represented by a triple $[P_1, \varphi, P_2]$ where P_1, P_2 are finitely generated projective $\mathbb{Z}[G]$-modules and $\varphi : P_1 \otimes_{\mathbb{Z}} \mathbb{R} \to P_2 \otimes_{\mathbb{Z}} \mathbb{R}$ is an isomorphism of $\mathbb{R}[G]$-modules. The involution ψ^*_G is defined by

$$\psi^*_G([P_1, \varphi, P_2]) := [\mathrm{Hom}_{\mathbb{Z}}(P_1, \mathbb{Z}), \mathrm{Hom}_{\mathbb{R}}(\varphi, \mathbb{R})^{-1}, \mathrm{Hom}_{\mathbb{Z}}(P_2, \mathbb{Z})].$$

One can show that ψ^*_G is compatible with the change of group homomorphisms defined in § 2.1.3, i.e. for a subgroup H of G one has $\mathrm{res}^G_H \circ \psi^*_G = \psi^*_H \circ \mathrm{res}^G_H$, $\mathrm{ind}^G_H \circ \psi^*_H = \psi^*_G \circ \mathrm{ind}^G_H$ and $\mathrm{q}^G_{G/H} \circ \psi^*_G = \psi^*_{G/H} \circ \mathrm{q}^G_{G/H}$ if H is normal in G.

Let $\mathrm{Z}(\mathbb{C}[G])$ denote the centre of $\mathbb{C}[G]$ and note that there is a canonical isomorphism $\mathrm{Z}(\mathbb{C}[G]) = \prod_{\chi \in \mathrm{Irr}(G)} \mathbb{C}$ where we write $\mathrm{Irr}(G)$ for the set of irreducible $\mathbb{C}$-valued characters of G. On $\mathrm{Z}(\mathbb{C}[G])$ there exists a natural involution $x \mapsto x^\#$ which is induced by the $\mathbb{C}$-linear anti-involution of $\mathbb{C}[G]$ which sends each element of G to its inverse. If $x = (x_\chi)_{\chi \in \mathrm{Irr}(G)}$ under the isomorphism $\mathrm{Z}(\mathbb{C}[G]) = \prod_{\chi \in \mathrm{Irr}(G)} \mathbb{C}$ then $x^\# = (x_{\overline{\chi}})_{\chi \in \mathrm{Irr}(G)}$. This involution of $\mathrm{Z}(\mathbb{C}[G])$ restricts to an involution of $\mathrm{Z}(\mathbb{R}[G])^\times$ which is compatible with the change of group homomorphisms from § 2.1.3.

Up to sign the involutions on $K_0(\mathbb{Z}[G], \mathbb{R})$ and $\mathrm{Z}(\mathbb{R}[G])^\times$ are compatible with the extended boundary homomorphism. More precisely, if $x \in \mathrm{Z}(\mathbb{R}[G])^\times$ then one has $\psi^*_G(\hat{\partial}^1_G(x)) = -\hat{\partial}^1_G(x^\#)$ in $K_0(\mathbb{Z}[G], \mathbb{R})$.

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2.2 Homological algebra

We fix the sign conventions used for the homological algebra constructions in this paper. Furthermore we prove an important lemma concerning extension classes and recall the notion of an Euler characteristic with values in a relative algebraic K-group.

2.2.1 *Complexes.* Let R be a ring. By a *complex* we mean a cochain complex of left R-modules. For any complex A we write $A[1]$ for the shifted complex that is given by $A[1]^i = A^{i+1}$ with differential $d_{A[1]}(a) = -d_A(a)$. The mapping cone $\mathrm{cone}(\omega)$ of a map of complexes $\omega : U \to V$ is the complex defined by $\mathrm{cone}(\omega)^i = V^i \oplus U^{i+1}$ with differential $d_{\mathrm{cone}(\omega)}(v,u) = (d_V(v) + \omega(u), -d_U(u))$. Let $\mathcal{D}(R)$ denote the derived category of the abelian category of R-modules. A triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$$

in $\mathcal{D}(R)$ is called *distinguished* if it is isomorphic to a triangle of the form

$$U \xrightarrow{\omega} V \xrightarrow{\iota} \mathrm{cone}(\omega) \xrightarrow{\pi} U[1],$$

where ω is a map of complexes, $\iota : V \to \mathrm{cone}(\omega)$ is the canonical inclusion $\iota(v) = (v,0)$, and $\pi : \mathrm{cone}(\omega) \to U[1]$ is the negative of the canonical projection, i.e. $\pi(v,u) = -u$. For typographic reasons we often write a distinguished triangle as $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma}$.

For every short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ there exists a canonical map $\gamma : C \to A[1]$ in the derived category such that $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a distinguished triangle. Our choice of triangulation guarantees that the cohomology sequences of the short exact sequence and of the distinguished triangle are the same, i.e. that the map on cohomology induced by γ coincides with the connecting homomorphism of the short exact sequence.

2.2.2 *Yoneda extensions.* We always use injective resolutions of the second variable to identify Yoneda Ext-groups (as defined in [HS97, IV, § 9]) with derived functor Ext-groups. For a natural interpretation of the connecting homomorphism for derived functor Ext-groups in terms of Yoneda extensions see [BF98, Lemma 3].

We will frequently interpret certain complexes in the derived category in terms of Yoneda extension classes as in [BF98, p. 1353]. To a complex E which is acyclic outside degrees 0 and $n \geqslant 1$ one associates the class $e(E) \in \mathrm{Ext}_R^{n+1}(H^n(E), H^0(E))$ which is given by the truncated complex $E' := \tau^{\leqslant n}\tau^{\geqslant 0}E$ with the induced maps $H^0(E) \xrightarrow{\cong} H^0(E') \to (E')^0$ and $(E')^n \to H^n(E') \xrightarrow{\cong} H^n(E)$ considered as a Yoneda extension.

Lemma 2.4. *Let E and F be complexes which are acyclic outside degrees 0 and $n \geqslant 1$. Let $\alpha : H^0(E) \to H^0(F)$ and $\beta : H^n(E) \to H^n(F)$ be homomorphisms of R-modules inducing maps $\alpha_* : \mathrm{Ext}_R^{n+1}(H^n(E), H^0(E)) \to \mathrm{Ext}_R^{n+1}(H^n(E), H^0(F))$ and $\beta^* : \mathrm{Ext}_R^{n+1}(H^n(F), H^0(F)) \to \mathrm{Ext}_R^{n+1}(H^n(E), H^0(F))$. There exists a morphism $\varphi : E \to F$ in $\mathcal{D}(R)$ which induces α on H^0 and β on H^n if and only if $\alpha_*(e(E)) = \beta^*(e(F))$. If in addition $\mathrm{Ext}_R^n(H^n(E), H^0(F)) = 0$, then the morphism φ with this property is unique.*

Proof. Without loss of generality we can assume that $E^i = 0$ and $F^i = 0$ unless $0 \leqslant i \leqslant n$. By interpreting the maps α_* and β^* in terms of Yoneda extensions and by the definition of equivalence of Yoneda extensions it is easy to see that $\alpha_*(e(E)) = \beta^*(e(F))$ implies the existence of a morphism $E \to F$ in the derived category with the required property. Conversely, if $\varphi : E \to F$ is such a morphism in $\mathcal{D}(R)$ then there exists a complex G with $G^i = 0$ unless $0 \leqslant i \leqslant n$, a quasi-isomorphism $\lambda : G \to E$ and a map of complexes $\mu : G \to F$ such that $\varphi = \mu \circ \lambda^{-1}$. This easily implies $\alpha_*(e(E)) = \beta^*(e(F))$.

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To show the uniqueness it suffices to prove that if $\varphi : E \to F$ induces the zero map on H^0 and H^n then $\varphi = 0$ in $\mathcal{D}(R)$. We first observe that there is a distinguished triangle

$$H^n(F)[-n-1] \longrightarrow H^0(F)[0] \longrightarrow F \longrightarrow H^n(F)[-n], \tag{6}$$

which on cohomology induces the canonical maps. Indeed, if $\widetilde{F}$ denotes the complex $F^0 \to F^1 \to \cdots \to F^n \to H^n(F)$ with F^0 in degree 0 then (6) arises from the short exact sequence of complexes $0 \to H^n(F)[-n-1] \to \widetilde{F} \to F \to 0$ and the quasi-isomorphism $H^0(F)[0] \to \widetilde{F}$. From (6) we obtain an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{D}(R)}(E, H^0(F)[0]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(E, F) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(E, H^n(F)[-n]).$$

The image of φ in $\mathrm{Hom}_{\mathcal{D}(R)}(E, H^n(F)[-n]) = \mathrm{Hom}_R(H^n(E), H^n(F))$ is trivial, thus φ is the image of a map $\psi \in \mathrm{Hom}_{\mathcal{D}(R)}(E, H^0(F)[0])$. There is a distinguished triangle for E similar to (6) which gives an exact sequence

$$\mathrm{Hom}_{\mathcal{D}(R)}(H^n(E)[-n], H^0(F)[0]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(E, H^0(F)[0]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(H^0(E)[0], H^0(F)[0]).$$

The image of ψ in $\mathrm{Hom}_{\mathcal{D}(R)}(H^0(E)[0], H^0(F)[0]) = \mathrm{Hom}_R(H^0(E), H^0(F))$ is trivial and by assumption $\mathrm{Hom}_{\mathcal{D}(R)}(H^n(E)[-n], H^0(F)[0]) = \mathrm{Ext}^n_R(H^n(E), H^0(F)) = 0$. Thus $\psi = 0$ and hence also $\varphi = 0$. □

2.2.3 *Euler characteristics.* Let G be a finite group. For any object C of $\mathcal{D}(\mathbb{Z}[G])$ we write $H^{\mathrm{ev}}(C)$ and $H^{\mathrm{od}}(C)$ for the direct sums $\bigoplus_{i\,\mathrm{even}} H^i(C)$ and $\bigoplus_{i\,\mathrm{odd}} H^i(C)$ where i runs over all even and all odd integers respectively.

We write $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ for the full triangulated subcategory of $\mathcal{D}(\mathbb{Z}[G])$ consisting of those complexes that are perfect (i.e. isomorphic in $\mathcal{D}(\mathbb{Z}[G])$ to a bounded complex of finitely generated projective $\mathbb{Z}[G]$-modules). Let C be an object in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ and t a trivialization of C (over $\mathbb{R}$), that is, an isomorphism of $\mathbb{R}[G]$-modules $t : H^{\mathrm{ev}}(C) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\cong} H^{\mathrm{od}}(C) \otimes_{\mathbb{Z}} \mathbb{R}$. We write $\chi_G(C, t)$ for the Euler characteristic in $K_0(\mathbb{Z}[G], \mathbb{R})$ that is defined in [BrB05, Definition 5.5] (where it is denoted by $\chi_{\mathbb{Z}[G],\mathbb{R}[G]}(C, t)$). This Euler characteristic depends on C only up to isomorphism. More precisely, if $\varphi : C \to C'$ is an isomorphism in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ and t' is the composite isomorphism of $\mathbb{R}[G]$-modules

$$H^{\mathrm{ev}}(C') \otimes \mathbb{R} \xrightarrow{H^{\mathrm{ev}}(\varphi^{-1})\otimes\mathbb{R}} H^{\mathrm{ev}}(C) \otimes \mathbb{R} \xrightarrow{t} H^{\mathrm{od}}(C) \otimes \mathbb{R} \xrightarrow{H^{\mathrm{od}}(\varphi)\otimes\mathbb{R}} H^{\mathrm{od}}(C') \otimes \mathbb{R}$$

then $\chi_G(C, t) = \chi_G(C', t')$.

To compute certain Euler characteristics and to compare our constructions to related results in the literature we will occasionally use the explicit approach described in [Bur04] and [BrB05, § 6]. By this approach we obtain an element $\chi_G^{\mathrm{old}}(C, t^{-1}) \in K_0(\mathbb{Z}[G], \mathbb{R})$ for a complex C and trivialization t as above. For the precise relation of $\chi_G(C, t)$ and $\chi_G^{\mathrm{old}}(C, t^{-1})$ see [BrB05, Theorem 6.2].

Finally we recall the relevant case of the additivity criterion for Euler characteristics [BrB05, Corollary 6.6].

Lemma 2.5. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ be a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ and let t_A, t_B and t_C be trivializations of A, B and C over $\mathbb{R}$. The long exact cohomology sequence of the distinguished triangle gives rise to exact sequences of $\mathbb{R}[G]$-modules*

$$0 \to \ker(H^{\mathrm{ev}}(\alpha) \otimes \mathbb{R}) \to H^{\mathrm{ev}}(A) \otimes \mathbb{R} \to H^{\mathrm{ev}}(B) \otimes \mathbb{R} \to H^{\mathrm{ev}}(C) \otimes \mathbb{R} \to \ker(H^{\mathrm{od}}(\alpha) \otimes \mathbb{R}) \to 0,$$
$$0 \to \ker(H^{\mathrm{od}}(\alpha) \otimes \mathbb{R}) \to H^{\mathrm{od}}(A) \otimes \mathbb{R} \to H^{\mathrm{od}}(B) \otimes \mathbb{R} \to H^{\mathrm{od}}(C) \otimes \mathbb{R} \to \ker(H^{\mathrm{ev}}(\alpha) \otimes \mathbb{R}) \to 0.$$

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We consider the $\mathbb{R}[G]$-modules and isomorphisms in the following (in general non-commutative) diagram

$$\begin{array}{ccc} H^{\mathrm{ev}}(B)\otimes\mathbb{R}\oplus\ker(H^{\mathrm{ev}}(\alpha)\otimes\mathbb{R})\oplus\ker(H^{\mathrm{od}}(\alpha)\otimes\mathbb{R}) & \xrightarrow{s^{\mathrm{ev}}} & H^{\mathrm{ev}}(A)\otimes\mathbb{R}\oplus H^{\mathrm{ev}}(C)\otimes\mathbb{R} \\ \downarrow{\scriptstyle t_B\oplus\mathrm{id}\oplus(-\mathrm{id})} & & \downarrow{\scriptstyle t_A\oplus t_C} \\ H^{\mathrm{od}}(B)\otimes\mathbb{R}\oplus\ker(H^{\mathrm{ev}}(\alpha)\otimes\mathbb{R})\oplus\ker(H^{\mathrm{od}}(\alpha)\otimes\mathbb{R}) & \xrightarrow{s^{\mathrm{od}}} & H^{\mathrm{od}}(A)\otimes\mathbb{R}\oplus H^{\mathrm{od}}(C)\otimes\mathbb{R} \end{array}$$

where s^{ev} and s^{od} are obtained from the exact sequences above by choosing splittings. If the automorphism $(t_B\oplus\mathrm{id}\oplus(-\mathrm{id}))^{-1}\circ(s^{\mathrm{od}})^{-1}\circ(t_A\oplus t_C)\circ s^{\mathrm{ev}}$ of $H^{\mathrm{ev}}(B)\otimes\mathbb{R}\oplus\ker(H^{\mathrm{ev}}(\alpha)\otimes\mathbb{R})\oplus\ker(H^{\mathrm{od}}(\alpha)\otimes\mathbb{R})$ has reduced norm equal to 1 then

$$\chi_G(B,t_B)=\chi_G(A,t_A)+\chi_G(C,t_C)$$

in $K_0(\mathbb{Z}[G],\mathbb{R})$.

2.3 The equivariant Dedekind zeta-function

We fix some basic notations for number fields which are used in the rest of this paper and define the equivariant zeta-function of a Galois extension of number fields.

2.3.1 *Notation for number fields.* Let L be a number field. We write $\mathcal{O}_L$ for the ring of integers of L and $S(L)$ for the set of all places of L. For any place $w\in S(L)$ we denote the completion of L at w by L_w. For a non-archimedean place w we write $\mathcal{O}_w$ for the ring of integers of L_w, $\mathfrak{m}_w$ for the maximal ideal of $\mathcal{O}_w$, $\lambda(w):=\mathcal{O}_w/\mathfrak{m}_w$ for the residue field and $Nw:=|\lambda(w)|$ for the cardinality of the residue field. Furthermore for $i\geqslant 1$ we let $U^{(i)}_{L_w}$ denote the group of ith principal units in L_w, i.e. $U^{(i)}_{L_w}:=1+\mathfrak{m}_w^i$.

If L is an extension of K and $v\in S(K)$ then $S_v(L)$ is the set of all places of L above v. This applies in particular to $K=\mathbb{Q}$ where either $v=p$ is a prime number or $v=\infty$ is the archimedean place. We also use the notation $S_f(L)$ for the set of all non-archimedean places, $S_{\mathbb{R}}(L)$ for the set of real archimedean places and $S_{\mathbb{C}}(L)$ for the set of complex archimedean places.

From now on let L/K be a Galois extension of number fields with Galois group G. For $w\in S(L)$ we write G_w for the decomposition group of w. For a non-archimedean place w we denote the inertia group by I_w and we let $\sigma_w\in G_w$ be any lift of the (arithmetic) Frobenius in G_w/I_w. For any place $v\in S(K)$ we set $L_v:=\prod_{w\in S_v(L)}L_w$ and (if $v\in S_f(K)$) $\mathcal{O}_{L,v}:=\prod_{w\in S_v(L)}\mathcal{O}_w$ and $\mathfrak{m}_{L,v}:=\prod_{w\in S_v(L)}\mathfrak{m}_w$. Note that L_v, $\mathcal{O}_{L,v}$ and $\mathfrak{m}_{L,v}$ are G-modules in an obvious way.

Let S be a finite subset of $S(K)$. The G-stable set of places of L that lie above a place in S will be denoted by the same letter S. This should not cause any confusion because places of K will be called v and places of L will be called w. For a finite subset S of $S(K)$ which contains all archimedean places we let $\mathcal{O}_{L,S}$ be the ring of S-integers in L. Note that $\mathcal{O}_{L,S}$ is again a G-module and that in the case $S=S_\infty(K)$ one has $\mathcal{O}_L=\mathcal{O}_{L,S}$.

2.3.2 *The equivariant zeta-function.* As in § 2.1.4 we denote the set of all irreducible $\mathbb{C}$-valued characters of a finite group G by $\mathrm{Irr}(G)$ and identify the centre $\mathrm{Z}(\mathbb{C}[G])$ of $\mathbb{C}[G]$ with $\prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C}$. Recall that a meromorphic $\mathrm{Z}(\mathbb{C}[G])$-valued function is a function of a complex variable s of the form $s\mapsto g(s)=(g(\chi,s))_{\chi\in\mathrm{Irr}(G)}\in\prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C}=\mathrm{Z}(\mathbb{C}[G])$ where each function $s\mapsto g(\chi,s)$ is meromorphic.

From now on let L/K be a Galois extension of number fields with Galois group G. The equivariant Dedekind zeta-function of L/K is a meromorphic $\mathrm{Z}(\mathbb{C}[G])$-valued function which is first defined

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on a right half plane as a product of equivariant Euler factors and then extended to the whole complex plane by meromorphic continuation.

Let $v \in S(K)$. Choose a place $w \in S_v(L)$ and denote its decomposition group by G_w. We define a meromorphic $\mathrm{Z}(\mathbb{C}[G_w])$-valued function $L_{L_w/K_v}(s)$ by

$$L_{L_w/K_v}(s) := (L_{L_w/K_v}(\psi, s))_{\psi \in \mathrm{Irr}(G_w)}.$$

Here for each $\psi \in \mathrm{Irr}(G_w)$ (and more generally for each not necessarily irreducible character ψ) the meromorphic function $L_{L_w/K_v}(\psi, s)$ is defined as follows. First we choose a $\mathbb{C}[G_w]$-module V_ψ with character ψ. In the case $v \in S_f(K)$ we then define

$$L_{L_w/K_v}(\psi, s) := \det_{\mathbb{C}}(1 - \sigma_w(Nv)^{-s} | V_\psi^{I_w})^{-1},$$

where (as in § 2.3.1) I_w denotes the inertia group of w, $\sigma_w \in G_w$ is a lift of the arithmetic Frobenius in G_w/I_w and Nv is the cardinality of the residue field of v. In the case $v \in S_\infty(K)$ we set $n_\psi := \dim_{\mathbb{C}}(V_\psi)$, $n_\psi^+ := \dim_{\mathbb{C}}(V_\psi^{G_w})$ and $n_\psi^- := n_\psi - n_\psi^+$, and define

$$L_{L_w/K_v}(\psi, s) := \begin{cases} [2(2\pi)^{-s}\Gamma(s)]^{n_\psi} & \text{if } v \in S_{\mathbb{C}}(K), \\ [\pi^{-s/2}\Gamma(s/2)]^{n_\psi^+}[\pi^{-(s+1)/2}\Gamma((s+1)/2)]^{n_\psi^-} & \text{if } v \in S_{\mathbb{R}}(K). \end{cases}$$

We note that $L_{L_w/K_v}(\psi + \psi', s) = L_{L_w/K_v}(\psi, s) \cdot L_{L_w/K_v}(\psi', s)$ for two characters ψ, ψ'; thus one can in fact define $L_{L_w/K_v}(\psi, s)$ for any virtual character ψ of G_w.

