ON PERMUTATIONAL PRODUCTS OF GROUPS

R. J. GREGORAC

(Received 1 December 1967)

1. Introduction

We here study permutational products of groups, a study which was initiated by B. H. Neumann [8], [9].

In defining a permutational product a transversal of the amalgamated subgroup must be chosen in each of the constituents of the amalgam and, in general, different transversals give rise to non-isomorphic permutational products [8]. One of the main results here is that an epimorphism from an amalgam onto a factor amalgam determines an epimorphism between selected permutational products on the amalgams (Theorem 3.1); these permutational products are chosen by making natural choices of their defining transversals.

It is also shown that some permutational products (again chosen by picking transversals) are embeddable in certain permutational wreath products, provided some restrictions are placed on the defining amalgams (Theorems 4.1, 5.2, and 6.1). These embeddings immediately relate properties of the amalgam to the permutational products, thus yielding several amalgam embedding theorems, where permutational products are chosen as the embedding groups.

Finally some amalgam embedding results are included which are not directly related to permutational products.

2. Preliminaries

Given two groups $(A, +)$ and $(B, \cdot)$, the set $A \cup B$ is the *amalgam* of these groups if and only if $A \cap B = H$ is a subgroup of both A and B and, for all $h, h_1 \in H$, $h+h_1 = h \cdot h_1$. The common subgroup H is called the *amalgamated subgroup* and the groups A and B are the *constituents of the amalgam*. The notation $A \cup B|H = \mathfrak{A}$ will be used to denote an amalgam of A and B with H amalgamated. Amalgams may be constructed as follows. Suppose A and B are given groups, disjoint as sets, containing isomorphic subgroups H and K, respectively. Suppose $\alpha : H \cong K$ is a given isomorphism from H onto K. If H and K are identified by setting $h = (h)\alpha$, $(h \in H)$, then the union $A \cup B$ becomes an amalgam of A and B. In general, different isomorphisms α used in this construction yield different amalgams.

A group G *embeds* the amalgam $A \cup B|H = \mathfrak{A}$, if G contains subgroups A^* and B^* which are isomorphic to A and B, say $A^* = A\mu$ and $B^* = B\nu$, such that $H\mu = H\nu = A^* \cap B^*$. A group G is *generated* by the amalgam $\mathfrak{A}$, if G embeds $\mathfrak{A}$ and the image of $\mathfrak{A}$ in G generates G.

Given an arbitrary amalgam $\mathfrak{A}$ of two groups there always exists a group embedding $\mathfrak{A}$. One such group, a *permutational product on the amalgam*, may be constructed as follows. Let $\mathfrak{A} = A \cup B|H$ be the given amalgam. Choose a transversal T_A of H in A and a transversal T_B of H in B, where a *transversal* of H in A is a complete set of left coset representatives of H in A.

Let the set $D = T_A \times T_B \times H$ and for $a \in A$, $b \in B$ define permutations on D as follows:

Let $d = (q, r, h)$, $q \in T_A$, $r \in T_B$, and $h \in H$. Then put

$$(d)\rho(a) = (q', r', h'),$$

where

$$q'h' = qha \quad (q' \in T_A,\, h' \in H), \quad \text{and} \quad r' = r.$$

Similarly, put

$$(d)\rho(b) = (q'', r'', h''),$$

where

$$r''h'' = rhb \quad (r'' \in T_B,\, h'' \in H), \quad \text{and} \quad q'' = q.$$

If $h \in H$, the above permutations $\rho(h)$ are easily seen to be the same by either definition. The permutations $\rho(a)$ and $\rho(b)$ may be considered right multiplications by a and b, so that if we compose functions from left to right, then $\rho(A) \cong A$ and $\rho(B) \cong B$, where $\rho(A) = \{\rho(a) | a \in A\}$. In the group of all permutations of D, $\rho(A) \cap \rho(B) = \rho(H)$ [8], so that the amalgam $\mathfrak{A} = A \cup B|H$ can be embedded in the subgroup $P(\mathfrak{A}; T_A, T_B)$ generated by $\rho(A)$ and $\rho(B)$ in the group of permutations on D. The group $P(\mathfrak{A}; T_A, T_B)$ is the *permutational product of* $\mathfrak{A} = A \cup B|H$ *depending on* T_A *and* T_B.

The notation $\langle \cdots \rangle$ denotes the subgroup generated by $\cdots$, where $\cdots$ are elements or subsets of a given group. If x and y are elements of a group, write $y^x = x^{-1}yx$.

Suppose the amalgamated subgroup of a given amalgam $\mathfrak{A} = A \cup B|H$ is central in both constituents of $\mathfrak{A}$. Then $\mathfrak{A}$ is clearly embeddable in the group $G = A \times B / \langle (h, h^{-1}) | h \in H \rangle$ [7] such that the constituents of the embedded copy of $\mathfrak{A}$ commute elementwise and generate G. The group G is called the *generalized direct product* on $\mathfrak{A}$.