Let $\mathrm{ind}_{G_w}^G : \mathrm{Z}(\mathbb{C}[G_w]) \to \mathrm{Z}(\mathbb{C}[G])$ be the map

$$(\alpha_\psi)_{\psi \in \mathrm{Irr}(G_w)} \mapsto \left(\prod_{\psi \in \mathrm{Irr}(G_w)} \alpha_\psi^{\langle \psi, \chi|_{G_w} \rangle} \right)_{\chi \in \mathrm{Irr}(G)},$$

where $\langle \psi, \chi|_{G_w} \rangle$ denotes the usual inner product of the characters ψ and $\chi|_{G_w}$ of G_w. The restriction of this map to $\mathrm{Z}(\mathbb{C}[G_w])^\times$ is the induction map $\mathrm{ind}_{G_w}^G : \mathrm{Z}(\mathbb{C}[G_w])^\times \to \mathrm{Z}(\mathbb{C}[G])^\times$ defined in § 2.1.3 so that using the same name for these maps is justified. The meromorphic $\mathrm{Z}(\mathbb{C}[G])$-valued function $s \mapsto \mathrm{ind}_{G_w}^G(L_{L_w/K_v}(s))$ depends only on the place v and not on the choice of $w \in S_v(L)$.

For a finite subset S of $S(K)$, the S-truncated equivariant Dedekind zeta-function $\zeta_{L/K,S}(s)$ is the meromorphic $\mathrm{Z}(\mathbb{C}[G])$-valued function which for $\mathrm{Re}(s) > 1$ is defined by the product

$$\zeta_{L/K,S}(s) := \prod_{v \in S(K) \setminus S} \mathrm{ind}_{G_w}^G(L_{L_w/K_v}(s)). \tag{7}$$

For $S = \emptyset$ the empty set we also write $Z_{L/K}(s) := \zeta_{L/K,\emptyset}(s)$.

Remark 2.6. In the past various names and notations have been used for the equivariant Dedekind zeta-function of L/K. The function $Z_{L/K}(s)$ defined above agrees with the 'completed equivariant Artin L-function' $\Lambda(s)$ in [BlB03, § 3.1], and the function $\zeta_{L/K,S}(s)$ defined in (7) agrees with the function $L_S(s)$ in [Bur01]. For simplicity we will sometimes abbreviate 'equivariant Dedekind zeta-function of L/K' to 'zeta-function of L/K'.

2.3.3 *The functional equation.* One has $Z_{L/K}(s) = (\Lambda_{L/K}(\chi, s))_{\chi \in \mathrm{Irr}(G)}$ where $\Lambda_{L/K}(\chi, s)$ is the completed Artin L-function of the character χ as defined for example in [Frö83, ch. I, § 5] (where it is denoted by $\tilde{L}(s, L/K, \chi)$). This implies that the function $Z_{L/K}(s)$ and more generally the functions $\zeta_{L/K,S}(s)$ for any finite set S have meromorphic continuations to the whole complex plane.

We recall that $\Lambda_{L/K}(\chi, s)$ satisfies the functional equation

$$\Lambda_{L/K}(\overline{\chi}, s) = \varepsilon_{L/K}(\chi, s)\Lambda_{L/K}(\chi, 1 - s),$$

where $\varepsilon_{L/K}(\chi, s) := W(\overline{\chi}) \cdot (|d_{K/\mathbb{Q}}|^{\deg(\chi)} Nf(\chi))^{1/2-s}$ with $W(\overline{\chi})$ denoting the Artin root number, $d_{K/\mathbb{Q}}$ the discriminant of the extension $K/\mathbb{Q}$, $\deg(\chi)$ the degree of the character χ and $f(\chi)$

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the conductor of χ (cf. [Frö83, ch. I, p. 38]). We define a $\mathrm{Z}(\mathbb{C}[G])$-valued epsilon function by $\varepsilon_{L/K}(s) := (\varepsilon_{L/K}(\chi, s))_{\chi\in\mathrm{Irr}(G)}$. From the functional equations for $\Lambda_{L/K}(\chi, s)$ one obtains the equivariant functional equation

$$Z_{L/K}(s)^{\#} = \varepsilon_{L/K}(s) Z_{L/K}(1-s), \tag{8}$$

where $\#$ denotes the involution from § 2.1.4.

2.3.4 *The leading terms.* For a meromorphic $\mathbb{C}$-valued function $g(s)$ of a complex variable s which has algebraic order d at a point s_0 we set $g^*(s_0) := \lim_{s\to s_0}(s-s_0)^{-d} g(s) \in \mathbb{C}^\times$. For a meromorphic $\mathrm{Z}(\mathbb{C}[G])$-valued function $g(s) = (g(\chi, s))_{\chi\in\mathrm{Irr}(G)}$ we set $g^*(s_0) := (g^*(\chi, s_0))_{\chi\in\mathrm{Irr}(G)} \in \mathrm{Z}(\mathbb{C}[G])^\times$.

Lemma 2.7. *Let L/K be a Galois extension of number fields with Galois group G. Let $s_0 \in \mathbb{R}$.*

(i) *For each $v \in S(K)$ and $w \in S_v(L)$ one has $L^*_{L_w/K_v}(s_0) \in \mathrm{Z}(\mathbb{R}[G_w])^{\times +}$.*

(ii) *Let S be a finite subset of $S(K)$. Then $\zeta^*_{L/K,S}(s_0) \in \mathrm{Z}(\mathbb{R}[G])^\times$ and moreover $\zeta^*_{L/K,S}(s_0) \in \mathrm{Z}(\mathbb{R}[G])^{\times +}$ if $s_0 \geqslant 1$.*

Proof. Recall that an element $x = (x_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times = \mathrm{Z}(\mathbb{C}[G])^\times$ belongs to $\mathrm{Z}(\mathbb{R}[G])^\times$, respectively $\mathrm{Z}(\mathbb{R}[G])^{\times +}$, if and only if $\overline{x_\chi} = x_{\overline{\chi}}$ for all χ, respectively $x \in \mathrm{Z}(\mathbb{R}[G])^\times$ and x_χ is a positive real number whenever χ is symplectic.

To prove claim (i) we first note that $\overline{L_{L_w/K_v}(\psi, s)} = L_{L_w/K_v}(\overline{\psi}, \overline{s})$ which implies $L^*_{L_w/K_v}(s_0) \in \mathrm{Z}(\mathbb{R}[G_w])^\times$ for $s_0 \in \mathbb{R}$. In the case $v \in S_\infty(K)$ the group G_w has no irreducible symplectic characters. If $v \in S_f(K)$ and $\psi \in \mathrm{Irr}(G_w)$ is an irreducible symplectic character then $V_\psi^{I_w} = 0$ and therefore $L^*_{L_w/K_v}(\psi, s_0) = 1 > 0$.

One has $\zeta_{L/K,S}(s) = (L_{L/K,S}(\chi, s))_{\chi\in\mathrm{Irr}(G)}$ where $L_{L/K,S}(\chi, s)$ denotes the usual S-truncated Artin L-function of the character χ. The statement $\zeta^*_{L/K,S}(s_0) \in \mathrm{Z}(\mathbb{R}[G])^\times$ for $s_0 \in \mathbb{R}$ in claim (ii) therefore follows from $\overline{L_{L/K,S}(\chi, s)} = L_{L/K,S}(\overline{\chi}, \overline{s})$ as above. Now let $\chi \in \mathrm{Irr}(G)$ be a symplectic character. For $s_0 > 1$ one has $\mathrm{ind}^G_{G_w}(L_{L_w/K_v}(s_0)) = \mathrm{ind}^G_{G_w}(L^*_{L_w/K_v}(s_0)) \in \mathrm{Z}(\mathbb{R}[G])^{\times +}$ by part (i) and the fact that $\mathrm{ind}^G_{G_w}$ maps $\mathrm{Z}(\mathbb{R}[G_w])^{\times +}$ to $\mathrm{Z}(\mathbb{R}[G])^{\times +}$. (This holds because $\mathrm{ind}^G_{G_w} : \mathrm{Z}(\mathbb{C}[G_w])^\times \to \mathrm{Z}(\mathbb{C}[G])^\times$ corresponds to the induction map $\mathrm{ind}^G_{G_w} : K_1(\mathbb{C}[G_w]) \to K_1(\mathbb{C}[G])$ defined in § 2.1.3 and therefore restricts to the induction map $\mathrm{ind}^G_{G_w} : \mathrm{Z}(\mathbb{R}[G_w])^{\times +} \to \mathrm{Z}(\mathbb{R}[G])^{\times +}$ corresponding to $\mathrm{ind}^G_{G_w} : K_1(\mathbb{R}[G_w]) \to K_1(\mathbb{R}[G])$.) Since the product (7) converges for $s_0 > 1$ this implies $L^*_{L/K,S}(\chi, s_0) = L_{L/K,S}(\chi, s_0) > 0$. It is well known that $L_{L/K,S}(\chi, s)$ has no zero or pole at $s = 1$, hence it follows that $L^*_{L/K,S}(\chi, 1) = \lim_{s\to 1, s>1} L_{L/K,S}(\chi, s) > 0$. This shows the result for $s_0 = 1$. □

3. The leading term at $s = 1$

Let L/K be a Galois extension of number fields with Galois group G. Let S be a finite subset of $S(K)$ which contains all archimedean places and all places which ramify in L/K and is such that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. In this section we formulate an explicit conjectural description of $\hat{\partial}^1_G(\zeta^*_{L/K,S}(1))$, and then describe some of its basic properties.

3.1 Statement of the conjecture

Recall that for a place $v \in S(K)$ we write $L_v := \prod_{w\in S_v(L)} L_w$ and for v non-archimedean $\mathcal{O}_{L,v} := \prod_{w\in S_v(L)} \mathcal{O}_w$ and $\mathfrak{m}_{L,v} := \prod_{w\in S_v(L)} \mathfrak{m}_w$. For each $v \in S_\infty(K)$ we let $\exp : L_v \to L_v^\times$ denote the product of the (real or complex) exponential maps $L_w \to L_w^\times$ for $w \in S_v(L)$. If $v \in S_f(K)$, then for

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sufficiently large i the exponential map $\exp : \mathfrak{m}_{L,v}^i \to L_v^\times$ is the product of the p-adic exponential maps $\mathfrak{m}_w^i \to L_w^\times$ for $w \in S_v(L)$.

To state our conjecture we need to choose certain lattices. For each $v \in S_f := S \cap S_f(K)$, with residue characteristic p, we choose a full projective $\mathbb{Z}_p[G]$-lattice $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ which is contained in a sufficiently large power of $\mathfrak{m}_{L,v}$ to ensure that the exponential map is defined on $\mathcal{L}_v$. Let $\mathcal{L}$ be the full projective $\mathbb{Z}[G]$-sublattice of $\mathcal{O}_L$ which has p-adic completions

$$\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \Big(\prod_{v \in S_p(K) \setminus S} \mathcal{O}_{L,v} \Big) \times \Big(\prod_{v \in S_p(K) \cap S} \mathcal{L}_v \Big). \tag{9}$$

We set $L_S := \prod_{v \in S} L_v$ and $\mathcal{L}_S := \prod_{v \in S} \mathcal{L}_v$ (where $\mathcal{L}_v := L_v$ for each $v \in S_\infty(K)$) and we let $\exp_S$ denote the map $\mathcal{L}_S \to L_S^\times$ that is induced by the product of the respective exponential maps. We also write Δ_S for the natural diagonal embedding from $L^\times$ to $L_S^\times$.

Following the notation of [NSW00, ch. VIII] we write I_L for the group of idèles of L and regard $L^\times$ as embedded diagonally in I_L. The idèle class group is $C_L := I_L/L^\times$ and the S-idèle class group is $C_S(L) := I_L/(L^\times U_{L,S})$, where $U_{L,S} := \prod_{w \in S} \{1\} \times \prod_{w \notin S} \mathcal{O}_w^\times$. We remark that since $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $L_S^\times \to C_S(L)$ is surjective with kernel $\Delta_S(\mathcal{O}_{L,S}^\times)$.

There exists a canonical invariant isomorphism

$$\mathrm{inv}_{L/K,S} : H^2(G, C_S(L)) \xrightarrow{\cong} \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}.$$

Indeed, since $U_{L,S}$ is cohomologically trivial the short exact sequence $0 \to U_{L,S} \to C_L \to C_S(L) \to 0$ induces an isomorphism $H^2(G, C_L) \xrightarrow{\cong} H^2(G, C_S(L))$, and from class field theory one has a canonical invariant isomorphism $\mathrm{inv}_{L/K} : H^2(G, C_L) \xrightarrow{\cong} (1/|G|)\mathbb{Z}/\mathbb{Z}$ (as defined, for example, in [NSW00, p. 379]). We let $e_{L/K,S}^{\mathrm{glob}}$ (or e_S^{glob} when L/K is clear from context) denote the global canonical class, i.e. the element of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)) = H^2(G, C_S(L))$ that is sent by $\mathrm{inv}_{L/K,S}$ to $1/|G|$.

Let E_S be a complex in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (via § 2.2.2) to e_S^{glob}. By Lemma 2.4 there exists a unique morphism $\alpha_S : \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ for which $H^0(\alpha_S)$ is the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ and $H^1(\alpha_S)$ is the restriction of the trace map $\mathrm{tr} : L \to \mathbb{Q}$ to $\mathcal{L}$. Let $E_S(\mathcal{L})$ be any complex which lies in a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\alpha_S} E_S \xrightarrow{\beta_S} E_S(\mathcal{L}) \xrightarrow{\gamma_S} . \tag{10}$$

To describe the cohomology of $E_S(\mathcal{L})$ we use the following notation. Let $L_\infty := \prod_{w \in S_\infty(L)} L_w$ and define $\mathrm{tr}_\infty : L_\infty \to \mathbb{R}$ by $(l_w)_{w \in S_\infty(L)} \mapsto \sum_{w \in S_\infty(L)} \mathrm{tr}_{L_w/\mathbb{R}}(l_w)$. We set $L_\infty^0 := \ker(\mathrm{tr}_\infty)$ and $L^0 := \ker(\mathrm{tr} : L \to \mathbb{Q})$, and observe that one has a commutative diagram of short exact sequences of $\mathbb{R}[G]$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^0 \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\subset} & L \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\mathrm{tr} \otimes_{\mathbb{Q}} \mathbb{R}} & \mathbb{R} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \mu'_L} & & \| & & \\
0 & \longrightarrow & L_\infty^0 & \xrightarrow{\subset} & L_\infty & \xrightarrow{\mathrm{tr}_\infty} & \mathbb{R} & \longrightarrow & 0
\end{array}$$

where μ'_L denotes the canonical isomorphism and μ_L its restriction to $L^0 \otimes_{\mathbb{Q}} \mathbb{R}$. We also use the notation $\exp_\infty : L_\infty \to L_\infty^\times$ for the product of the exponential maps and $\Delta_\infty : L^\times \to L_\infty^\times$ for the diagonal embedding. Finally we set $\log_\infty(\mathcal{O}_L^\times) := \{x \in L_\infty : \exp_\infty(x) \in \Delta_\infty(\mathcal{O}_L^\times)\}$. Using the proof of Dirichlet's unit theorem (see [Neu92, Kapitel I, § 7, in particular Satz (7.3)]) it is not difficult to show that $\log_\infty(\mathcal{O}_L^\times)$ is a full lattice in L_∞^0.

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Our conjectural formula for $\hat{\partial}^1_G(\zeta^*_{L/K,S}(1))$ uses the Euler characteristic defined in part (iii) of the following lemma. In the sequel we shall abbreviate 'cohomologically trivial' to 'c-t'.

Lemma 3.1. *The complex $E_S(\mathcal{L})$ defined by the distinguished triangle (10) has the following properties.*

(i) *$E_S(\mathcal{L})$ is a perfect complex of $\mathbb{Z}[G]$-modules.*

(ii) *$E_S(\mathcal{L}) \otimes \mathbb{Q}$ is acyclic outside degrees -1 and 0, and there exist canonical identifications of $\mathbb{Q}[G]$-modules $H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q} \cong \{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}_L^\times)\} \otimes \mathbb{Q} \cong \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$ and $H^0(E_S(\mathcal{L})) \otimes \mathbb{Q} \cong L^0$.*

(iii) *The identifications from claim (ii) and the canonical isomorphism $\log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R} \cong L^0_\infty$ allow us to consider μ_L as a trivialization of $E_S(\mathcal{L})$. The Euler characteristic $\chi_G(E_S(\mathcal{L}), \mu_L)$ in $K_0(\mathbb{Z}[G], \mathbb{R})$ depends only upon L/K and S.*

Proof. Since e_S^{glob} is a generator of $H^2(G, C_S(L))$, cup-product with e_S^{glob} induces isomorphisms between the (Tate) cohomology groups of $\mathbb{Z}$ and $C_S(L)$ (with a dimension shift of 2) and hence in $\mathcal{D}(\mathbb{Z}[G])$ the complex E_S is isomorphic to a bounded complex of G-modules each of which is c-t. Since this is also obviously true for $\mathcal{L}_S[0] \oplus \mathcal{L}[-1]$, the triangle (10) implies that $E_S(\mathcal{L})$ is isomorphic to a bounded complex of G-modules each of which is c-t. We will show that the cohomology of $E_S(\mathcal{L})$ is finitely generated. Claim (i) then follows because a bounded complex of c-t $\mathbb{Z}[G]$-modules with finitely generated cohomology is perfect by a standard argument (similar to the constructions in the proofs of [Lan02, ch. XXI, Propositions 1.1 and 1.2]). The exact cohomology sequence associated to (10) implies that $E_S(\mathcal{L})$ is acyclic outside degrees $-1, 0$ and 1 and that there is an exact sequence

$$0 \to H^{-1}(E_S(\mathcal{L})) \to \mathcal{L}_S \to C_S(L) \to H^0(E_S(\mathcal{L})) \to \mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z} \to H^1(E_S(\mathcal{L})) \to 0.$$

This immediately shows that $H^1(E_S(\mathcal{L}))$ is finite and $H^1(E_S(\mathcal{L})) \otimes \mathbb{Q} = 0$. The map $\mathcal{L}_S \to C_S(L)$ in the exact sequence is the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ whose cokernel is easily seen to be finite; hence $H^0(E_S(\mathcal{L}))$ is finitely generated and there is an identification $H^0(E_S(\mathcal{L})) \otimes \mathbb{Q} \cong L^0$. Finally $H^{-1}(E_S(\mathcal{L})) \cong \{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}_L^\times)\}$ and the projection map $\mathcal{L}_S \to L_\infty$ induces an isomorphism between this set and $\log_\infty(U) := \{x \in L_\infty : \exp_\infty(x) \in \Delta_\infty(U)\}$, where U is a subgroup of finite index in $\mathcal{O}_L^\times$. Since $\log_\infty(U)$ is a full lattice in L^0_∞ we see that $H^{-1}(E_S(\mathcal{L}))$ is finitely generated, which completes the proof of claim (i). Moreover $\log_\infty(U) \otimes \mathbb{Q} = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$, which completes the proof of claim (ii).

The element $\chi_G(E_S(\mathcal{L}), \mu_L)$ does not depend on the choice of E_S or of the distinguished triangle (10) because up to isomorphism in $\mathcal{D}(\mathbb{Z}[G])$ the complex $E_S(\mathcal{L})$ is independent of these choices. It remains to prove that $\chi_G(E_S(\mathcal{L}), \mu_L)$ is independent of the choice of $\mathcal{L}$. For each $v \in S_f$ we let $\mathcal{L}'_v \subseteq \mathcal{O}_{L,v}$ be lattices giving rise to a lattice $\mathcal{L}' \subseteq \mathcal{O}_L$ as above. We assume (as we may) that $\mathcal{L}'_v \subseteq \mathcal{L}_v$ for all $v \in S_f$ so $\mathcal{L}' \subseteq \mathcal{L}$. We set $\mathcal{L}'_S := \prod_{v \in S} \mathcal{L}'_v$ and consider the following commutative diagram of distinguished triangles in $\mathcal{D}(\mathbb{Z}[G])$ (the existence of such a diagram follows for example from [BBD82, Proposition 1.1.11]).

$$\begin{array}{ccccccc}
\mathcal{L}'_S[0] \oplus \mathcal{L}'[-1] & \longrightarrow & E_S & \longrightarrow & E_S(\mathcal{L}') & \longrightarrow & \\
\downarrow \subseteq & & \| & & \downarrow & & \\
\mathcal{L}_S[0] \oplus \mathcal{L}[-1] & \longrightarrow & E_S & \longrightarrow & E_S(\mathcal{L}) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{L}_S/\mathcal{L}'_S[0] \oplus \mathcal{L}/\mathcal{L}'[-1] & \longrightarrow & 0 & \longrightarrow & \mathcal{L}_S/\mathcal{L}'_S[1] \oplus \mathcal{L}/\mathcal{L}'[0] & = & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}$$

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In this diagram the first two rows are distinguished triangles as in (10) and the first column is induced by the obvious short exact sequence. Now the G-modules $\mathcal{L}_S/\mathcal{L}'_S$ and $\mathcal{L}/\mathcal{L}'$ are c-t, finite and isomorphic. Hence the zero map is a trivialization of $\mathcal{L}_S/\mathcal{L}'_S[1]\oplus\mathcal{L}/\mathcal{L}'[0]$ and the associated Euler characteristic is $\chi_G(\mathcal{L}_S/\mathcal{L}'_S[1]\oplus\mathcal{L}/\mathcal{L}'[0],0)=0\in K_0(\mathbb{Z}[G],\mathbb{R})$. Upon applying [BrB05, Theorem 5.7] to the third column of the above diagram we thus deduce that $\chi_G(E_S(\mathcal{L}'),\mu_L)=\chi_G(E_S(\mathcal{L}),\mu_L)$, as required. □

Remark 3.2. It is occasionally convenient to give an explicit representative of $E_S(\mathcal{L})$ in the following way. Fix an extension $0\to C_S(L)\xrightarrow{\subseteq} A\xrightarrow{d} B\to\mathbb{Z}\to 0$, which represents $e_S^{\mathrm{glob}}\in\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z},C_S(L))$ and in which the G-modules A and B are c-t. Then E_S can be taken to be the complex $A\xrightarrow{d}B$ in degrees 0 and 1, and the morphism $\mathcal{L}_S[0]\oplus\mathcal{L}[-1]\to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ is represented by the map α of complexes which in degree 0 is $\mathcal{L}_S\xrightarrow{\exp_S}L_S^\times\to C_S(L)\subseteq A$ and in degree 1 is any lift $\mathcal{L}\xrightarrow{\mathrm{tr}'}B$ of $\mathcal{L}\xrightarrow{\mathrm{tr}}\mathbb{Z}$ through the given surjection $B\to\mathbb{Z}$.

The complex $E_S(\mathcal{L})$ can be taken to be the mapping cone of this map α of complexes, that is

$$\mathcal{L}_S\xrightarrow{(d^{-1},0)}A\oplus\mathcal{L}\xrightarrow{d^0}B,$$

where $\mathcal{L}_S$ is placed in degree -1, $d^{-1}:\mathcal{L}_S\to A$ is as above and $d^0=(d,\mathrm{tr}')$. Note that for these complexes the distinguished triangle (10) which gives rise to the identification of the cohomology of $E_S(\mathcal{L})\otimes\mathbb{Q}$ in Lemma 3.1(ii) is

$$\mathcal{L}_S[0]\oplus\mathcal{L}[-1]\xrightarrow{\alpha}E_S\xrightarrow{\beta}E_S(\mathcal{L})\xrightarrow{\gamma}\mathcal{L}_S[1]\oplus\mathcal{L}[0],$$

where α is as described above, β is the identity on A and B, and γ is minus the identity on $\mathcal{L}_S$ and $\mathcal{L}$.

We now formulate our conjectural description of $\hat\partial^1_G(\zeta^*_{L/K,S}(1))$.

Conjecture 3.3. In $K_0(\mathbb{Z}[G],\mathbb{R})$ one has $\hat\partial^1_G(\zeta^*_{L/K,S}(1))=-\chi_G(E_S(\mathcal{L}),\mu_L)$.

3.2 Basic properties

To describe some basic properties of Conjecture 3.3 it is convenient to set

$$T\Omega(L/K,1):=\hat\partial^1_G(\zeta^*_{L/K,S}(1))+\chi_G(E_S(\mathcal{L}),\mu_L)\in K_0(\mathbb{Z}[G],\mathbb{R}).$$

Proposition 3.4. *The element $T\Omega(L/K,1)$ depends only upon L/K.*

Proof. Since $\chi_G(E_S(\mathcal{L}),\mu_L)$ depends only upon L/K and S (by Lemma 3.1(iii)) it suffices to prove that $T\Omega(L/K,1)$ is unchanged if one replaces S by $S':=S\cup\{v'\}$ where v' is any element of $S(K)\setminus S$. But $\zeta^*_{L/K,S}(1)=\zeta^*_{L/K,S'}(1)\cdot\mathrm{ind}^G_{G_w}(L^*_{L_w/K_{v'}}(1))$ where as in § 2.3.2 we fix $w\in S_{v'}(L)$ and let G_w be the decomposition group of w. Hence we must show that

$$\chi_G(E_{S'}(\mathcal{L}'),\mu_L)-\chi_G(E_S(\mathcal{L}),\mu_L)=\hat\partial^1_G(\mathrm{ind}^G_{G_w}(L^*_{L_w/K_{v'}}(1))).\tag{11}$$

To simplify the notation we set $\mathcal{O}':=\mathcal{O}_{L,v'}$, $U':=\mathcal{O}^\times_{L,v'}$ and, for each integer $i\geqslant 1$, $U'^{(i)}:=\prod_{w\in S_{v'}(L)}U^{(i)}_{L_w}$. Let $\mathcal{O}_{v'}$ be the ring of integers in the completion $K_{v'}$ and let π be a uniformizing parameter for $\mathcal{O}_{v'}$. We choose a lattice $\mathcal{L}'\subseteq\mathcal{L}$ for S' such that $\mathcal{L}'_v=\mathcal{L}_v$ for $v\in S$ and $\mathcal{L}'_{v'}=\pi^m\mathcal{O}'$ where m is a sufficiently large integer. Note that $\pi^m\mathcal{O}'$ is a projective $\mathbb{Z}_p[G]$-lattice since v' is unramified in L/K and that $\exp(\mathcal{L}'_{v'})=\exp(\pi^m\mathcal{O}')=U'^{(m)}$.

There is a canonical short exact sequence $0\to U'\to C_{S'}(L)\to C_S(L)\to 0$, and since the G-module U' is c-t one obtains an isomorphism $H^2(G,C_{S'}(L))\xrightarrow{\cong}H^2(G,C_S(L))$. This isomorphism

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maps $e^{\mathrm{glob}}_{S'}$ to e^{glob}_{S} and hence Lemma 2.4 implies the existence of a morphism $\kappa : E_{S'} \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ which induces the identity on H^1 and the map $C_{S'}(L) \to C_S(L)$ on H^0. We consider the following commutative diagram of distinguished triangles in $\mathcal{D}(\mathbb{Z}[G])$.

$$\begin{array}{ccccccc}
\mathcal{L}'_{v'}[0] \oplus (\mathcal{L}/\mathcal{L}')[-2] & \longrightarrow & \mathcal{L}'_{S'}[0] \oplus \mathcal{L}'[-1] & \longrightarrow & \mathcal{L}_S[0] \oplus \mathcal{L}[-1] & \longrightarrow \\
\downarrow & & \downarrow{\scriptstyle \alpha_{S'}} & & \downarrow{\scriptstyle \alpha_S} & \\
U'[0] & \longrightarrow & E_{S'} & \xrightarrow{\kappa} & E_S & \longrightarrow \\
\downarrow & & \downarrow{\scriptstyle \beta_{S'}} & & \downarrow{\scriptstyle \beta_S} & \\
(U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1] & \longrightarrow & E_{S'}(\mathcal{L}') & \longrightarrow & E_S(\mathcal{L}) & \longrightarrow \\
\downarrow & & \downarrow{\scriptstyle \gamma_{S'}} & & \downarrow{\scriptstyle \gamma_S} &
\end{array}$$

In this diagram the upper row is induced by the obvious short exact sequences $0 \to \mathcal{L}'_{v'} \to \mathcal{L}'_{S'} \to \mathcal{L}_S \to 0$ and $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}/\mathcal{L}' \to 0$; the first column is the distinguished triangle induced by the exact sequence $0 \to \mathcal{L}'_{v'} \xrightarrow{\exp} U' \to U'/U'^{(m)} \to 0$; and the second and third columns are the distinguished triangles defined as in (10). The existence of the diagram follows by applying [BBD82, Proposition 1.1.11] to the upper right square, then rotating the resulting diagram and observing that the arrow $\mathcal{L}'_{v'}[0] \oplus (\mathcal{L}/\mathcal{L}')[-2] \to U'[0]$ is indeed as stated. Now the complex $(U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1]$ has finite cohomology groups and so by applying [BrB05, Theorem 5.7] to the third row of the diagram we find that

$$\chi_G(E_{S'}(\mathcal{L}'), \mu_L) - \chi_G(E_S(\mathcal{L}), \mu_L) = \chi_G((U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1], 0). \tag{12}$$

To compute further we recall that any finite c-t G-module M gives rise to a canonical element (M) of $K_0(\mathbb{Z}[G], \mathbb{Q}) \subseteq K_0(\mathbb{Z}[G], \mathbb{R})$ and that if we consider $M[-i]$ as an object of $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$, then $\chi_G(M[-i], 0) = (-1)^{i+1}(M)$. Hence one has

$$\begin{aligned}
\chi_G((U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1], 0) &= -(U'/U'^{(m)}) + (\mathcal{L}/\mathcal{L}') \\
&= -(U'/U'^{(1)}) - (U'^{(1)}/U'^{(m)}) + (\mathcal{O}'/\pi\mathcal{O}') + (\pi\mathcal{O}'/\pi^m\mathcal{O}').
\end{aligned}$$

From the isomorphisms $U'^{(i)}/U'^{(i+1)} \cong \pi^i\mathcal{O}'/\pi^{i+1}\mathcal{O}'$ for all $i \geqslant 1$ one deduces $(U'^{(1)}/U'^{(m)}) = (\pi\mathcal{O}'/\pi^m\mathcal{O}')$. In addition, writing $\lambda(w)$ for the residue field of $\mathcal{O}_w$, one has the isomorphisms $\mathcal{O}'/\pi\mathcal{O}' \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} \lambda(w)$ and $U'/U'^{(1)} \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} \lambda(w)^\times$, and so the last displayed formula implies that

$$\chi_G((U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1], 0) = \mathrm{ind}^G_{G_w}(-(\lambda(w)^\times) + (\lambda(w))), \tag{13}$$

where $(\lambda(w)^\times)$ and $(\lambda(w))$ are considered as elements of $K_0(\mathbb{Z}[G_w], \mathbb{R})$. Now the exact sequence

$$0 \to \mathbb{Z}[G_w] \xrightarrow{Nv' - \sigma_w} \mathbb{Z}[G_w] \to \lambda(w)^\times \to 0$$

implies

$$(\lambda(w)^\times) = \hat{\partial}^1_{G_w}((\det_{\mathbb{C}}(Nv' - \sigma_w \mid V_\psi))_{\psi \in \mathrm{Irr}(G_w)}),$$

where Nv', σ_w and V_ψ are as in § 2.3.2. Also, if v' has residue characteristic p and residue degree f (i.e. $Nv' = p^f$) then $\lambda(w)$ is a free $(\mathbb{Z}/p)[G_w]$-module of rank f and so

$$\begin{aligned}
(\lambda(w)) &= [\mathbb{Z}[G_w]^f, \cdot p, \mathbb{Z}[G_w]^f] \\
&= [\mathbb{Z}[G_w], \cdot Nv', \mathbb{Z}[G_w]] = \hat{\partial}^1_{G_w}((\det_{\mathbb{C}}(Nv' \mid V_\psi))_{\psi \in \mathrm{Irr}(G_w)}).
\end{aligned}$$

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It follows that

$$-(\lambda(w)^{\times}) + (\lambda(w)) = \hat{\partial}^1_{G_w}((\det_{\mathbb{C}}(1 - \sigma_w(Nv')^{-1} \mid V_\psi)^{-1})_{\psi \in \mathrm{Irr}(G_w)})$$
$$= \hat{\partial}^1_{G_w}(L^*_{L_w/K_{v'}}(1)). \tag{14}$$

Equations (12), (13) and (14) imply (11) which completes the proof. □

We next describe the behaviour of $T\Omega(L/K, 1)$ under the maps discussed in § 2.1.3.

Proposition 3.5. *Let M be an intermediate field of L/K and $H = \mathrm{Gal}(L/M)$. Then we have the following:*

(i) $\mathrm{res}^G_H(T\Omega(L/K, 1)) = T\Omega(L/M, 1)$; *and*

(ii) *if H is normal in G, then* $\mathrm{q}^G_{G/H}(T\Omega(L/K, 1)) = T\Omega(M/K, 1)$.

Proof. Let $S = S_K$ be a finite set of places of K satisfying all conditions necessary to formulate Conjecture 3.3 for L/K. Then the set S_M consisting of all places of M lying above a place in S satisfies the corresponding conditions with respect to L/M. Further, if $v \in S_f$ has residue characteristic p, then one can choose a full projective $\mathbb{Z}_p[H]$-lattice $\mathcal{L}_u \subset \mathcal{O}_{L,u}$ for every $u \in S_v(M)$ such that $\mathcal{L}_v := \prod_{u \in S_v(M)} \mathcal{L}_u \subseteq \prod_{u \in S_v(M)} \mathcal{O}_{L,u} = \mathcal{O}_{L,v}$ is a full projective $\mathbb{Z}_p[G]$-lattice. We thus obtain the same lattice $\mathcal{L} \subseteq \mathcal{O}_L$ for the extensions L/K and L/M, and also $\mathcal{L}_{S_K} = \mathcal{L}_{S_M}$. Now $e^{\mathrm{glob}}_{L/M,S_M}$ is the image of $e^{\mathrm{glob}}_{L/K,S_K}$ under the restriction map $H^2(G, C_S(L)) \to H^2(H, C_S(L))$ and so the complex $E_{L/M,S_M}$ can be taken to be equal to $E_{L/K,S_K}$ with the group action restricted from G to H. It follows that the trivialized complex $(E_{L/M,S_M}(\mathcal{L}), \mu_L)$ for the extension L/M can be taken to equal $(E_{L/K,S_K}(\mathcal{L}), \mu_L)$ for the extension L/K (again with the group action restricted from G to H) and so

$$\mathrm{res}^G_H(\chi_G(E_{L/K,S_K}(\mathcal{L}), \mu_L)) = \chi_H(E_{L/M,S_M}(\mathcal{L}), \mu_L) \in K_0(\mathbb{Z}[H], \mathbb{R}).$$

Under the identifications $\mathrm{Z}(\mathbb{C}[G])^{\times} = \prod_{\chi \in \mathrm{Irr}(G)} \mathbb{C}^{\times}$ and $\mathrm{Z}(\mathbb{C}[H])^{\times} = \prod_{\psi \in \mathrm{Irr}(H)} \mathbb{C}^{\times}$, the restriction map $\mathrm{res}^G_H : \mathrm{Z}(\mathbb{C}[G])^{\times} \to \mathrm{Z}(\mathbb{C}[H])^{\times}$ is given by

$$(\alpha_\chi)_{\chi \in \mathrm{Irr}(G)} \mapsto \left(\prod_{\chi \in \mathrm{Irr}(G)} \alpha_\chi^{\langle \chi, \mathrm{ind}^G_H(\psi) \rangle} \right)_{\psi \in \mathrm{Irr}(H)},$$

where $\mathrm{ind}^G_H(\psi)$ denotes the induced character (see [Bre04c, Lemma 2.4]). Furthermore one has $\zeta^*_{L/K,S_K}(1) = (L^*_{L/K,S_K}(\chi, 1))_{\chi \in \mathrm{Irr}(G)}$ and $\zeta^*_{L/M,S_M}(1) = (L^*_{L/M,S_M}(\psi, 1))_{\psi \in \mathrm{Irr}(H)}$. Hence the induction property of truncated Artin L-functions (which is similar to the non-truncated version in [Neu92, Kapitel VII, Satz (10.4)]) implies that $\mathrm{res}^G_H(\zeta^*_{L/K,S_K}(1)) = \zeta^*_{L/M,S_M}(1) \in \mathrm{Z}(\mathbb{R}[H])^{\times}$, and that therefore $\mathrm{res}^G_H(\hat{\partial}^1_G(\zeta^*_{L/K,S_K}(1))) = \hat{\partial}^1_H(\zeta^*_{L/M,S_M}(1)) \in K_0(\mathbb{Z}[H], \mathbb{R})$. From this, claim (i) follows.

To prove claim (ii) we fix a finite set S of places of K containing all archimedean places and all places which ramify in L/K (and hence all which ramify in M/K) and which is sufficiently large to ensure that both $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$ and $\mathrm{Pic}(\mathcal{O}_{M,S}) = 0$. Set $Q := G/H$. One has $\mathrm{q}^G_Q(\zeta^*_{L/K,S}(1)) = \zeta^*_{M/K,S}(1) \in \mathrm{Z}(\mathbb{R}[Q])^{\times}$ and hence $\mathrm{q}^G_Q(\hat{\partial}^1_G(\zeta^*_{L/K,S}(1))) = \hat{\partial}^1_Q(\zeta^*_{M/K,S}(1)) \in K_0(\mathbb{Z}[Q], \mathbb{R})$. In addition, if we have chosen lattices $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ and $\mathcal{L} \subseteq \mathcal{O}_L$ as in (9) with respect to $(L/K, S)$, then $\mathcal{L}^H_v \subseteq \mathcal{O}^H_{L,v} = \mathcal{O}_{M,v}$ and $\mathcal{L}^H \subseteq \mathcal{O}_M$ satisfy (9) with respect to $(M/K, S)$, and so we need only show that $\mathrm{q}^G_Q(\chi_G(E_{L/K,S}(\mathcal{L}), \mu_L)) = \chi_Q(E_{M/K,S}(\mathcal{L}^H), \mu_M) \in K_0(\mathbb{Z}[Q], \mathbb{R})$.

We first make the following general observation which follows easily from the description of Euler characteristics given in [BrB05, § 6]. If $C \in \mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ is a complex of c-t G-modules and $t : H^{\mathrm{ev}}(C \otimes \mathbb{R}) \to H^{\mathrm{od}}(C \otimes \mathbb{R})$ a trivialization of C then $\mathrm{q}^G_Q(\chi_G(C, t)) = \chi_Q(C^H, t^H)$ in $K_0(\mathbb{Z}[Q], \mathbb{R})$ where $C^H \in \mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[Q])$ is the complex of H-invariants and t^H is the trivialization $H^{\mathrm{ev}}(C^H \otimes \mathbb{R}) \cong (H^{\mathrm{ev}}(C \otimes \mathbb{R}))^H \xrightarrow{t} (H^{\mathrm{od}}(C \otimes \mathbb{R}))^H \cong H^{\mathrm{od}}(C^H \otimes \mathbb{R})$.

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To apply this we note first that if we represent $e^{\mathrm{glob}}_{L/K,S}$ by an extension of $\mathbb{Z}[G]$-modules

$$0 \to C_S(L) \xrightarrow{\subseteq} A \xrightarrow{d} B \xrightarrow{\kappa} \mathbb{Z} \to 0 \tag{15}$$

in which A and B are c-t, then $e^{\mathrm{glob}}_{M/K,S}$ is represented by the induced extension of $\mathbb{Z}[Q]$-modules

$$0 \to C_S(M) \xrightarrow{\subseteq} A^H \xrightarrow{d} B^H \xrightarrow{\kappa'} \mathbb{Z} \to 0, \tag{16}$$

where $\kappa' := (1/|H|)\kappa$. Indeed, since the injective inflation map $H^2(Q, C_S(M)) \to H^2(G, C_S(L))$ sends $e^{\mathrm{glob}}_{M/K,S}$ to $|H| \cdot e^{\mathrm{glob}}_{L/K,S}$, to conclude that (16) represents $e^{\mathrm{glob}}_{M/K,S}$ it is enough to note that there is an equivalence of Yoneda extensions between the extension obtained by first regarding (16) as an extension of G-modules and then taking the push-out under the inclusion $C_S(M) \to C_S(L)$ and the extension obtained by taking the pull-back of (15) under the 'multiplication by $|H|$' homomorphism $\mathbb{Z} \to \mathbb{Z}$. (It is a straightforward exercise to prove the claimed equivalence of Yoneda extensions and so we omit details.) Thus we can take the complexes $E_{L/K,S}$ and $E_{M/K,S}$ to be $A \xrightarrow{d} B$ and $A^H \xrightarrow{d} B^H$ respectively (where in both cases the modules are placed in degrees 0 and 1). When combined with the fact that $\mathrm{tr}_{M/\mathbb{Q}} = (1/|H|)\mathrm{tr}_{L/\mathbb{Q}}$ (on M) and the explicit construction of $E_{L/K,S}(\mathcal{L})$ and $E_{M/K,S}(\mathcal{L}^H)$ in Remark 3.2 we see that $E_{M/K,S}(\mathcal{L}^H) = (E_{L/K,S}(\mathcal{L}))^H$ and $\mu_M = (\mu_L)^H$. This gives the required equality $\mathrm{q}^G_Q(\chi_G(E_{L/K,S}(\mathcal{L}), \mu_L)) = \chi_Q((E_{L/K,S}(\mathcal{L}))^H, (\mu_L)^H) = \chi_Q(E_{M/K,S}(\mathcal{L}^H), \mu_M)$. □

3.3 The conjectures of Stark and Chinburg

We show that Conjecture 3.3 refines both Stark's conjecture at $s = 1$ (as discussed by Tate in [Tat84, ch. I, Conjecture 8.2]) and also Chinburg's 'Ω_1-conjecture' (as formulated in [Chi85, Question 3.2] and [CCFT91, § 4.2, Conjecture 3]).

Proposition 3.6. *Let $T\Omega(L/K, 1)$ be the element of $K_0(\mathbb{Z}[G], \mathbb{R})$ defined in § 3.2 (so Conjecture 3.3 is equivalent to an equality $T\Omega(L/K, 1) = 0$). Then both of the following assertions are valid:*

(i) *$T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if Stark's conjecture at $s = 1$ is valid for L/K;*

(ii) *$T\Omega(L/K, 1) \in \ker(K_0(\mathbb{Z}[G], \mathbb{R}) \xrightarrow{\partial^0_G} K_0(\mathbb{Z}[G]))$ if and only if Chinburg's 'Ω_1-conjecture' is valid for L/K.*

Proof. Since the $\mathbb{R}[G]$-modules $L^0 \otimes_{\mathbb{Q}} \mathbb{R}$ and $\log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R}$ are isomorphic we may choose an isomorphism $\kappa : L^0 \xrightarrow{\cong} \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$ of $\mathbb{Q}[G]$-modules. Then $\chi_G(E_S(\mathcal{L}), \kappa \otimes_{\mathbb{Q}} \mathbb{R}) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ and so $T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if $\chi_G(E_S(\mathcal{L}), \kappa \otimes_{\mathbb{Q}} \mathbb{R}) - T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$. But [BrB05, Proposition 5.6.2] implies that

$$\chi_G(E_S(\mathcal{L}), \kappa \otimes_{\mathbb{Q}} \mathbb{R}) - \chi_G(E_S(\mathcal{L}), \mu_L) = \partial^1_G([L^0 \otimes_{\mathbb{Q}} \mathbb{R}, \lambda])$$

with $\lambda := \mu_L^{-1} \circ (\kappa \otimes_{\mathbb{Q}} \mathbb{R})$. Since $\mathrm{Z}(\mathbb{Q}[G])^\times$ is the full pre-image of $K_0(\mathbb{Z}[G], \mathbb{Q})$ under the map $\hat{\partial}^1_G : \mathrm{Z}(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ and $\partial^1_G([L^0 \otimes_{\mathbb{Q}} \mathbb{R}, \lambda]) = \hat{\partial}^1_G(\mathrm{nr}(\lambda))$, it follows that $T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if $\zeta^*_{L/K,S}(1)^{-1} \cdot \mathrm{nr}(\lambda) \in \mathrm{Z}(\mathbb{Q}[G])^\times$.

Now $\mathrm{Z}(\mathbb{Q}[G])^\times$ is equal to the subgroup of elements $(z_\psi)_{\psi \in \mathrm{Irr}(G)}$ of $\mathrm{Z}(\mathbb{C}[G])^\times = \prod_{\psi \in \mathrm{Irr}(G)} \mathbb{C}^\times$ with the property that $\omega(z_\psi) = z_{\omega \circ \psi}$ for all $\omega \in \mathrm{Aut}(\mathbb{C})$ and all $\psi \in \mathrm{Irr}(G)$. Since for each $\psi \in \mathrm{Irr}(G)$ one has $\zeta^*_{L/K,S}(1)_\psi = L^*_{L/K,S}(\psi, 1)$ and $\mathrm{nr}(\lambda)_\psi = \det_{\mathbb{C}}(\lambda \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, L^0 \otimes_{\mathbb{Q}} \mathbb{C}))$, where we write λ for the $\mathbb{C}$-linear automorphism of $\mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, L^0 \otimes_{\mathbb{Q}} \mathbb{C})$ that is induced by $\lambda \otimes_{\mathbb{R}} \mathbb{C}$, we deduce that $T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if for each $\psi \in \mathrm{Irr}(G)$ and each $\omega \in \mathrm{Aut}(\mathbb{C})$ one has

$$\omega\left(\frac{\det_{\mathbb{C}}(\lambda \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, L^0 \otimes_{\mathbb{Q}} \mathbb{C}))}{L^*_{L/K,S}(\psi, 1)}\right) = \frac{\det_{\mathbb{C}}(\lambda \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_{\omega \circ \psi}, L^0 \otimes_{\mathbb{Q}} \mathbb{C}))}{L^*_{L/K,S}(\omega \circ \psi, 1)}.$$

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To show that this condition is equivalent to the validity of [Tat84, ch. I, Conjecture 8.2] for each $\psi \in \mathrm{Irr}(G)$ one need only mimic the proof of [Tat84, ch. I, Proposition 6.1]. Indeed, the last displayed equality is the variant of [Tat84, ch. I, Conjecture 8.2] that is alluded to in [Tat84, top of p. 35]. This proves claim (i).

Chinburg's 'Ω_1-conjecture' asserts the vanishing of the element $\Omega(L/K,1)$ of $\mathrm{Cl}(\mathbb{Z}[G])$ that is defined in [Chi85, Definition 3.1]. Since Lemma 2.7(ii) implies that $\partial^0_G(\hat{\partial}^1_G(\zeta^*_{L/K,S}(1))) = 0$, claim (ii) will follow if $\partial^0_G(\chi_G(E_S(\mathcal{L}),\mu_L)) = \Omega(L/K,1)$. After enlarging S if necessary, we may assume that $\mathcal{L}$ is a free $\mathbb{Z}[G]$-module (indeed one can take $\mathcal{L}$ to be a suitable integer multiple of any free $\mathcal{O}_K[G]$-submodule of $\mathcal{O}_L$, and then define $\mathcal{L}_v$ by (9)). We choose an extension $0 \to C_S(L) \xrightarrow{\subseteq} A \xrightarrow{d} B \to \mathbb{Z} \to 0$ representing e^{glob}_S as in Remark 3.2 but with the additional condition that B is a finitely generated free $\mathbb{Z}[G]$-module. We set $\mathcal{L}_f := \prod_{v\in S_f}\mathcal{L}_v$, we write $\exp(\mathcal{L}_f)$ for the image of $\mathcal{L}_f \subset \mathcal{L}_S$ under the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ and note that the induced map $\exp : \mathcal{L}_f \to \exp(\mathcal{L}_f)$ is an isomorphism. We also consider $\exp(\mathcal{L}_f)$ as a submodule of A and set $A_{\mathcal{L}} := A/\exp(\mathcal{L}_f)$. Then one has a short exact sequence of complexes (with vertical differentials) of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & B & = & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d^0} & & \uparrow & & \\
0 & \longrightarrow & \exp(\mathcal{L}_f) & \longrightarrow & A\oplus\mathcal{L} & \longrightarrow & A_{\mathcal{L}}\oplus\mathcal{L} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle (d^{-1},0)} & & \uparrow{\scriptstyle (d',0)} & & \\
0 & \longrightarrow & \mathcal{L}_f & \xrightarrow{\subset} & \mathcal{L}_S & \longrightarrow & L_\infty & \longrightarrow & 0
\end{array}$$

where the central column is the representative of $E_S(\mathcal{L})$ described in Remark 3.2. Since the left complex is acyclic this sequence implies that $\chi_G(E_S(\mathcal{L}),\mu_L) = \chi_G(E_S(\mathcal{L})',\mu_L)$ where $E_S(\mathcal{L})'$ denotes the complex given by the right column of the diagram. From [BrB05, Proposition 5.6.1] we may therefore deduce that $\partial^0_G(\chi_G(E_S(\mathcal{L}),\mu_L)) \equiv \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d'} A_{\mathcal{L}}) \bmod F(\mathbb{Z}[G])$ where we write $\chi_{\mathbb{Z}[G]}$ for the Euler characteristic with values in $K_0(\mathbb{Z}[G])$ and $F(\mathbb{Z}[G])$ for the subgroup of $K_0(\mathbb{Z}[G])$ which is generated by the class $[\mathbb{Z}[G]]$.

For each $v \in S_\infty(K)$ we now fix a place w of L above v and a finitely generated $\mathbb{Z}[G_w]$-submodule W_v of $L_w^\times$ which satisfies the conditions of [Chi85, Lemma 2.1(i), (ii)]. We let W_∞ denote the $\mathbb{Z}[G]$-submodule $\prod_{v\in S_\infty(K)}\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]} W_v$ of $L_\infty^\times$. Both $\Delta_S(\mathcal{O}^\times_{L,S})$ and $\exp_S(\mathcal{L}_f)$ lie in the subgroup $W_\infty \times \prod_{w\in S_f} L_w^\times$ of $L_S^\times$, and we set $C^f_{L,S} := (W_\infty \times \prod_{w\in S_f} L_w^\times)/(\Delta_S(\mathcal{O}^\times_{L,S})\cdot\exp_S(\mathcal{L}_f))$. Then we may construct a commutative diagram of the following form.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C^f_{L,S} & \longrightarrow & A' & \longrightarrow & B & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \subset} & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & C_S(L)/\exp(\mathcal{L}_f) & \xrightarrow{e} & A'_{\mathcal{L}} & \longrightarrow & B & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & L_\infty^\times/W_\infty & = & L_\infty^\times/W_\infty & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array} \tag{17}$$

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In this diagram the exactness of the left-hand column is clear and hence, since (by definition of W_∞) the G-module $L_\infty^\times/W_\infty$ is c-t, this column induces an isomorphism $\iota : \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C^f_{L,S}) \cong \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)/\exp(\mathcal{L}_f))$. The upper row is chosen to be a representative of the pre-image under ι of the element corresponding to the exact sequence

$$0 \to C_S(L)/\exp(\mathcal{L}_f) \to A_\mathcal{L} \xrightarrow{d} B \to \mathbb{Z} \to 0, \tag{18}$$

and the second row of (17) is then constructed via the push-out of the given maps $C^f_{L,S} \to A'$ and $C^f_{L,S} \to C_S(L)/\exp(\mathcal{L}_f)$. The second column of (17) is thus exact and hence, since $A_\mathcal{L}$ and $L_\infty^\times/W_\infty$ are both c-t, the G-module A', respectively $A'_\mathcal{L}$, is finitely generated and c-t, respectively c-t. Now the complexes $L_\infty \xrightarrow{d'} A_\mathcal{L}$ and $L_\infty \xrightarrow{d''} A'_\mathcal{L}$ (where d'' is the composite of $L_\infty \xrightarrow{\exp_\infty} L_\infty^\times \to C_S(L)/\exp(\mathcal{L}_f)$ and the map e from (17)) are isomorphic in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ since (18) and the second row of (17) represent the same element of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)/\exp(\mathcal{L}_f))$, and hence $\chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d'} A_\mathcal{L}) = \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d''} A'_\mathcal{L})$. In addition, by its very definition, $\Omega(L/K, 1) \equiv \chi_{\mathbb{Z}[G]}(A' \to B) \equiv \chi_{\mathbb{Z}[G]}(A'[0])$ mod $F(\mathbb{Z}[G])$. From the exact sequence of perfect complexes (with vertical differentials)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A'_\mathcal{L} & \longrightarrow & L_\infty^\times/W_\infty & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle \exp_\infty} & & \\
0 & \longrightarrow & 0 & \longrightarrow & L_\infty & = & L_\infty & \longrightarrow & 0
\end{array}$$

(where the upper row comes from (17)) we may therefore deduce that

$$\begin{aligned}
\partial^0_G(\chi_G(E_S(\mathcal{L}), \mu_L)) &\equiv \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d'} A_\mathcal{L}) \\
&= \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d''} A'_\mathcal{L}) \\
&= \chi_{\mathbb{Z}[G]}(A'[0]) + \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{\exp_\infty} L_\infty^\times/W_\infty) \\
&\equiv \Omega(L/K, 1) + \sum_{v \in S_\infty(K)} \mathrm{ind}^G_{G_w}(\chi_{\mathbb{Z}[G_w]}(L_w \xrightarrow{\exp} L_w^\times/W_v)),
\end{aligned}$$

where, for each $v \in S_\infty(K)$, $\mathrm{ind}^G_{G_w}$ is the natural induction map $K_0(\mathbb{Z}[G_w]) \to K_0(\mathbb{Z}[G])$. But $\chi_{\mathbb{Z}[G_w]}(L_w \xrightarrow{\exp} L_w^\times/W_v) \equiv 0$ mod $F(\mathbb{Z}[G_w])$ for every $v \in S_\infty(K)$ because $\mathrm{Cl}(\mathbb{Z}[G_w]) = 0$ (since $|G_w| \leqslant 2$). So the last displayed formula implies that $\partial^0_G(\chi_G(E_S(\mathcal{L}), \mu_L)) \equiv \Omega(L/K, 1)$ mod $F(\mathbb{Z}[G])$, and since both sides lie in $\mathrm{Cl}(\mathbb{Z}[G])$ this shows that $\partial^0_G(\chi_G(E_S(\mathcal{L}), \mu_L)) = \Omega(L/K, 1)$ as required. □

4. The leading term at $s = 0$

As in § 3 we consider a Galois extension L/K of number fields with Galois group G and a finite set S of places of K containing all archimedean places, all places ramified in L/K, and for which $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. In this section we formulate a conjectural description of $\hat\partial^1_G(\zeta^*_{L/K,S}(0))$ in terms of a natural Euler characteristic. An equivalent form of this conjecture has already been studied in [Bur01], see Remark 4.3 below.

We shall use the following standard notation. If T is any finite set of places of K then Y_T denotes the G-module $Y_T := \prod_{w \in T} \mathbb{Z}$ where the product is over all places w of L lying above a place in T. There is a natural augmentation map $\mathrm{aug} : Y_T \to \mathbb{Z}$ and we define X_T to be its kernel.

Let P_S be a complex in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds to Tate's canonical extension class in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, \mathcal{O}^\times_{L,S})$ that is defined in [Tat66] (see also [Chi85] and [Tat84, ch. II] for a discussion of this class). Then P_S is a perfect complex which is acyclic outside degrees 0 and 1, and there are isomorphisms $H^0(P_S) \cong \mathcal{O}^\times_{L,S}$ and $H^1(P_S) \cong X_S$. Let $\mathrm{Reg}_S : \mathcal{O}^\times_{L,S} \to X_S \otimes_\mathbb{Z} \mathbb{R}$ be the regulator

map $\mathrm{Reg}_S(u) := (\log|u|_w)_{w\in S}$, where the absolute values $|\cdot|_w$ are normalized as in [Tat84, ch. 0, § 0]. It induces an isomorphism $\mathcal{O}_{L,S}^\times \otimes_{\mathbb{Z}} \mathbb{R} \to X_S \otimes_{\mathbb{Z}} \mathbb{R}$ of $\mathbb{R}[G]$-modules which we again denote by Reg_S. We take the negative of this regulator as trivialization of the complex P_S. Recall that in § 2.1.4 we defined an involution ψ_G^* of $K_0(\mathbb{Z}[G],\mathbb{R})$.

Conjecture 4.1. One has $\hat\partial_G^1(\zeta^*_{L/K,S}(0)) = -\psi_G^*(\chi_G(P_S, -\mathrm{Reg}_S))$ in $K_0(\mathbb{Z}[G],\mathbb{R})$.

To study Conjecture 4.1 it is convenient to set

$$T\Omega(L/K,0) := \psi_G^*(\hat\partial_G^1(\zeta^*_{L/K,S}(0)^\#) - \chi_G(P_S,-\mathrm{Reg}_S)) \in K_0(\mathbb{Z}[G],\mathbb{R}).$$

Since $\psi_G^*(\hat\partial_G^1(\zeta^*_{L/K,S}(0)^\#)) = -\hat\partial_G^1(\zeta^*_{L/K,S}(0))$ (see § 2.1.4), Conjecture 4.1 is equivalent to the equality $T\Omega(L/K,0)=0$. The element $T\Omega(L/K,0)$ and therefore also Conjecture 4.1 depend only on the extension L/K, as can be seen by an argument similar to the proof of Proposition 3.4. Alternatively this follows from [Bur01, Theorem 2.1.2] and Remark 4.3 below.

Remark 4.2. The invariant $T\Omega(L/K,0)$ is functorial in the field extension, i.e. if M is an intermediate field of L/K and $H = \mathrm{Gal}(L/M)$ then:

(i) $\mathrm{res}_H^G(T\Omega(L/K,0)) = T\Omega(L/M,0)$; and

(ii) $\mathrm{q}_{G/H}^G(T\Omega(L/K,0)) = T\Omega(M/K,0)$ if H is normal in G.

To show this one can apply an argument similar to the proof of Proposition 3.5, or alternatively use [Bur01, Proposition 2.1.4] and the following remark.

Remark 4.3. In [Bur01, Theorem 2.1.2] an invariant $T\Omega(L/K,0) \in K_0(\mathbb{Z}[G],\mathbb{R})$ is defined by (in our notation) $T\Omega(L/K,0) := \psi_G^*(\hat\partial_G^1(\zeta^*_{L/K,S}(0)^\#) + \chi_G^{\mathrm{old}}(\Psi_S, (-\mathrm{Reg}_S)^{-1}))$. The complex Ψ_S used here is defined in [BF98, Proposition 3.1] (see also [Bur01, Proposition 2.1.1]). Now the extension class of Ψ_S in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, \mathcal{O}_{L,S}^\times)$ with respect to an injective resolution of $\mathcal{O}_{L,S}^\times$ is the negative of Tate's canonical class used to define P_S (this follows from [Bur01, Lemma 2.3.5] since there the extension class is computed with respect to a projective resolution of X_S). Therefore

$$\chi_G^{\mathrm{old}}(\Psi_S,(-\mathrm{Reg}_S)^{-1}) = -\chi_G(\Psi_S,-\mathrm{Reg}_S) + \partial_G^1[X_S\otimes\mathbb{R},-\mathrm{id}] = -\chi_G(P_S,-\mathrm{Reg}_S),$$

which shows that our definition of the invariant $T\Omega(L/K,0)$ agrees with the definition in [Bur01]. Thus from [Bur01, § 2.2 and § 2.3] we can deduce the following results.

Proposition 4.4. *Let L/K be any Galois extension of number fields of group G.*

(i) *The invariant $T\Omega(L/K,0)$ belongs to $K_0(\mathbb{Z}[G],\mathbb{Q})$ if and only if the main conjecture of Stark at $s=0$ (as interpreted by Tate in [Tat84, ch. I, Conjecture 5.1]) is valid for L/K.*

(ii) *The invariant $T\Omega(L/K,0)$ belongs to the torsion subgroup of $K_0(\mathbb{Z}[G],\mathbb{Q})$ if and only if the strong Stark conjecture (as formulated by Chinburg in [Chi83, Conjecture 2.2]) is valid for L/K.*

(iii) *One has $\partial_G^0(\psi_G^*(T\Omega(L/K,0))) = W_{L/K} - \Omega(L/K,3)$ where $W_{L/K}$ is the 'Cassou-Noguès–Fröhlich root number class' and $\Omega(L/K,3)$ is the element defined by Chinburg in [Chi85] (and denoted by $\Omega_m(L/K)$ in [Chi83]). In particular, the vanishing of $\partial_G^0(T\Omega(L/K,0))$ is equivalent to the 'Ω_3-conjecture' that is formulated in [Chi85].*

(iv) *The invariant $T\Omega(L/K,0)$ vanishes if and only if the 'lifted root number conjecture' of Gruenberg, Ritter and Weiss [GRW99] is valid for L/K.*

Remark 4.5. In [Bur01, § 2.4] it is shown that Conjecture 4.1 is equivalent to the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}\,L),\mathbb{Z}[G])$. Although we expect this result to be true without restriction it should be noted that in its proof and in the necessary constructions in [BF98]

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some relevant sign conventions relating to the Artin–Verdier duality theorem are not specified. It is easy to check that there are no such sign ambiguities in the proof of Proposition 4.4. In this paper we avoid these sign issues by working with complexes corresponding to canonical extension classes.

5. Functional equation compatibility

In the previous two sections we formulated conjectures for the leading terms of the equivariant Dedekind zeta-function at $s = 0$ and $s = 1$. In this section we show that the compatibility of these conjectures with respect to the functional equation of the zeta-function gives rise to a natural conjecture for the epsilon constant.

5.1 Statement of the main result

Let L/K be a Galois extension of number fields, G its Galois group and S a finite set of places of K as in §§ 3 and 4. Before we can formulate the main result we must introduce an invariant encoding certain semilocal information about the extension L/K.

5.1.1 *Definition of the semilocal terms.* The definition of the following invariants is motivated by similar constructions in [BlB03]; see Remark 5.4 for a detailed comparison.

Let v be a place in S_f and denote its residue characteristic by p. We choose $w \in S_v(L)$ and let M_w be a complex in $\mathcal{D}(\mathbb{Z}[G_w])$ which corresponds to the local canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z}, L_w^\times) = H^2(G_w, L_w^\times)$, i.e. the pre-image of $1/|G_w|$ under the local invariant isomorphism

$$H^2(G_w, L_w^\times) \xrightarrow{\mathrm{inv}_{L_w/K_v}} \frac{1}{|G_w|}\mathbb{Z}/\mathbb{Z}.$$

Furthermore we choose a full projective $\mathbb{Z}_p[G_w]$-sublattice $\mathcal{L}_w$ of $\mathcal{O}_w$ which is contained in a sufficiently large power of $\mathfrak{m}_w$. The exponential map $\exp : \mathcal{L}_w \to L_w^\times = H^0(M_w)$ induces a morphism $\exp : \mathcal{L}_w[0] \to M_w$ in $\mathcal{D}(\mathbb{Z}[G_w])$ and we define a complex $M_w(\mathcal{L}_w)$ by the distinguished triangle

$$\mathcal{L}_w[0] \xrightarrow{\exp} M_w \longrightarrow M_w(\mathcal{L}_w) \longrightarrow .$$

From the corresponding cohomology sequence we see that $M_w(\mathcal{L}_w)$ is acyclic outside degrees 0 and 1, and that there are identifications $H^0(M_w(\mathcal{L}_w)) \cong L_w^\times/\exp(\mathcal{L}_w)$ and $H^1(M_w(\mathcal{L}_w)) \cong \mathbb{Z}$. Moreover one easily sees that the complex $M_w(\mathcal{L}_w)$ is perfect. The (normalized) valuation $L_w^\times \to \mathbb{Z}$ induces a trivialization $\nu_w : H^0(M_w(\mathcal{L}_w)\otimes\mathbb{R}) \xrightarrow{\cong} H^1(M_w(\mathcal{L}_w)\otimes\mathbb{R})$ and we can consider the Euler characteristic $\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) \in K_0(\mathbb{Z}[G_w], \mathbb{R})$. We also define

$$m_w := \frac{\alpha_w \cdot L^*_{L_w/K_v}(0)^\#}{L^*_{L_w/K_v}(1)} \in \mathrm{Z}(\mathbb{R}[G_w])^{\times +}, \tag{19}$$

where $\alpha_w = (\alpha_{w,\chi})_{\chi\in\mathrm{Irr}(G_w)} \in \prod_{\chi\in\mathrm{Irr}(G_w)} \mathbb{C}^\times = \mathrm{Z}(\mathbb{C}[G_w])^\times$ is the element with $\alpha_{w,\chi} = \log(Nw)$ if χ is the trivial character and $\alpha_{w,\chi} = 1$ otherwise.

From the local lattices $\mathcal{L}_w \subseteq \mathcal{O}_w$ we obtain a global lattice $\mathcal{L} \subseteq \mathcal{O}_L$ as follows. For each $v \in S_f$, with residue characteristic p, we first define a $\mathbb{Z}_p[G]$-lattice $\mathcal{L}_v$ by $\mathcal{L}_v := \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G_w]} \mathcal{L}_w \subseteq \mathcal{O}_{L,v}$. We then define the full projective $\mathbb{Z}[G]$-lattice $\mathcal{L} \subseteq \mathcal{O}_L$ by specifying its completions as in (9).

Let $\Sigma(L)$ denote the set of all embeddings $L \to \mathbb{C}$. Then $H_L := \prod_{\sigma\in\Sigma(L)} \mathbb{Z}$ is a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module and we write $\rho_L : L \otimes_{\mathbb{Q}} \mathbb{C} \to H_L \otimes_{\mathbb{Z}} \mathbb{C}$ for the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant isomorphism $l \otimes z \mapsto (\sigma(l)z)_{\sigma\in\Sigma(L)}$ (note that $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts only on the second factor of $L \otimes_{\mathbb{Q}} \mathbb{C}$ but on both factors of $H_L \otimes_{\mathbb{Z}} \mathbb{C}$). For any $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module X we write X^+ and X^- for the submodules on which complex conjugation acts by $+1$ and -1 respectively. We define π_L to be the composite

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isomorphism of $\mathbb{R}[G]$-modules

$$\begin{aligned} L \otimes_{\mathbb{Q}} \mathbb{R} &= (L \otimes_{\mathbb{Q}} \mathbb{C})^+ \xrightarrow{\rho_L} (H_L \otimes_{\mathbb{Z}} \mathbb{C})^+ \\ &= (H_L^+ \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (H_L^- \otimes_{\mathbb{Z}} (i\mathbb{R})) \xrightarrow{\mathrm{id}\oplus(\cdot i)} (H_L^+ \otimes_{\mathbb{Z}} \mathbb{R}) \oplus (H_L^- \otimes_{\mathbb{Z}} \mathbb{R}) \\ &= H_L \otimes_{\mathbb{Z}} \mathbb{R}. \end{aligned}$$

The map π_L depends on the choice of $i = \sqrt{-1} \in \mathbb{C}$, but one can check that the element $[\mathcal{L}, \pi_L, H_L] \in K_0(\mathbb{Z}[G], \mathbb{R})$ is independent of this choice.

We now set

$$R\Omega^{\mathrm{loc}}(L/K, 1) := [\mathcal{L}, \pi_L, H_L] + \sum_{v \in S_f} \mathrm{ind}_{G_w}^G(\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) + \partial^1_{G_w}(m_w))$$

and

$$T\Omega^{\mathrm{loc}}(L/K, 1) := \hat{\partial}^1_G(\varepsilon_{L/K}(0)) - R\Omega^{\mathrm{loc}}(L/K, 1)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$. One can show that $R\Omega^{\mathrm{loc}}(L/K, 1)$ and $T\Omega^{\mathrm{loc}}(L/K, 1)$ depend only on the extension L/K.

Remark 5.1. The invariant $T\Omega^{\mathrm{loc}}(L/K, 1)$ is functorial in the field extension, i.e. if M is an intermediate field of L/K and $H = \mathrm{Gal}(L/M)$ then:

(i) $\mathrm{res}^G_H(T\Omega^{\mathrm{loc}}(L/K, 1)) = T\Omega^{\mathrm{loc}}(L/M, 1)$; and

(ii) $\mathrm{q}^G_{G/H}(T\Omega^{\mathrm{loc}}(L/K, 1)) = T\Omega^{\mathrm{loc}}(M/K, 1)$ if H is normal in G.

5.1.2 *The comparison result.* We can now state the main result which describes the relation of the invariants $T\Omega(L/K, 0)$ and $T\Omega(L/K, 1)$, and therefore of the conjectures for the leading terms at $s = 0$ and $s = 1$.

THEOREM 5.2. *One has*

$$\psi^*_G(T\Omega(L/K, 0)) - T\Omega(L/K, 1) = T\Omega^{\mathrm{loc}}(L/K, 1)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

We shall use the functional equation of the equivariant zeta-function and computations of various Euler characteristics to prove Theorem 5.2 in § 5.2. By this theorem the leading term Conjectures 3.3 and 4.1 force the following conjecture for the epsilon constant.

CONJECTURE 5.3. One has $T\Omega^{\mathrm{loc}}(L/K, 1) = 0$ in $K_0(\mathbb{Z}[G], \mathbb{R})$. Equivalently, one has $\hat{\partial}^1_G(\varepsilon_{L/K}(0)) = R\Omega^{\mathrm{loc}}(L/K, 1)$ in $K_0(\mathbb{Z}[G], \mathbb{R})$.

Remark 5.4. Conjecture 5.3 is equivalent to the conjecture formulated by Bley and the second named author in [BlB03, Conjecture 4.1]. To see this we first recall that, in the notation of that paper, their Conjecture 4.1 is the conjectural equality $\mathcal{E}_{L/K} = \delta_{L/K}(\mathcal{L}) + \sum_{v \in S_f} I_G(v, \mathcal{L})$ in $K_0(\mathbb{Z}[G], \mathbb{R})$ for S and $\mathcal{L}$ as in § 5.1.1. We claim that $T\Omega^{\mathrm{loc}}(L/K, 1) = \mathcal{E}_{L/K} - \delta_{L/K}(\mathcal{L}) - \sum_{v \in S_f} I_G(v, \mathcal{L})$, which then immediately implies the equivalence of the conjectures. It is straightforward to verify that $\hat{\partial}^1_G(\varepsilon_{L/K}(0)) = \mathcal{E}_{L/K}$ and $[\mathcal{L}, \pi_L, H_L] = \delta_{L/K}(\mathcal{L})$.

Next we show that m_w as defined in (19) agrees with m_w defined in [BlB03, p. 561]. Recall that in [BlB03, p. 561] m_w is defined to be

$$m_w := \frac{{}^*(|G_w/I_w| e_{G_w})^*((1 - \sigma_w(Nv)^{-1}) e_{I_w})}{{}^*((1 - \sigma_w^{-1}) e_{I_w})}, \tag{20}$$

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where the notation ${}^*x$ for $x \in \mathrm{Z}(\mathbb{Q}[G_w])$ is introduced in [BlB03, p. 560]. Now for $\chi \in \mathrm{Irr}(G_w)$ a direct computation of the leading term of $L_{L_w/K_v}(\chi, s)$ at $s = 0$ gives

$$L^*_{L_w/K_v}(\chi, 0) = \begin{cases} (\log(Nv))^{-1} & \text{if } \chi \text{ is trivial,} \\ (1 - \chi(\sigma_w))^{-1} & \text{if } \chi \text{ is non-trivial but } \chi|_{I_w} \text{ is trivial,} \\ 1 & \text{if } \chi|_{I_w} \text{ is non-trivial.} \end{cases}$$

Therefore

$$\begin{aligned} (\alpha_w \cdot L^*_{L_w/K_v}(0)^{\#})_\chi &= \alpha_{w,\chi} \cdot L^*_{L_w/K_v}(\overline{\chi}, 0) \\ &= \begin{cases} |G_w/I_w| & \text{if } \chi \text{ is trivial,} \\ (1 - \chi(\sigma_w^{-1}))^{-1} & \text{if } \chi \text{ is non-trivial but } \chi|_{I_w} \text{ is trivial,} \\ 1 & \text{if } \chi|_{I_w} \text{ is non-trivial,} \end{cases} \\ &= \left(\frac{{}^*(|G_w/I_w| e_{G_w})}{{}^*((1 - \sigma_w^{-1}) e_{I_w})} \right)_\chi \end{aligned}$$

for all $\chi \in \mathrm{Irr}(G_w)$ which implies that

$$\alpha_w \cdot L^*_{L_w/K_v}(0)^{\#} = \frac{{}^*(|G_w/I_w| e_{G_w})}{{}^*((1 - \sigma_w^{-1}) e_{I_w})}.$$

A similar argument shows that $(L^*_{L_w/K_v}(1))^{-1} = {}^*((1 - \sigma_w(Nv)^{-1}) e_{I_w})$. Hence the definitions of m_w in (19) and (20) coincide.

Since by definition $I_G(v, \mathcal{L}) = \mathrm{ind}_{G_w}^G(\partial^1_{G_w}(m_w) - \chi^{\mathrm{old}}_{G_w}(K^\bullet_w(\exp(\mathcal{L}_w)), \nu_w^{-1}))$ with $K^\bullet_w(\exp(\mathcal{L}_w))$ as in [BlB03, (18)], it remains to show that $\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) = -\chi^{\mathrm{old}}_{G_w}(K^\bullet_w(\exp(\mathcal{L}_w)), \nu_w^{-1})$. Since the complex $\mathrm{Cone}(\mu)$ in [BlB03, (18)] corresponds to the negative of the local canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z}, L_w^\times)$ (in [BlB03, p. 558] it is stated that $\mathrm{Cone}(\mu)$ corresponds to the local canonical class, however, this difference is explained by the fact that in [BlB03] extension classes are computed with respect to a projective resolution of the first variable), it follows that the extension class of $K^\bullet_w(\exp(\mathcal{L}_w))$ in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z}, L_w^\times/\exp(\mathcal{L}_w))$ is the negative of the class of $M_w(\mathcal{L}_w)$. Hence $\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) = -\chi^{\mathrm{old}}_{G_w}(K^\bullet_w(\exp(\mathcal{L}_w)), \nu_w^{-1})$ as required.

Remark 5.5. By combining Remark 5.4 with [BlB03, Remark 4.2(iv)] we may deduce the equality $\partial^0_G(T\Omega^{\mathrm{loc}}(L/K, 1)) = W_{L/K} - \Omega(L/K, 2)$ in $\mathrm{Cl}(\mathbb{Z}[G])$ where $\Omega(L/K, 2)$ is the element defined by Chinburg in [Chi85] and so Conjecture 5.3 is a refinement of the 'Ω_2-conjecture' that is formulated in [Chi85, Question 3.1].

Now if L/K is tamely ramified, then $\mathcal{O}_L$ is a projective $\mathbb{Z}[G]$-module and one has $\Omega(L/K, 2) = [\mathcal{O}_L] - [K : \mathbb{Q}] \cdot [\mathbb{Z}[G]] \in K_0(\mathbb{Z}[G])$ (see [Chi85, Theorem 3.2]). The study of $\Omega(L/K, 2)$ can therefore be regarded as a natural generalization of the Galois structure theory of rings of algebraic integers that is described by Fröhlich in [Frö83] (indeed, a similarly explicit interpretation of $\Omega(L/K, 2)$ is also valid for wildly ramified extensions [HW94, Theorem 4.1]). On the other hand, the study of $\Omega(L/K, 3)$ is a natural generalization of the explicit study of the Galois structures of unit groups and ideal class groups that was undertaken by Fröhlich in [Frö89] and [Frö92] (in this regard see, for example, the formulas for $\Omega(L/K, 3)$ that are obtained in [Bur95, Theorems 1.2 and 1.7]).

From this viewpoint, Proposition 4.4(iii) shows that the 'multiplicative' Galois structure results obtained by Fröhlich in [Frö89] and [Frö92] are explicit consequences (for special families of extensions) of the natural leading term conjecture for equivariant Dedekind zeta-functions at $s = 0$, whilst Theorem 5.2 shows that the 'additive' Galois structure results that Fröhlich discusses in [Frö83] and [Frö89] reflect the compatibility of the leading term conjectures at $s = 0$ and $s = 1$ with respect to the functional equation of the equivariant zeta-function. We thereby resolve the problem posed

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by Fröhlich in [Frö89, Introduction] of using the functional equation to give a natural explanation of the 'amazing analogy' between the Galois structure theories of unit groups and ideal class groups and of rings of algebraic integers that he stresses in both [Frö89] and [CCFT91, § 3].

Remark 5.6. Conjecture 5.3 is essentially of a local nature. More precisely, in [Bre04b] a conjecture for the equivariant local epsilon constant of a Galois extension of p-adic fields is formulated. It is then shown that the validity of this local conjecture for all non-archimedean completions L_w/K_v of L/K implies the validity of Conjecture 5.3. Such a local approach lies (implicitly or explicitly) behind the proof of the known cases mentioned in the following proposition.

Proposition 5.7. *For every Galois extension L/K, the invariant $T\Omega^{\mathrm{loc}}(L/K,1)$ lies in the finite group $K_0(\mathbb{Z}[G],\mathbb{Q})_{\mathrm{tors}}$, the torsion subgroup of $K_0(\mathbb{Z}[G],\mathbb{Q}) \subset K_0(\mathbb{Z}[G],\mathbb{R})$. Moreover $T\Omega^{\mathrm{loc}}(L/K,1)$ is known to vanish in each of the following cases:*

(i) *L/K is a tamely ramified extension;*

(ii) *L is an abelian extension of $\mathbb{Q}$ with odd conductor;*

(iii) *L is a non-abelian extension of $\mathbb{Q}$ of degree 6.*

Proof. The first statement is [BlB03, Corollary 6.3(i)]. Case (i) is [BlB03, Corollary 7.7]. Cases (ii) and (iii) follow by combining Remark 5.1 with [BlB03, Corollary 5.4(ii)] and [Bre04a, Theorem 1.1] respectively. □

Recall that if χ is a symplectic character of G then the Artin root number $W(\chi)$ is either 1 or -1. In the case where L/K is tamely ramified, this sign has been determined by Cassou-Noguès and Taylor in terms of a natural algebraic invariant (see [CT83a, CT83b]). Assuming the validity of Conjecture 5.3, one has the following generalization of this result to wildly ramified extensions.

Theorem 5.8. *If Conjecture 5.3 is valid, i.e. if the leading term conjectures at $s=0$ and $s=1$ are compatible, then for every symplectic character χ of G the Artin root number $W(\chi)$ is determined by the algebraic invariant $R\Omega^{\mathrm{loc}}(L/K,1)$ in $K_0(\mathbb{Z}[G],\mathbb{R})$.*

Theorem 5.8 was first shown in [Bre04c, § 7] but for easier reference we have included the proof in § 5.3.

5.2 Proof of the main result

By the functorial properties of the invariants (see Proposition 3.5 and Remarks 4.2 and 5.1) it suffices to show Theorem 5.2 for K totally real and L totally complex. We will assume this for the rest of this section. We fix a finite set S of places of K containing $S_\infty(K)$, all places ramified in L/K and for which $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. Furthermore for each $v \in S_f$ we fix a place $w \in S_v(L)$, a complex M_w and a lattice $\mathcal{L}_w \subseteq \mathcal{O}_w$ as in § 5.1.1. These lattices give rise to $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ and $\mathcal{L} \subseteq \mathcal{O}_L$ as above. In § 5.2.1 we will use the functional equation of the equivariant zeta-function to compute the quotient of the leading terms at $s=0$ and $s=1$. Then in §§ 5.2.2 to 5.2.5 we will apply the additivity of Euler characteristics in distinguished triangles and some explicit computations to express the sum $\chi_G(P_S, -\mathrm{Reg}_S) + \chi_G(E_S(\mathcal{L}), \mu_L)$ in terms of certain semilocal invariants. After these preliminary steps the proof of Theorem 5.2 will be given in § 5.2.6.

5.2.1 *Functional equation of the equivariant zeta-function.* We now consider the behaviour of the leading terms of the S-truncated zeta-function with respect to the functional equation. To simplify the notation we use the following convention. If W is a finitely generated $\mathbb{Z}[G]$-module (respectively $\mathbb{R}[G]$-module) and $\alpha \in \mathbb{R}^\times$ then $[W,\alpha]$ denotes the element in $K_1(\mathbb{R}[G])$ which is represented by the $\mathbb{R}[G]$-module $W \otimes_{\mathbb{Z}} \mathbb{R}$ (respectively W) with automorphism given by multiplication with α.

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Lemma 5.9. *One has*

$$\hat\partial^1_G\left(\frac{\zeta^*_{L/K,S}(0)^\#}{\zeta^*_{L/K,S}(1)}\right) = \hat\partial^1_G(\varepsilon_{L/K}(0)) + \sum_{v\in S_f}\mathrm{ind}^G_{G_w}\,\hat\partial^1_{G_w}\left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#}\right) + \partial^1_G(-[H^-_L,\pi] - [H^+_L,2] + [\mathbb{R},-1])$$

in $K_0(\mathbb{Z}[G],\mathbb{R})$.

Proof. Taking the leading term at $s_0 = 0$ of both sides of the functional equation (8) we obtain the equality

$$Z^*_{L/K}(0)^\# = \varepsilon_{L/K}(0)\cdot\alpha\cdot Z^*_{L/K}(1)$$

in $\mathrm{Z}(\mathbb{R}[G])^\times$ where $\alpha = (\alpha_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C}^\times = \mathrm{Z}(\mathbb{C}[G])^\times$ is the element with $\alpha_\chi = -1$ if χ is the trivial character and $\alpha_\chi = 1$ otherwise.

The relation to the leading term of the S-truncated zeta-function is given by

$$Z^*_{L/K}(s_0) = \zeta^*_{L/K,S}(s_0)\cdot\prod_{v\in S}\mathrm{ind}^G_{G_w}(L^*_{L_w/K_v}(s_0)).$$

Since the induction $\mathrm{ind}^G_{G_w}$ and the involution $x\mapsto x^\#$ commute we find

$$\frac{\zeta^*_{L/K,S}(0)^\#}{\zeta^*_{L/K,S}(1)} = \varepsilon_{L/K}(0)\cdot\alpha\cdot\prod_{v\in S}\mathrm{ind}^G_{G_w}\left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#}\right). \tag{21}$$

The product of the leading terms of the archimedean Euler factors can be written in the following more explicit form. Let $v\in S_\infty(K)$, $\psi\in\mathrm{Irr}(G_w)$ and let n^+_ψ, n^-_ψ be as in § 2.3.2. The well-known properties of the Γ-function imply that $L^*_{L_w/K_v}(\psi,0) = 2^{n^+_\psi}$ and $L^*_{L_w/K_v}(\psi,1) = \pi^{-n^-_\psi}$. From this one easily deduces that

$$\prod_{v\in S_\infty(K)}\mathrm{ind}^G_{G_w}\left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#}\right) = -[H^-_L,\pi] - [H^+_L,2]$$

in $\mathrm{Z}(\mathbb{R}[G])^{\times+} \cong K_1(\mathbb{R}[G])$. Therefore the lemma follows by applying $\hat\partial^1_G$ to (21). □

5.2.2 *The distinguished triangle.* As in the case of a non-archimedean place in § 5.1.1, for each $v\in S_\infty(K)$ we fix $w\in S_v(L)$ and let M_w be a complex in $\mathcal{D}(\mathbb{Z}[G_w])$ which represents the canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z},L^\times_w) = H^2(G_w,L^\times_w) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Then for every $v\in S$ the complex $\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w$ in $\mathcal{D}(\mathbb{Z}[G])$ is acyclic outside degrees 0 and 1, and there are isomorphisms $H^0(\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w) \cong \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}L^\times_w \cong \prod_{w\in S_v(L)}L^\times_w$ and $H^1(\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w) \cong \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}\mathbb{Z} \cong \prod_{w\in S_v(L)}\mathbb{Z}$. Here and in the following the letter w stands either for the fixed place in $S_v(L)$ (in expressions like $\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}L^\times_w$) or for all places in $S_v(L)$ (in $\prod_{w\in S_v(L)}L^\times_w$); this should not cause any confusion. We set $M_{S_f} := \bigoplus_{v\in S_f}\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w$ and $M_{S_\infty} := \bigoplus_{v\in S_\infty(K)}\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w$.

The local and global invariant maps are compatible, that is, for every $w\in S(L)$ the following diagram commutes.

$$\begin{array}{ccc} H^2(G_w,L^\times_w) & \xrightarrow{\mathrm{inv}_{L_w/K_v}} & \frac{1}{|G_w|}\mathbb{Z}/\mathbb{Z} \\ \downarrow & & \| \\ H^2(G_w,C_S(L)) & \xrightarrow{\mathrm{inv}_{L/L^{G_w},S}} & \frac{1}{|G_w|}\mathbb{Z}/\mathbb{Z} \end{array}$$

Here the left vertical arrow is induced by the map $L^\times_w\to C_S(L)$ which is the composite of the inclusion $L^\times_w\to I_L$ and the canonical map $I_L\to C_S(L)$. Therefore by Lemma 2.4 there exists a

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morphism $M_w \to E_S$ in $\mathcal{D}(\mathbb{Z}[G_w])$ which induces the map $L_w^\times \to C_S(L)$ on H^0 and the identity $\mathbb{Z} \xrightarrow{=} \mathbb{Z}$ on H^1. From this we obtain a morphism $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ and adding over all $v \in S$ gives a map $M_{S_f} \oplus M_{S_\infty} \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$. One can show that the complex P_S lies in the distinguished triangle

$$P_S \longrightarrow M_{S_f} \oplus M_{S_\infty} \longrightarrow E_S \longrightarrow \tag{22}$$

whose cohomology sequence identifies with the canonical sequence

$$0 \to \mathcal{O}_{L,S}^\times \to L_S^\times \to C_S(L) \xrightarrow{0} X_S \to Y_S \to \mathbb{Z} \to 0$$

(compare [BF98, § 3.1]).

5.2.3 *Replacing the trace map by the zero map.* From now on we simply write $\exp : \mathcal{L}_S[0] \to E_S$ for the map in $\mathcal{D}(\mathbb{Z}[G])$ which induces $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L) \cong H^0(E_S)$ on H^0. Similarly we write $\mathrm{tr} : \mathcal{L}[-1] \to E_S$ for the map which induces $\mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z} \cong H^1(E_S)$ on H^1. Recall that the complex $E_S(\mathcal{L})$ is defined by the distinguished triangle

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\exp \oplus \mathrm{tr}} E_S \longrightarrow E_S(\mathcal{L}) \longrightarrow \tag{23}$$

whose cohomology sequence induces the identifications

$$H^i(E_S(\mathcal{L}) \otimes \mathbb{R}) = \begin{cases} L_\infty^0 & \text{if } i = -1, \\ L^0 \otimes \mathbb{R} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that we consider the canonical isomorphism $\mu_L : L^0 \otimes \mathbb{R} \to L_\infty^0$ as trivialization of the complex $E_S(\mathcal{L})$.

Instead of the trace map $\mathrm{tr} : \mathcal{L} \to \mathbb{Z} = H^1(E_S)$ we now consider the zero map $0 : \mathcal{L} \to H^1(E_S)$. We define a complex $F_S(\mathcal{L})$ by the distinguished triangle

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\exp \oplus 0} E_S \longrightarrow F_S(\mathcal{L}) \longrightarrow \tag{24}$$

whose cohomology sequence induces identifications

$$H^i(F_S(\mathcal{L}) \otimes \mathbb{R}) = \begin{cases} L_\infty^0 & \text{if } i = -1, \\ L \otimes \mathbb{R} & \text{if } i = 0, \\ \mathbb{R} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define a trivialization $t_F : L \otimes \mathbb{R} \to L_\infty^0 \oplus \mathbb{R}$ of $F_S(\mathcal{L})$ by $L \otimes \mathbb{R} \xrightarrow{\mu_L'} L_\infty \cong L_\infty^0 \oplus \mathbb{R}$, where the last isomorphism is induced by the canonical splitting of the surjection $L_\infty \xrightarrow{\mathrm{tr}_\infty} \mathbb{R}$, i.e. by $\mathbb{R} \to L_\infty$, $x \mapsto (x/[L : \mathbb{Q}])_{w \in S_\infty(L)}$.

LEMMA 5.10. *One has*

$$\chi_G(E_S(\mathcal{L}), \mu_L) = \chi_G(F_S(\mathcal{L}), t_F)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

Proof. We will show below that there exists a distinguished triangle

$$\mathcal{L}[0] \oplus \mathcal{L}[-1] \xrightarrow{\alpha} F_S(\mathcal{L}) \xrightarrow{\beta} E_S(\mathcal{L}) \xrightarrow{\gamma} \tag{25}$$

in $\mathcal{D}(\mathbb{Z}[G])$ whose cohomology sequence after tensoring with $\mathbb{R}$ identifies with the exact sequence (starting with $H^{-1}(F_S(\mathcal{L}) \otimes \mathbb{R})$)

$$L_\infty^0 \xrightarrow{\mathrm{id}} L_\infty^0 \xrightarrow{0} L \otimes \mathbb{R} \xrightarrow{\mathrm{id}} L \otimes \mathbb{R} \xrightarrow{0} L^0 \otimes \mathbb{R} \xrightarrow{\subset} L \otimes \mathbb{R} \xrightarrow{\mathrm{tr}} \mathbb{R}.$$

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On the complex $\mathcal{L}[0] \oplus \mathcal{L}[-1]$ we take the trivialization $\mathrm{id} : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$. We want to show that the distinguished triangle (25) with the trivializations id, t_F and μ_L satisfies the additivity criterion in Lemma 2.5. Note that $\ker(H^{\mathrm{ev}}(\alpha) \otimes \mathbb{R}) = 0$ and that we can therefore omit this term when applying Lemma 2.5. We consider the $\mathbb{R}[G]$-modules and isomorphisms in the following (in general non-commutative) diagram

$$\begin{array}{ccc} (L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R}) & \xrightarrow{s^{\mathrm{ev}}} & (L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R}) \\ \downarrow{\scriptstyle t_F \oplus (-\mathrm{id})} & & \downarrow{\scriptstyle \mathrm{id} \oplus \mu_L} \\ (L^0_\infty \oplus \mathbb{R}) \oplus (L^0 \otimes \mathbb{R}) & \xrightarrow{s^{\mathrm{od}}} & (L \otimes \mathbb{R}) \oplus (L^0_\infty) \end{array}$$

and we must show that the automorphism $(t_F \oplus (-\mathrm{id}))^{-1} \circ (s^{\mathrm{od}})^{-1} \circ (\mathrm{id} \oplus \mu_L) \circ s^{\mathrm{ev}}$ of the module $(L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R})$ has reduced norm equal to 1.

The map $s^{\mathrm{ev}} : (L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R}) \to (L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R})$ is induced by splittings of the exact sequence

$$0 \to 0 \to L \otimes \mathbb{R} \xrightarrow{\mathrm{id}} L \otimes \mathbb{R} \xrightarrow{0} L^0 \otimes \mathbb{R} \xrightarrow{\mathrm{id}} L^0 \otimes \mathbb{R} \to 0.$$

Hence we have $s^{\mathrm{ev}}(a, b) = (a, b)$. The map $s^{\mathrm{od}} : (L^0_\infty \oplus \mathbb{R}) \oplus (L^0 \otimes \mathbb{R}) \to (L \otimes \mathbb{R}) \oplus (L^0_\infty)$ is induced by splittings of the exact sequence

$$0 \to L^0 \otimes \mathbb{R} \xrightarrow{\subset} L \otimes \mathbb{R} \xrightarrow{(0,\mathrm{tr})} L^0_\infty \oplus \mathbb{R} \xrightarrow{(\mathrm{id},0)} L^0_\infty \to 0 \to 0.$$

As splitting of the surjection $\mathrm{tr} : L \otimes \mathbb{R} \to \mathbb{R}$ we take the map $\mathbb{R} \to L \otimes \mathbb{R}$, $b \mapsto 1 \otimes b/[L : \mathbb{Q}]$. With this choice of splitting we have $s^{\mathrm{od}}(a, b, c) = (c + 1 \otimes b/[L : \mathbb{Q}], a)$.

To simplify the description of the automorphism $(t_F \oplus (-\mathrm{id}))^{-1} \circ (s^{\mathrm{od}})^{-1} \circ (\mathrm{id} \oplus \mu_L) \circ s^{\mathrm{ev}}$ of $(L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R})$, we replace $(L \otimes \mathbb{R}) \oplus (L^0 \otimes \mathbb{R})$ by the module $(L^0 \otimes \mathbb{R}) \oplus \mathbb{R} \oplus (L^0 \otimes \mathbb{R})$ via the isomorphism $(a, b) \mapsto (a_1, a_2, b)$ with $a_2 = \mathrm{tr}(a)$ and $a_1 = a - 1 \otimes a_2/[L : \mathbb{Q}]$. Then $(t_F \oplus (-\mathrm{id}))^{-1} \circ (s^{\mathrm{od}})^{-1} \circ (\mathrm{id} \oplus \mu_L) \circ s^{\mathrm{ev}}$ becomes the automorphism $(a_1, a_2, b) \mapsto (b, a_2, -a_1)$ of $(L^0 \otimes \mathbb{R}) \oplus \mathbb{R} \oplus (L^0 \otimes \mathbb{R})$, and obviously this has reduced norm equal to 1. So we can apply Lemma 2.5 and find that

$$\chi_G(F_S(\mathcal{L}), t_F) = \chi_G(\mathcal{L}[0] \oplus \mathcal{L}[-1], \mathrm{id}) + \chi_G(E_S(\mathcal{L}), \mu_L).$$

Since $\chi_G(\mathcal{L}[0] \oplus \mathcal{L}[-1], \mathrm{id}) = 0$ in $K_0(\mathbb{Z}[G], \mathbb{R})$, the required equality $\chi_G(F_S(\mathcal{L}), t_F) = \chi_G(E_S(\mathcal{L}), \mu_L)$ follows.

It remains to show the existence of the distinguished triangle (25) with the claimed cohomology sequence. Consider the continuous arrows in the following diagram.

$$\begin{array}{ccccccc} \mathcal{L}[0] \oplus \mathcal{L}[-1] & \overset{\alpha}{\dashrightarrow} & F_S(\mathcal{L}) & \overset{\beta}{\dashrightarrow} & E_S(\mathcal{L}) & \xrightarrow{\gamma} & \mathcal{L}[1] \oplus \mathcal{L}[0] \\ \| & & \downarrow & & \downarrow & & \| \\ \mathcal{L}[0] \oplus \mathcal{L}[-1] & \xrightarrow{a} & \mathcal{L}_S[1] \oplus \mathcal{L}[0] & \xrightarrow{\mathrm{id} \oplus 0} & \mathcal{L}_S[1] \oplus \mathcal{L}[0] & \xrightarrow{0 \oplus \mathrm{id}} & \mathcal{L}[1] \oplus \mathcal{L}[0] \\ & & \downarrow{\scriptstyle \exp \oplus 0} & & \downarrow{\scriptstyle \exp \oplus \mathrm{tr}} & & \\ & & E_S[1] & = & E_S[1] & & \\ & & \downarrow & & \downarrow & & \\ & & F_S(\mathcal{L})[1] & \overset{\beta[1]}{\dashrightarrow} & E_S(\mathcal{L})[1] & & \end{array} \tag{26}$$

Here $a : \mathcal{L}[0] \oplus \mathcal{L}[-1] \to \mathcal{L}_S[1] \oplus \mathcal{L}[0]$ is the map $(k, l) \mapsto (0, k)$, so clearly the second row is a distinguished triangle. The two central columns are the distinguished triangles obtained from (24) and (23) by rotation (without changing the signs of the maps; these are still distinguished triangles

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because we rotated twice), and γ is the map making the top right-hand square commutative. It follows from the octahedral axiom (more precisely, the version TR $4'$ of the octahedral axiom in [BBD82, bottom of p. 21]) that diagram (26) can be completed by the dashed arrows in such a way that one obtains an octahedral diagram, i.e. such that the first row is also a distinguished triangle, the diagram is commutative, and furthermore the square

$$\begin{array}{ccc} \mathcal{L}_S[1] \oplus \mathcal{L}[0] & \xrightarrow{0\oplus \mathrm{id}} & \mathcal{L}[1] \oplus \mathcal{L}[0] \\ \downarrow{\scriptstyle \exp \oplus \mathrm{tr}} & & \downarrow{\scriptstyle \alpha[1]} \\ E_S[1] & \longrightarrow & F_S(\mathcal{L})[1] \end{array} \tag{27}$$

commutes.

The first row in diagram (26) is the required triangle (25). After tensoring with $\mathbb{R}$, its cohomology sequence has the form (starting with $H^{-1}(F_S(\mathcal{L}) \otimes \mathbb{R})$)

$$L^0_\infty \xrightarrow{u_1} L^0_\infty \xrightarrow{v_1} L \otimes \mathbb{R} \xrightarrow{w_1} L \otimes \mathbb{R} \xrightarrow{u_2} L^0 \otimes \mathbb{R} \xrightarrow{v_2} L \otimes \mathbb{R} \xrightarrow{w_2} \mathbb{R}.$$

We still have to compute the maps in this sequence. In diagram (26) we have a morphism from the first row to the second row. After tensoring with $\mathbb{R}$, the associated morphism of cohomology sequences (starting with $H^{-1}(F_S(\mathcal{L}) \otimes \mathbb{R})$ in the first row and $H^{-1}((\mathcal{L}_S[1] \oplus \mathcal{L}[0]) \otimes \mathbb{R})$ in the second row) is as shown below.

$$\begin{array}{ccccccccccccc} L^0_\infty & \xrightarrow{u_1} & L^0_\infty & \xrightarrow{v_1} & L \otimes \mathbb{R} & \xrightarrow{w_1} & L \otimes \mathbb{R} & \xrightarrow{u_2} & L^0 \otimes \mathbb{R} & \xrightarrow{v_2} & L \otimes \mathbb{R} & \xrightarrow{w_2} & \mathbb{R} \\ \downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle \subset} & & \| & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \subset} & & \| & & \downarrow \\ \mathcal{L}_S \otimes \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathcal{L}_S \otimes \mathbb{R} & \xrightarrow{0} & L \otimes \mathbb{R} & \xrightarrow{\mathrm{id}} & L \otimes \mathbb{R} & \xrightarrow{0} & L \otimes \mathbb{R} & \xrightarrow{\mathrm{id}} & L \otimes \mathbb{R} & \longrightarrow & 0 \end{array}$$

We deduce that $u_1 = \mathrm{id}$, $v_1 = 0$, $w_1 = \mathrm{id}$, $u_2 = 0$ and that v_2 is the inclusion $L^0 \otimes \mathbb{R} \subset L \otimes \mathbb{R}$. It remains to compute the map $w_2 : L \otimes \mathbb{R} \to \mathbb{R}$. Taking H^0 of the commutative square (27) and tensoring with $\mathbb{R}$ gives the following commutative square.

$$\begin{array}{ccc} L \otimes \mathbb{R} & \xrightarrow{\mathrm{id}} & L \otimes \mathbb{R} \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle w_2} \\ \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} \end{array}$$

This implies that $w_2 = \mathrm{tr}$ and completes the proof. □

5.2.4 *The semilocal complexes.* We now construct two complexes M^{finite} and M^{arch} in $\mathcal{D}(\mathbb{Z}[G])$ with trivializations $t_{M^{\mathrm{finite}}}$ and $t_{M^{\mathrm{arch}}}$ respectively, and show that their Euler characteristics are closely related to the terms defined in § 5.1.1.

Recall that M_{S_f} is the complex $\bigoplus_{v \in S_f} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w$ and that $H^0(M_{S_f}) = \prod_{w \in S_f} L_w^\times$ and $H^1(M_{S_f}) = Y_{S_f}$. Notice that for every $v \in S_f$ the lattice $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ decomposes as $\prod_{w \in S_v(L)} \mathcal{L}_w \subseteq \prod_{w \in S_v(L)} \mathcal{O}_w$ and that we can therefore talk about lattices $\mathcal{L}_w \subseteq \mathcal{O}_w$ for all $w \in S_f$ (not just for the fixed places w). We set $\mathcal{L}_{S_f} := \prod_{w \in S_f} \mathcal{L}_w$ and define M^{finite} by the distinguished triangle

$$\mathcal{L}_{S_f}[0] \xrightarrow{\exp} M_{S_f} \longrightarrow M^{\mathrm{finite}} \longrightarrow . \tag{28}$$

From its cohomology sequence we see that for the non-zero cohomology groups there are identifications

$$H^0(M^{\mathrm{finite}}) \cong \prod_{w \in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)}$$

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and $H^1(M^{\mathrm{finite}}) \cong Y_{S_f}$. On the complex M^{finite} we consider the trivialization

$$t_{M^{\mathrm{finite}}} : \prod_{w \in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \otimes \mathbb{R} \to Y_{S_f} \otimes \mathbb{R}, \quad (x_w)_{w \in S_f} \mapsto (v_w(x_w) \cdot \log Nw)_{w \in S_f},$$

where v_w is the normalized valuation of $L_w^\times$ and Nw is the cardinality of the residue field of w.

Lemma 5.11. *One has*

$$\chi_G(M^{\mathrm{finite}}, t_{M^{\mathrm{finite}}}) = \sum_{v \in S_f} \mathrm{ind}_{G_w}^G(\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) + \partial^1_{G_w}[\mathbb{R}, \log Nw])$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

Proof. This follows easily from [BrB05, Proposition 5.6.2]. □

Before defining M^{arch} we must introduce some notation. We shall write S_∞ for both $S_\infty(K)$ and $S_\infty(L)$; the meaning will always be clear by our convention to write places as v, respectively w. For every $w \in S_\infty$ we denote by $\mathbf{R}(L_w)$ and $\mathbf{I}(L_w)$ the real and imaginary axis in L_w respectively, i.e. the $\mathbb{R}$-line generated by 1 in L_w and the $\mathbb{R}$-line generated by a square root of -1 in L_w. Furthermore we set $\mathbf{R}(L_\infty) := \prod_{w \in S_\infty} \mathbf{R}(L_w)$ and $\mathbf{I}(L_\infty) := \prod_{w \in S_\infty} \mathbf{I}(L_w)$. Then $\mathbf{R}(L_\infty)$ and $\mathbf{I}(L_\infty)$ are $\mathbb{R}[G]$-submodules of L_∞ and one has $L_\infty = \mathbf{I}(L_\infty) \oplus \mathbf{R}(L_\infty)$. Note that $\mathbf{I}(L_\infty)$ lies in the kernel of the trace map $\mathrm{tr}_\infty : L_\infty \to \mathbb{R}$. We let $\mathbf{R}^0(L_\infty)$ be the kernel of the restriction of tr_∞ to $\mathbf{R}(L_\infty)$; thus there is a short exact sequence $0 \to \mathbf{R}^0(L_\infty) \to \mathbf{R}(L_\infty) \xrightarrow{\mathrm{tr}_\infty} \mathbb{R} \to 0$. Using the canonical splitting $\mathbb{R} \to \mathbf{R}(L_\infty)$, $x \mapsto (x/[L : \mathbb{Q}])_{w \in S_\infty}$ we obtain the direct sum decomposition $\mathbf{R}(L_\infty) = \mathbf{R}^0(L_\infty) \oplus \mathbb{R}$.

We identify $\mathbf{R}(L_\infty)$ with $Y_{S_\infty} \otimes_{\mathbb{Z}} \mathbb{R}$ by the isomorphism of $\mathbb{R}[G]$-modules which sends $(x_w)_{w \in S_\infty} \in \mathbf{R}(L_\infty)$ to $(2x_w)_{w \in S_\infty} \in Y_{S_\infty} \otimes \mathbb{R}$. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{R}^0(L_\infty) & \longrightarrow & \mathbf{R}(L_\infty) & \xrightarrow{\mathrm{tr}_\infty} & \mathbb{R} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \| & & \\
0 & \longrightarrow & X_{S_\infty} \otimes \mathbb{R} & \longrightarrow & Y_{S_\infty} \otimes \mathbb{R} & \xrightarrow{\mathrm{aug}} & \mathbb{R} & \longrightarrow & 0
\end{array}$$

and the above splitting of $\mathbf{R}(L_\infty) \xrightarrow{\mathrm{tr}_\infty} \mathbb{R}$ corresponds to the canonical splitting of $Y_{S_\infty} \otimes \mathbb{R} \xrightarrow{\mathrm{aug}} \mathbb{R}$.

Recall that M_{S_∞} is the complex $\bigoplus_{v \in S_\infty} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w$. One has $H^0(M_{S_\infty}) = L_\infty^\times$ and we define M^{arch} by the distinguished triangle

$$L_\infty[0] \oplus \mathcal{L}[-1] \xrightarrow{\exp \oplus 0} M_{S_\infty} \longrightarrow M^{\mathrm{arch}} \longrightarrow . \tag{29}$$

From its cohomology sequence we see that for the non-zero cohomology groups there are identifications $H^{-1}(M^{\mathrm{arch}}) \cong \prod_{w \in S_\infty} 2\pi\sqrt{-1}\,\mathbb{Z} \subset L_\infty$, $H^0(M^{\mathrm{arch}}) \cong \mathcal{L}$ and $H^1(M^{\mathrm{arch}}) \cong Y_{S_\infty}$. Note that $\prod_{w \in S_\infty} 2\pi\sqrt{-1}\,\mathbb{Z}$ is a full lattice in $\mathbf{I}(L_\infty)$ and therefore $(\prod_{w \in S_\infty} 2\pi\sqrt{-1}\,\mathbb{Z}) \otimes \mathbb{R} \cong \mathbf{I}(L_\infty)$. We define the trivialization $t_{M^{\mathrm{arch}}}$ by

$$\begin{aligned}
\mathcal{L} \otimes \mathbb{R} \xrightarrow{\mu'_L} L_\infty = \mathbf{I}(L_\infty) \oplus \mathbf{R}^0(L_\infty) \oplus \mathbb{R} &\xrightarrow{\mathrm{id} \oplus (-\mathrm{id}) \oplus \mathrm{id}} \mathbf{I}(L_\infty) \oplus \mathbf{R}^0(L_\infty) \oplus \mathbb{R} \\
&= \mathbf{I}(L_\infty) \oplus \mathbf{R}(L_\infty) \xrightarrow{\cong} \mathbf{I}(L_\infty) \oplus Y_{S_\infty} \otimes \mathbb{R}.
\end{aligned}$$

Lemma 5.12. *One has*

$$\chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) = [\mathcal{L}, \pi_L, H_L] + \partial^1_G(-[H_L^-, -\pi] - [H_L^+, 2] + [\mathbb{R}, -1])$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

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Proof. First we choose a special representative of the complex M^{arch} in $\mathcal{D}(\mathbb{Z}[G])$. We note that for each $v \in S_\infty$ the complex $M_w(L_w)$ in $\mathcal{D}(\mathbb{Z}[G_w])$ defined by the distinguished triangle

$$L_w[0] \xrightarrow{\exp} M_w \longrightarrow M_w(L_w) \longrightarrow$$

has non-zero cohomology modules $H^{-1}(M_w(L_w)) = 2\pi\sqrt{-1}\,\mathbb{Z} \subset L_w$ and $H^1(M_w(L_w)) = \mathbb{Z}$, and that its extension class in $\mathrm{Ext}^3_{\mathbb{Z}[G_w]}(\mathbb{Z}, 2\pi\sqrt{-1}\,\mathbb{Z}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is non-trivial. On the other hand let $H_w := \prod_{\sigma\in\Sigma(L_w)} \mathbb{Z}$ where $\Sigma(L_w)$ is the set of continuous isomorphisms $L_w \to \mathbb{C}$, and consider the complex of G_w-modules $N_w : H_w \xrightarrow{1+\tau} H_w \xrightarrow{1-\tau} H_w$ with non-zero terms in degrees -1, 0 and 1, where $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denotes complex conjugation. Then N_w is acyclic outside degrees -1 and 1, and one has $H^{-1}(N_w) = H_w^-$, $H^1(N_w) = H_w/H_w^-$. Moreover N_w has non-trivial extension class in $\mathrm{Ext}^3_{\mathbb{Z}[G_w]}(H_w/H_w^-, H_w^-) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Thus by fixing isomorphisms $2\pi\sqrt{-1}\,\mathbb{Z} \cong H_w^-$ and $\mathbb{Z} \cong H_w/H_w^-$ we obtain an induced isomorphism $M_w(L_w) \cong N_w$ in $\mathcal{D}(\mathbb{Z}[G_w])$. Applying $\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}$ and summing over all $v \in S_\infty$ we see that M^{arch} is isomorphic to $N \oplus \mathcal{L}[0]$ in $\mathcal{D}(\mathbb{Z}[G])$ where N is the complex

$$H_L \xrightarrow{1+\tau} H_L \xrightarrow{1-\tau} H_L.$$

The trivialization $t_{M^{\mathrm{arch}}}$ corresponds to the following trivialization t on $N \oplus \mathcal{L}[0]$:

$$\begin{aligned}
\mathcal{L}\otimes\mathbb{R} &\xrightarrow{t_{M^{\mathrm{arch}}}} \mathbf{I}(L_\infty) \oplus Y_{S_\infty}\otimes\mathbb{R}\\
&\xrightarrow{\cong} \prod_{v\in S_\infty} (\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]} (2\pi\sqrt{-1}\,\mathbb{Z}))\otimes\mathbb{R} \oplus \prod_{v\in S_\infty} (\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]} \mathbb{Z})\otimes\mathbb{R}\\
&\xrightarrow{\cong} \prod_{v\in S_\infty} (\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]} H_w^-)\otimes\mathbb{R} \oplus \prod_{v\in S_\infty} (\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]} H_w/H_w^-)\otimes\mathbb{R}\\
&= H_L^-\otimes\mathbb{R} \oplus H_L/H_L^-\otimes\mathbb{R}.
\end{aligned}$$

We compute the Euler characteristic $\chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) = \chi_G(N\oplus\mathcal{L}[0], t)$ by using splittings, i.e. we use the refined Euler characteristic χ_G^{old} discussed in [BrB05, § 6]. The relation of χ_G and χ_G^{old} is given in [BrB05, Theorem 6.2]. With the canonical splittings (given by the direct sum decomposition $H_L\otimes\mathbb{R} = H_L^-\otimes\mathbb{R} \oplus H_L^+\otimes\mathbb{R}$) we find that $\chi_G^{\mathrm{old}}(N\oplus\mathcal{L}[0], t^{-1}) = [H_L\oplus H_L, g, H_L\oplus\mathcal{L}]$, where $g : (H_L\oplus H_L)\otimes\mathbb{R} \to (H_L\oplus\mathcal{L})\otimes\mathbb{R}$ is the isomorphism

$$\begin{aligned}
H_L\otimes\mathbb{R}\oplus H_L\otimes\mathbb{R} &= (H_L^-\otimes\mathbb{R}\oplus H_L^+\otimes\mathbb{R}) \oplus (H_L^-\otimes\mathbb{R}\oplus H_L^+\otimes\mathbb{R})\\
&\xrightarrow{1\oplus 2\oplus 1/2\oplus 1} H_L^-\otimes\mathbb{R} \oplus (H_L^+\otimes\mathbb{R}\oplus H_L^-\otimes\mathbb{R}) \oplus H_L^+\otimes\mathbb{R}\\
&\cong H_L^-\otimes\mathbb{R}\oplus H_L\otimes\mathbb{R}\oplus (H_L/H_L^-)\otimes\mathbb{R}\\
&\xrightarrow{\mathrm{id}\oplus t^{-1}} H_L\otimes\mathbb{R}\oplus\mathcal{L}\otimes\mathbb{R}.
\end{aligned}$$

Here the isomorphism $H_L^+\otimes\mathbb{R} \cong (H_L/H_L^-)\otimes\mathbb{R}$ is induced by the embedding $H_L^+ \to H_L/H_L^-$ and for the arrow marked $\mathrm{id}\oplus t^{-1}$ we have first used the obvious permutation and then the identity on $H_L\otimes\mathbb{R}$ and the isomorphism t^{-1}. Let $h : \mathcal{L}\otimes\mathbb{R} \to H_L^-\otimes\mathbb{R}\oplus H_L/H_L^-\otimes\mathbb{R} \cong H_L\otimes\mathbb{R}$ be defined as t except that we omit $-\mathrm{id}$ on $\mathbf{R}^0(L_\infty)$. One then has

$$[H_L\oplus H_L, g, H_L\oplus\mathcal{L}] = -[\mathcal{L}, h, H_L] + \partial_G^1([X_{S_\infty}, -1] - [H_L^-, 2] + [H_L^+, -2]).$$

Finally we claim that

$$[\mathcal{L}, h, H_L] = [\mathcal{L}, \pi_L, H_L] - \partial_G^1[H_L^-, 2\pi]. \tag{30}$$

Indeed, the map $L_\infty \xrightarrow{(\mu_L')^{-1}} L\otimes\mathbb{R} \subset L\otimes\mathbb{C} \xrightarrow{\rho_L} H_L\otimes\mathbb{C}$ gives an identification of L_∞ with $(H_L\otimes\mathbb{C})^+$. Under this identification one has $\mathbf{R}(L_\infty) \cong H_L^+\otimes\mathbb{R}$ and $\mathbf{I}(L_\infty) \cong H_L^-\otimes i\mathbb{R}$. In the case $K = \mathbb{Q}$ it easily follows that for the correct choice of $i = \sqrt{-1} \in \mathbb{C}$ in the definition of π_L the

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map $h : \mathcal{L} \otimes \mathbb{R} \to H_L \otimes \mathbb{R}$ agrees with $\mathcal{L} \otimes \mathbb{R} \xrightarrow{\pi_L} H_L^+ \otimes \mathbb{R} \oplus H_L^- \otimes \mathbb{R} \xrightarrow{\mathrm{id} \oplus 1/(2\pi)} H_L^+ \otimes \mathbb{R} \oplus H_L^- \otimes \mathbb{R}$, which implies (30). In the case of a general totally real field K the two maps still agree if one chooses the correct i in the definition of π_L for each v-component of $H_L^- = \prod_{v \in S_\infty(K)} (\prod_{\sigma \in \Sigma(L), \sigma|_K = v} \mathbb{Z})^-$ separately; one can check that $[\mathcal{L}, \pi_L, H_L]$ is independent of all such choices.

Summarizing we find

$$\begin{aligned}\chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) &= \chi_G(N \oplus \mathcal{L}[0], t) \\ &= -\chi_G^{\mathrm{old}}(N \oplus \mathcal{L}[0], t^{-1}) + \partial_G^1[H_L^-, -1] \\ &= [\mathcal{L}, h, H_L] + \partial_G^1(-[X_{S_\infty}, -1] + [H_L^-, -2] - [H_L^+, -2]) \\ &= [\mathcal{L}, \pi_L, H_L] + \partial_G^1(-[H_L^-, -\pi] - [H_L^+, 2] + [\mathbb{R}, -1]).\end{aligned}$$

This completes the proof of Lemma 5.12. □

5.2.5 *Relation of Euler characteristics.* Here we prove two lemmas.

Lemma 5.13. *One has*

$$\chi_G(M^{\mathrm{finite}}, t_{M^{\mathrm{finite}}}) + \chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) = \chi_G(P_S, -\mathrm{Reg}_S) + \chi_G(F_S(\mathcal{L}), t_F)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

For the proof of Lemma 5.13 we need the following result.

Lemma 5.14. *There exists a distinguished triangle*

$$P_S \xrightarrow{\alpha} M^{\mathrm{finite}} \oplus M^{\mathrm{arch}} \xrightarrow{\beta} F_S(\mathcal{L}) \xrightarrow{\gamma} \tag{31}$$

in $\mathcal{D}(\mathbb{Z}[G])$ *whose cohomology sequence after tensoring with* $\mathbb{R}$ *identifies with the exact sequence (starting with* $H^{-1}((M^{\mathrm{finite}} \oplus M^{\mathrm{arch}}) \otimes \mathbb{R})$*)*

$$\mathbf{I}(L_\infty) \xrightarrow{\subset} L_\infty^0 \xrightarrow{-\exp} \mathcal{O}_{L,S}^\times \otimes \mathbb{R} \xrightarrow{\eta} \left(\prod_{w \in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \right) \otimes \mathbb{R} \oplus L \otimes \mathbb{R}$$

$$\xrightarrow{\xi} L \otimes \mathbb{R} \xrightarrow{0} X_S \otimes \mathbb{R} \xrightarrow{\subset} Y_S \otimes \mathbb{R} \xrightarrow{\mathrm{aug}} \mathbb{R}.$$

Here the map $-\exp : L_\infty^0 \to \mathcal{O}_{L,S}^\times \otimes \mathbb{R}$ *is the composite*

$$L_\infty^0 = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R} \xrightarrow{-\mathrm{id}} \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R} \xrightarrow{\exp} \mathcal{O}_L^\times \otimes \mathbb{R} \xrightarrow{\subset} \mathcal{O}_{L,S}^\times \otimes \mathbb{R},$$

the map η *is induced by the canonical maps* $\mathcal{O}_{L,S}^\times \to L_w^\times$*, and the map* ξ *is the identity on* $L \otimes \mathbb{R}$ *and zero on the first summand.*

Proof. Consider the continuous arrows in the following diagram.

$$\begin{array}{ccccccc}
\mathcal{L}_S[0] \oplus \mathcal{L}[-1] & \xrightarrow{\exp \oplus 0} & M_{S_f} \oplus M_{S_\infty} & \longrightarrow & M^{\mathrm{finite}} \oplus M^{\mathrm{arch}} & \longrightarrow & \mathcal{L}_S[1] \oplus \mathcal{L}[0] \\
\| & & \downarrow & & \downarrow \beta & & \| \\
\mathcal{L}_S[0] \oplus \mathcal{L}[-1] & \xrightarrow{\exp \oplus 0} & E_S & \longrightarrow & F_S(\mathcal{L}) & \longrightarrow & \mathcal{L}_S[1] \oplus \mathcal{L}[0] \\
 & & \downarrow & & \downarrow \gamma & & \\
 & & P_S[1] & = & P_S[1] & & \\
 & & \downarrow - & & \downarrow -\alpha[1] & & \\
 & & M_{S_f}[1] \oplus M_{S_\infty}[1] & \longrightarrow & M^{\mathrm{finite}}[1] \oplus M^{\mathrm{arch}}[1] & &
\end{array} \tag{32}$$

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The first row is the direct sum of the distinguished triangles (28) and (29), and the second row is (24). The first vertical triangle is the rotation of (22) (the arrow labelled with a minus sign is the negative of the shift of the first arrow in (22)), and $-\alpha[1]$ is the map making the bottom square commutative. It follows from the octahedral axiom that the diagram can be completed by the dashed arrows in such a way that one obtains an octahedral diagram, i.e. such that the second vertical triangle is also distinguished, the diagram is commutative, and furthermore the square

$$\begin{array}{ccc} F_S(\mathcal{L}) & \longrightarrow & \mathcal{L}_S[1] \oplus \mathcal{L}[0] \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \exp \oplus 0} \\ P_S[1] & \xrightarrow{-} & M_{S_f}[1] \oplus M_{S_\infty}[1] \end{array} \tag{33}$$

commutes.

We obtain the required distinguished triangle (31) as the rotation of the triangle

$$M^{\text{finite}} \oplus M^{\text{arch}} \xrightarrow{\beta} F_S(\mathcal{L}) \xrightarrow{\gamma} P_S[1] \xrightarrow{-\alpha[1]} M^{\text{finite}}[1] \oplus M^{\text{arch}}[1] \tag{34}$$

in diagram (32). We will compute the cohomology sequence of (34). Taking into account the sign change coming from the rotation, it then follows that the cohomology sequence of (31) has the stated form.

After tensoring with $\mathbb{R}$ the cohomology sequence of (34) has the following form (starting with $H^{-1}((M^{\text{finite}} \oplus M^{\text{arch}}) \otimes \mathbb{R})$):

$$\mathbf{I}(L_\infty) \xrightarrow{u_1} L^0_\infty \xrightarrow{v_1} \mathcal{O}^\times_{L,S} \otimes \mathbb{R} \xrightarrow{w_1} \left(\prod_{w \in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \right) \otimes \mathbb{R} \oplus L \otimes \mathbb{R}$$
$$\xrightarrow{u_2} L \otimes \mathbb{R} \xrightarrow{v_2} X_S \otimes \mathbb{R} \xrightarrow{w_2} Y_S \otimes \mathbb{R} \xrightarrow{u_3} \mathbb{R}.$$

We must compute the maps in this sequence. In diagram (32) we have a morphism from the first row to the second row. After tensoring with $\mathbb{R}$, the associated morphism of cohomology sequences is as given below (starting with $H^{-1}((M^{\text{finite}} \oplus M^{\text{arch}}) \otimes \mathbb{R})$ in the first row and $H^{-1}(F_S(\mathcal{L}) \otimes \mathbb{R})$ in the second row).

$$\begin{array}{ccccccc} \mathbf{I}(L_\infty) & \xrightarrow{\subset} & \mathcal{L}_S \otimes \mathbb{R} & \xrightarrow{\exp} & L_S^\times \otimes \mathbb{R} & \longrightarrow & \left(\prod_{w \in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \right) \otimes \mathbb{R} \oplus L \otimes \mathbb{R} \\ \downarrow{\scriptstyle u_1} & & \| & & \downarrow & & \downarrow{\scriptstyle u_2} \\ L^0_\infty & \xrightarrow{\subset} & \mathcal{L}_S \otimes \mathbb{R} & \xrightarrow{\exp} & C_S(L) \otimes \mathbb{R} & \xrightarrow{0} & L \otimes \mathbb{R} \end{array}$$

$$\begin{array}{ccccccc} \xrightarrow{\xi} & L \otimes \mathbb{R} & \xrightarrow{0} & Y_S \otimes \mathbb{R} & \xrightarrow{\text{id}} & Y_S \otimes \mathbb{R} \\ & \| & & \downarrow{\scriptstyle \text{aug}} & & \downarrow{\scriptstyle u_3} \\ \xrightarrow{\text{id}} & L \otimes \mathbb{R} & \xrightarrow{0} & \mathbb{R} & \xrightarrow{\text{id}} & \mathbb{R} \end{array}$$

We deduce that $u_1 : \mathbf{I}(L_\infty) \to L^0_\infty$ is the inclusion, $u_2 = \xi$ and $u_3 : Y_S \otimes \mathbb{R} \to \mathbb{R}$ is the augmentation.

A similar argument using the morphism of the cohomology sequences of the two vertical triangles in diagram (32) shows that $w_1 = -\eta$ and that $w_2 : X_S \otimes \mathbb{R} \to Y_S \otimes \mathbb{R}$ is the negative of the inclusion map. It immediately follows that $v_2 = 0$. Finally, taking H^{-1} of the commutative square (33) and

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tensoring with $\mathbb{R}$ gives the following commutative square (where we use the maps $\exp_S$ and Δ_S from § 3.1).

$$\begin{array}{ccccc} L^0_\infty & & \xrightarrow{\quad\subset\quad} & & \mathcal{L}_S\otimes\mathbb{R} \\ \downarrow{\scriptstyle v_1} & & & & \downarrow{\scriptstyle \exp_S\otimes\mathbb{R}} \\ \mathcal{O}^\times_{L,S}\otimes\mathbb{R} & \xrightarrow{-\mathrm{id}} & \mathcal{O}^\times_{L,S}\otimes\mathbb{R} & \xrightarrow{\Delta_S\otimes\mathbb{R}} & L^\times_S\otimes\mathbb{R} \end{array}$$

Now we recall from the proof of Lemma 3.1 that the inclusion $L^0_\infty \subset \mathcal{L}_S\otimes\mathbb{R}$ is obtained via $L^0_\infty = \log_\infty(\mathcal{O}^\times_L)\otimes\mathbb{R} \cong \{x\in\mathcal{L}_S : \exp_S(x)\in\Delta_S(\mathcal{O}^\times_L)\}\otimes\mathbb{R}\subset\mathcal{L}_S\otimes\mathbb{R}$. It easily follows that $v_1 : L^0_\infty \to \mathcal{O}^\times_{L,S}\otimes\mathbb{R}$ is the composite $L^0_\infty = \log_\infty(\mathcal{O}^\times_L)\otimes\mathbb{R} \xrightarrow{-\mathrm{id}} \log_\infty(\mathcal{O}^\times_L)\otimes\mathbb{R} \xrightarrow{\exp} \mathcal{O}^\times_L\otimes\mathbb{R} \xrightarrow{\subset} \mathcal{O}^\times_{L,S}\otimes\mathbb{R}$. This completes the proof of Lemma 5.14. □

Proof of Lemma 5.13. We want to apply the additivity criterion of Lemma 2.5 to the distinguished triangle in Lemma 5.14. Note that $\ker(H^{\mathrm{od}}(\alpha)\otimes\mathbb{R}) = 0$ and that we can therefore omit this term when applying Lemma 2.5. We have to consider the $\mathbb{R}[G]$-modules and isomorphisms in the following (in general non-commutative) diagram.

$$\begin{array}{ccc} \left(\prod_{w\in S_f}\dfrac{L^\times_w}{\exp(\mathcal{L}_w)}\otimes\mathbb{R}\oplus L\otimes\mathbb{R}\right)\oplus(\mathcal{O}^\times_L\otimes\mathbb{R}) & \xrightarrow{s^{\mathrm{ev}}} & (\mathcal{O}^\times_{L,S}\otimes\mathbb{R})\oplus(L\otimes\mathbb{R}) \\ \downarrow{\scriptstyle t_{M^{\mathrm{finite}}}\oplus t_{M^{\mathrm{arch}}}\oplus\mathrm{id}} & & \downarrow{\scriptstyle(-\mathrm{Reg}_S)\oplus t_F} \\ (\mathbf{I}(L_\infty)\oplus Y_{S,\mathbb{R}})\oplus(\mathcal{O}^\times_L\otimes\mathbb{R}) & \xrightarrow{s^{\mathrm{od}}} & (X_{S,\mathbb{R}})\oplus(L^0_\infty\oplus\mathbb{R}) \end{array} \tag{35}$$

Here we write $Y_{S,\mathbb{R}} := Y_S\otimes\mathbb{R}$ etc., and for the left vertical map we have composed $t_{M^{\mathrm{finite}}}\oplus t_{M^{\mathrm{arch}}}$ with the canonical isomorphism $Y_{S_f,\mathbb{R}}\oplus\mathbf{I}(L_\infty)\oplus Y_{S_\infty,\mathbb{R}}\cong\mathbf{I}(L_\infty)\oplus Y_{S,\mathbb{R}}$. The maps s^{ev} and s^{od} are given by splittings of the exact sequences

$$0\to\mathcal{O}^\times_L\otimes\mathbb{R}\to\mathcal{O}^\times_{L,S}\otimes\mathbb{R}\xrightarrow{\eta}\prod_{w\in S_f}\frac{L^\times_w}{\exp(\mathcal{L}_w)}\otimes\mathbb{R}\oplus L\otimes\mathbb{R}\xrightarrow{\xi}L\otimes\mathbb{R}\to 0\to 0$$

and

$$0\to 0\to X_{S,\mathbb{R}}\to\mathbf{I}(L_\infty)\oplus Y_{S,\mathbb{R}}\to L^0_\infty\oplus\mathbb{R}\xrightarrow{-\exp}\mathcal{O}^\times_L\otimes\mathbb{R}\to 0$$

respectively. Lemma 5.13 will follow from Lemma 2.5 once we have shown that the automorphism

$$(t_{M^{\mathrm{finite}}}\oplus t_{M^{\mathrm{arch}}}\oplus\mathrm{id})^{-1}\circ(s^{\mathrm{od}})^{-1}\circ((-\mathrm{Reg}_S)\oplus t_F)\circ s^{\mathrm{ev}} \tag{36}$$

of

$$\left(\prod_{w\in S_f}\frac{L^\times_w}{\exp(\mathcal{L}_w)}\otimes\mathbb{R}\oplus L\otimes\mathbb{R}\right)\oplus(\mathcal{O}^\times_L\otimes\mathbb{R})$$

has reduced norm equal to 1.

To compute the reduced norm of (36) we use the following isomorphisms to replace various of the modules:

$$-\mathrm{Reg}_{S_\infty} : \mathcal{O}^\times_L\otimes\mathbb{R}\xrightarrow{\cong}X_{S_\infty,\mathbb{R}},$$
$$-\mathrm{Reg}_S : \mathcal{O}^\times_{L,S}\otimes\mathbb{R}\xrightarrow{\cong}X_{S,\mathbb{R}},$$
$$\prod_{w\in S_f}(v_w(\cdot)\cdot\log Nw) : \prod_{w\in S_f}\frac{L^\times_w}{\exp(\mathcal{L}_w)}\otimes\mathbb{R}\xrightarrow{\cong}Y_{S_f,\mathbb{R}},$$

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$$Y_{S,\mathbb{R}} = Y_{S_f,\mathbb{R}} \oplus Y_{S_\infty,\mathbb{R}} = Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R},$$
$$L_\infty^0 = \mathbf{I}(L_\infty) \oplus \mathbf{R}^0(L_\infty) \cong \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}}$$

and

$$L \otimes \mathbb{R} \xrightarrow{\mu_L'} L_\infty = \mathbf{I}(L_\infty) \oplus \mathbf{R}^0(L_\infty) \oplus \mathbb{R} \cong \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}.$$

Diagram (35) then becomes

$$\begin{array}{ccc} (Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \oplus (X_{S_\infty,\mathbb{R}}) & \xrightarrow{\tilde{s}^{\mathrm{ev}}} & (X_{S,\mathbb{R}}) \oplus (\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \\ \downarrow t_1 & & \downarrow t_2 \\ (\mathbf{I}(L_\infty) \oplus Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \oplus (X_{S_\infty,\mathbb{R}}) & \xrightarrow{\tilde{s}^{\mathrm{od}}} & (X_{S,\mathbb{R}}) \oplus (\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \end{array}$$

and we must show that the automorphism $t_1^{-1} \circ (\tilde{s}^{\mathrm{od}})^{-1} \circ t_2 \circ \tilde{s}^{\mathrm{ev}}$ of $(Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \oplus (X_{S_\infty,\mathbb{R}})$ has reduced norm equal to 1. For this it is necessary to give explicit descriptions of the maps $\tilde{s}^{\mathrm{ev}}$, t_2, $(\tilde{s}^{\mathrm{od}})^{-1}$ and t_1^{-1}. It is easy to see that $t_1(a,b,c,d,e) = (b,a,-c,d,e)$ and $t_2 = \mathrm{id}$.

Next we compute the map $\tilde{s}^{\mathrm{ev}}$. It is induced by splittings of the exact sequence

$$0 \to X_{S_\infty,\mathbb{R}} \to X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to 0 \to 0,$$

which consists of the canonical short exact sequence $X_{S_\infty,\mathbb{R}} \to X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}}$ and the identity $\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \xrightarrow{=} \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$. As splitting of the surjection $X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}}$ we choose the map $Y_{S_f,\mathbb{R}} \to X_{S,\mathbb{R}}$ given by $a = (a_w)_{w \in S_f} \mapsto (g_w)_{w \in S}$ where $g_w = a_w$ if $w \in S_f$ and $g_w = -\mathrm{aug}(a)/|S_\infty|$ if $w \in S_\infty$ (here $\mathrm{aug}(a) = \sum_{w \in S_f} a_w$ is the augmentation of a). With this splitting we obtain

$$\tilde{s}^{\mathrm{ev}}(a,b,c,d,e) = ((f_w)_{w \in S}, b, c, d)$$

with

$$f_w = \begin{cases} e_w - \mathrm{aug}(a)/|S_\infty| & \text{if } w \in S_\infty, \\ a_w & \text{if } w \in S_f, \end{cases}$$

where we write $a = (a_w)_{w \in S_f} \in Y_{S_f,\mathbb{R}}$ and $e = (e_w)_{w \in S_\infty} \in X_{S_\infty,\mathbb{R}}$.

Finally we compute the map $\tilde{s}^{\mathrm{od}}$. It is induced by splittings of the exact sequence

$$0 \to 0 \to X_{S,\mathbb{R}} \to \mathbf{I}(L_\infty) \oplus Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to X_{S_\infty,\mathbb{R}} \to 0,$$

which consists of the canonical short exact sequences $X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}$ (recall that $Y_{S,\mathbb{R}} = Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$) and $\mathbf{I}(L_\infty) \to \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \to X_{S_\infty,\mathbb{R}}$. As splitting of the surjection $Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} = Y_{S,\mathbb{R}} \to \mathbb{R}$ we choose the map $\mathbb{R} \to Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$ given by $d \mapsto ((d/|S|)_{w \in S_f}, (0)_{w \in S_\infty}, d|S_\infty|/|S|)$. With this splitting we obtain

$$(\tilde{s}^{\mathrm{od}})^{-1}(a,b,c,d) = (b, (a_w + d/|S|)_{w \in S_f}, (a_w - \mathrm{aug}_\infty(a)/|S_\infty|)_{w \in S_\infty}, \mathrm{aug}_\infty(a) + d|S_\infty|/|S|, c)$$

where we write $a = (a_w)_{w \in S} \in X_{S,\mathbb{R}}$ and $\mathrm{aug}_\infty(a) = \sum_{w \in S_\infty} a_w$.

It follows that the automorphism $t_1^{-1} \circ (\tilde{s}^{\mathrm{od}})^{-1} \circ t_2 \circ \tilde{s}^{\mathrm{ev}}$ of $(Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \oplus (X_{S_\infty,\mathbb{R}})$ is

$$(a,b,c,d,e) \mapsto (a + (d/|S|)_{w \in S_f}, b, -e, -\mathrm{aug}(a) + d|S_\infty|/|S|, c).$$

Therefore its reduced norm can be computed as the product of the reduced norm of the automorphism $(a,d) \mapsto (a + (d/|S|)_{w \in S_f}, -\mathrm{aug}(a) + d|S_\infty|/|S|)$ of $Y_{S_f,\mathbb{R}} \oplus \mathbb{R}$ and of the reduced norm of the automorphism $(b,c,e) \mapsto (b,-e,c)$ of $\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}}$. It is straightforward to verify that both of these reduced norms are equal to 1. □

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5.2.6 *Completion of the proof.* We now collect all the previous results to complete the proof of Theorem 5.2.

Proof of Theorem 5.2. By the definitions of $T\Omega(L/K,0)$ and $T\Omega(L/K,1)$, and by Lemmas 5.10 and 5.13 we have

$$\begin{aligned}&\psi_G^*(T\Omega(L/K,0)) - T\Omega(L/K,1)\\ &\quad= \hat\partial_G^1(\zeta^*_{L/K,S}(0)^\#) - \chi_G(P_S, -\mathrm{Reg}_S) - \hat\partial_G^1(\zeta^*_{L/K,S}(1)) - \chi_G(F_S(\mathcal{L}), t_F)\\ &\quad= \hat\partial_G^1\left(\frac{\zeta^*_{L/K,S}(0)^\#}{\zeta^*_{L/K,S}(1)}\right) - \chi_G(M^{\mathrm{finite}}, t_{M^{\mathrm{finite}}}) - \chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}).\end{aligned}$$

Lemmas 5.9, 5.11 and 5.12 show that this is equal to

$$\begin{aligned}&\hat\partial_G^1(\varepsilon_{L/K}(0)) - [\mathcal{L}, \pi_L, H_L]\\ &\quad- \sum_{v\in S_f} \mathrm{ind}_{G_w}^G\left(-\hat\partial_{G_w}^1\left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#}\right) + \chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) + \partial^1_{G_w}[\mathbb{R}, \log Nw]\right)\\ &\quad+ \partial_G^1(-[H_L^-, \pi] - [H_L^+, 2] + [\mathbb{R}, -1] + [H_L^-, -\pi] + [H_L^+, 2] - [\mathbb{R}, -1]),\end{aligned}$$

which is $T\Omega^{\mathrm{loc}}(L/K,1)$ since $\partial_G^1[H_L^-, -1] = 0$. □

5.3 Proof of Theorem 5.8

The following proof of Theorem 5.8 is taken from [Bre04c, § 7]. As a preliminary step of independent interest we show that the invariant $R\Omega^{\mathrm{loc}}(L/K,1)$ in $K_0(\mathbb{Z}[G],\mathbb{R})$ allows one to determine the absolute norm of the Artin conductor of every character of G.

5.3.1 *Determining conductors.* In the following result $|\cdot|$ denotes the usual absolute value on the complex numbers $\mathbb{C}$.

Lemma 5.15. *Let $\alpha = (\alpha_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times = \mathrm{Z}(\mathbb{C}[G])^\times$ and assume that $|\omega(\alpha_\chi)| = |\alpha_\chi|$ for all $\chi \in \mathrm{Irr}(G)$ and all automorphisms ω of $\mathbb{C}$. Then for every $\chi \in \mathrm{Irr}(G)$ the absolute value $|\alpha_\chi|$ is determined by $\partial^1_{\mathbb{Z}[G],\mathbb{C}}(\alpha) \in K_0(\mathbb{Z}[G],\mathbb{C})$.*

Proof. The hypothesis implies that all α_χ are algebraic over $\mathbb{Q}$. Therefore there exists a finite extension E of $\mathbb{Q}$ in $\mathbb{C}$ such that $\partial^1_{\mathbb{Z}[G],\mathbb{C}}(\alpha) \in K_0(\mathbb{Z}[G],E)$ and which is big enough to ensure that every irreducible representation of G is realizable over E. This implies that $\alpha \in \mathrm{Z}(E[G])^\times = \prod_{\chi\in\mathrm{Irr}(G)} E^\times$.

Using the hypothesis and the product formula for the field E we obtain

$$|\alpha_\chi|^{[E:\mathbb{Q}]} = \prod_{v\in S_\infty(E)} |\alpha_\chi|_v = \prod_{v\in S_f(E)} |\alpha_\chi|_v^{-1},$$

where the valuations $|\cdot|_v$ are normalized as usual. It therefore suffices to show that $\partial^1_{\mathbb{Z}[G],E}(\alpha)$ determines $|\alpha_\chi|_v$ for every non-archimedean place v of E.

Let v be a non-archimedean place of E and let p be the residue characteristic of v. We write $j : E \to E_v$ for the embedding of E into its completion at v. Then j induces maps j_* of the centres of the group rings and of the relative algebraic K-groups making the following diagram commutative.

$$\begin{array}{ccc}\mathrm{Z}(E[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}[G],E}} & K_0(\mathbb{Z}[G],E)\\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*}\\ \mathrm{Z}(E_v[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}_p[G],E_v}} & K_0(\mathbb{Z}_p[G],E_v)\end{array}$$

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But it is well known that $\partial^1_{\mathbb{Z}_p[G],E_v}(j_*(\alpha))$ determines $j_*(\alpha)_{j\circ\chi} = j(\alpha_\chi)$ up to a unit in E_v for every $\chi \in \mathrm{Irr}(G)$, hence it determines $|\alpha_\chi|_v$. □

COROLLARY 5.16. *For each character χ of G the absolute norm of the Artin conductor of χ is determined by $R\Omega^{\mathrm{loc}}(L/K,1) \in K_0(\mathbb{Z}[G],\mathbb{R})$.*

Proof. Let n be the order of the finite group $K_0(\mathbb{Z}[G],\mathbb{Q})_{\mathrm{tors}}$. Since the invariant $T\Omega^{\mathrm{loc}}(L/K,1) = \hat\partial^1_G(\varepsilon_{L/K}(0)) - R\Omega^{\mathrm{loc}}(L/K,1)$ lies in $K_0(\mathbb{Z}[G],\mathbb{Q})_{\mathrm{tors}}$ (cf. Proposition 5.7) and $\varepsilon_{L/K}(0)^{2n} \in \mathrm{Z}(\mathbb{R}[G])^{\times+}$ one has $2n\cdot R\Omega^{\mathrm{loc}}(L/K,1) = 2n\cdot\hat\partial^1_G(\varepsilon_{L/K}(0)) = \partial^1_{\mathbb{Z}[G],\mathbb{R}}(\varepsilon_{L/K}(0)^{2n})$. Furthermore

$$\varepsilon_{L/K}(0)^{2n} = (W(\overline{\chi})^{2n} Nf(\chi)^n |d_{K/\mathbb{Q}}|^{\deg(\chi)n})_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times$$

by definition, and since $Nf(\chi)$ and $|d_{K/\mathbb{Q}}|$ are both rational integers and $W(\overline{\chi})$ is an algebraic number with absolute value equal to 1 for every archimedean place (this follows for example from [Tat77, Remark on p. 110]), one sees that $\varepsilon_{L/K}(0)^{2n}$ satisfies the hypothesis of Lemma 5.15. Thus Lemma 5.15 shows that $2n\cdot R\Omega^{\mathrm{loc}}(L/K,1)$ determines

$$|W(\overline{\chi})^{2n} Nf(\chi)^n |d_{K/\mathbb{Q}}|^{\deg(\chi)n}| = Nf(\chi)^n |d_{K/\mathbb{Q}}|^{\deg(\chi)n}$$

for every $\chi \in \mathrm{Irr}(G)$. This allows us to find $|d_{K/\mathbb{Q}}|$ because the conductor of the trivial character is equal to 1. We then get $Nf(\chi)$ for every $\chi \in \mathrm{Irr}(G)$ and finally $Nf(\chi)$ for arbitrary characters χ because the Artin conductor is multiplicative. □

5.3.2 *Determining symplectic epsilon constants.* We denote the group of all complex-valued (virtual) characters of G by R_G and the subgroup of symplectic characters by R^s_G. Let $\alpha = (\alpha_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times = \mathrm{Z}(\mathbb{C}[G])^\times$. The map $\mathrm{Irr}(G) \to \mathbb{C}^\times$ given by $\chi \mapsto \alpha_\chi$ has a unique extension to a homomorphism of abelian groups $R_G \to \mathbb{C}^\times$ and thus defines α_χ for every $\chi \in R_G$.

LEMMA 5.17. *Let $\alpha \in \mathrm{Z}(\mathbb{R}[G])^\times$ be such that $\alpha_\chi \in \{\pm 1\}$ for all $\chi \in R^s_G$. If $\hat\partial^1_G(\alpha) = 0 \in K_0(\mathbb{Z}[G],\mathbb{R})$ then $\alpha_\chi = 1$ for all $\chi \in R^s_G$.*

Proof. We reduce to [CT83a, Proposition (6.1)] and use the same notation as there. In particular, J denotes the idèle group of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. We write $+1_\infty$ for the idèle with component 1 at all places and -1_∞ for the idèle with component -1 at all non-archimedean places and component 1 at all archimedean places. Let $f \in \mathrm{Hom}(R^s_G, \pm 1_\infty)$ be the homomorphism given by

$$\chi \mapsto \begin{cases} +1_\infty & \text{if } \alpha_\chi = +1, \\ -1_\infty & \text{if } \alpha_\chi = -1, \end{cases}$$

for $\chi \in R^s_G$. To apply [CT83a, Proposition (6.1)] we must show that $f \in \mathrm{Det}^s(\tilde{\mathbb{Z}}[G]^\times)$ where $\tilde{\mathbb{Z}}[G] = \mathbb{R}[G] \times \prod_p \mathbb{Z}_p[G]$ with p running through all rational prime numbers and Det^s denoting the restriction of the determinantal homomorphisms to R^s_G as discussed in [CT83a, § 3]. The archimedean component of f is obviously contained in $\mathrm{Det}^s(\mathbb{R}[G]^\times)$.

Let p be a prime number. Let $j : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ be any embedding and extend it to an embedding $j : \mathbb{C} \to \overline{\mathbb{Q}_p}$. Then j induces maps $j_* : \mathrm{Z}(\mathbb{R}[G])^\times \to \mathrm{Z}(\overline{\mathbb{Q}_p}[G])^\times$ and $j_* : K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p})$ such that

$$\begin{array}{ccc} \mathrm{Z}(\mathbb{R}[G])^\times & \xrightarrow{\hat\partial^1_G} & K_0(\mathbb{Z}[G],\mathbb{R}) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \mathrm{Z}(\overline{\mathbb{Q}_p}[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}}} & K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}) \end{array}$$

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commutes (cf. Lemma 2.2). From the assumption $\hat{\partial}^1_G(\alpha) = 0$ it follows that

$$j_*(\alpha) \in \mathrm{im}(K_1(\mathbb{Z}_p[G]) \to \mathrm{Z}(\overline{\mathbb{Q}_p}[G])^\times). \tag{37}$$

In particular $j_*(\alpha) \in \mathrm{Z}(\mathbb{Q}_p[G])^\times$ which implies $\alpha_{j^{-1}\circ\omega\circ j\circ\chi} = \alpha_\chi$ for all $\chi \in R^s_G$ and $\omega \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Since this is true for all p and embeddings j we find that $\alpha_{\omega\circ\chi} = \alpha_\chi$ for all $\chi \in R^s_G$ and $\omega \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The p-component of f is the map $f_p : \chi \mapsto \alpha_\chi \in (\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ for $\chi \in R^s_G$. By the argument above f_p lies in $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(R^s_G, (\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times)$. Denote the group of symplectic $\overline{\mathbb{Q}_p}$-valued characters by $R^s_{G,p}$. An embedding $j : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ induces an isomorphism

$$j_* : \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(R^s_G, (\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times) \to \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}(R^s_{G,p}, \overline{\mathbb{Q}_p}^\times),$$

and to show that f_p lies in $\mathrm{Det}^s(\mathbb{Z}_p[G]^\times)$ it suffices to show that $j_*(f_p)$ lies in $\mathrm{Det}^s(\mathbb{Z}_p[G]^\times)$ (compare the diagram in [CT83a, p. 254]). But $j_*(f_p)$ is the homomorphism which corresponds to $j_*(\alpha) \in \mathrm{Z}(\overline{\mathbb{Q}_p}[G])^\times$ and therefore lies in $\mathrm{Det}^s(\mathbb{Z}_p[G]^\times)$ by (37).

We have shown that $f \in \mathrm{Hom}(R^s_G, \pm 1_\infty) \cap \mathrm{Det}^s(\tilde{\mathbb{Z}}[G]^\times)$. By [CT83a, Proposition (6.1)] this intersection consists only of the trivial homomorphism; hence $\alpha_\chi = 1$ for all $\chi \in R^s_G$. □

Proof of Theorem 5.8. By the proof of Corollary 5.16 the element $\hat{\partial}^1_G(\varepsilon_{L/K}(0)) \in K_0(\mathbb{Z}[G], \mathbb{R})$ determines $Nf(\chi)|d_{K/\mathbb{Q}}|^{\deg(\chi)}$ for every $\chi \in \mathrm{Irr}(G)$. We set $\delta_\chi := \sqrt{Nf(\chi)|d_{K/\mathbb{Q}}|^{\deg(\chi)}}$ (positive square root), $\delta := (\delta_\chi)_{\chi\in\mathrm{Irr}(G)}$ and $\alpha := \varepsilon_{L/K}(0)\delta^{-1}$. Then δ and α lie in $\mathrm{Z}(\mathbb{R}[G])^\times$ and $\alpha = (W(\overline{\chi}))_{\chi\in\mathrm{Irr}(G)}$. Since $W(\overline{\chi}) \in \{\pm 1\}$ for every $\chi \in R^s_G$, we can apply Lemma 5.17 to α and conclude that the root numbers $W(\overline{\chi})$ for $\chi \in R^s_G$ are determined by $\hat{\partial}^1_G(\alpha) = \hat{\partial}^1_G(\varepsilon_{L/K}(0)) - \hat{\partial}^1_G(\delta)$ and therefore also by $\hat{\partial}^1_G(\varepsilon_{L/K}(0))$. Thus assuming the validity of Conjecture 5.3, the symplectic root numbers are determined by $R\Omega^{\mathrm{loc}}(L/K, 1)$. □

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Leading terms of Artin L-functions

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