LEMMA 2.1. (*B. H. Neumann* [8]). *Let* $\mathfrak{A} = A \cup B|H$ *be an amalgam. If* H *is central in both* A *and* B, *then for any choices of transversals* T_A *and* T_B, $P(\mathfrak{A}; T_A, T_B)$ *is isomorphic to the generalized direct product on* $\mathfrak{A}$.

Let Aut(H) denote the group of automorphisms of the group H. Let C and H be given groups and ϕ a homomorphism from C into Aut(H). Define multiplication of the ordered pairs $[c, h] \in C \times H$ by

$$[c, h][c_1, h_1] = [cc_1, h_1^{c\phi} h_1].$$

The resulting group of ordered pairs, G, is the *semi-direct product* of H by C depending on ϕ. It is also clear that $G = C \cdot H$, where $H \lhd G$ and $C \cap H = 1$, that is, G is a *split-extension* of H by C.

If G is a group and X a set, let G^X denote the *unrestricted direct power* of G taken $|X|$ times, that is, G^X is the set of all functions from X into G with co-ordinatewise multiplication, i.e., if $f_1, f_2 \in G^X$, then $f_1 \cdot f_2 = f_3$, where $f_1(x)f_2(x) = f_3(x)$ $(x \in X)$.

Finally, let G and H be groups and X a set on which H acts as a permutation group. The *wreath product* of G and H relative to X, denoted $GWr(H; X)$, is the semi-direct product $G^X H$, where the action of H on G^X is given by

$$f^h(x) = f(xh^{-1}) \quad (f \in G^X, h \in H, x \in X).$$

3. The epimorphism

Suppose $\mathfrak{A} = A \cup B|H$ is an amalgam, U and V are normal subgroups of A and B respectively, and $U \cap H = V \cap H$. Then a *factor amalgam*

$$\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V) = A/U \cup B/V|HU/U \tag{1}$$

can be formed by identifying HU/U and HV/V according to their natural isomorphisms with $H/H \cap U = H/H \cap V$. Write $N = H \cap U = H \cap V$.

In many places here transversals of the constituents A and B must be chosen which map onto transversals of A/U and B/V. These required transversals will always be assumed to be chosen as follows, unless stated otherwise in a specific theorem or section. A transversal of H in A is chosen by first choosing a transversal of H in HU consisting of elements of U, say $K = \{k_i \in U | i \in I\}$ and then a transversal of HU in A, say $S = \{s_j \in A | j \in J\}$. Then

$$SK = \{s_j k_i | i \in I, j \in J\} \tag{2}$$

and

$$S' = \{s_j U | j \in J\} \tag{3}$$

are the required transversals of H in A and of HU/U in A/U, respectively.

Similarly choose a transversal of H in B; let $L = \{l_i \in V | i \in I'\}$ be a transversal of H in HV, $T = \{t_j \in B | j \in J'\}$ a transversal of HV in B, and

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

$$TL = \{t_j l_i | i \in I', j \in J'\} \tag{4}$$

and

$$T' = \{t_j V | j \in J'\} \tag{5}$$

be transversals of H in B and HV/V in B/V, respectively.

Finally, let $P = P(\mathfrak{A}; SK, TL)$, $P' = P(\mathfrak{F}; S', T')$, $W = SK \times TL \times H$ and $W' = S' \times T' \times H/N$, where $\mathfrak{A}$ and $\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V)$ are given as above.

THEOREM 3.1. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam and $\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V)$ a factor amalgam. If transversals are chosen as in* (2) *to* (5), *then there exists an epimorphism f from $P = P(\mathfrak{A}; SK, TL)$ onto $P' = P(\mathfrak{F}; S', T')$.*

The kernel of f is the set of those $x \in P$ such that

$$x : (sk, tl, h) \to (sk', tl', hn) \tag{6}$$

where $s \in S$, $t \in T$, $k, k' \in K$, $l, l' \in L$, and $n \in U \cap H = N$.

PROOF. The function $g : W \to W'$ defined by $(sk, tl, h) \to (sU, tV, hN)$ is onto. It follows from the definitions of P and g that for each $a \in A$ and $b \in B$, $\rho(a)g = g\rho(aU)$ and $\rho(b)g = g\rho(bV)$. Thus, since P is generated by $\rho(A)$ and $\rho(B)$, to each $x \in P$ there corresponds a unique $f(x) \in P'$ such that

$$xg = gf(x); \tag{7}$$

$f(x)$ is unique because g is onto. The required epimorphism having the stated kernel is $f : P \to P'$, given by $x \to f(x)$.

The epimorphism $f : P \to P'$ will be referred to as the natural epimorphism from P to P' and if $x \in P$, then the image of x, $f(x)$, will be denoted by x'.

REMARK. If T_A and T_B are any transversals of H in A and B, respectively, which map onto transversals T'_A and T'_B of H/N in A/U and B/V, respectively, then the above proof also shows that there is an epimorphism f from $P(\mathfrak{A}; T_A, T_B)$ onto $P(\mathfrak{F}; T'_A, T'_B)$. (We do not use this here.)

4. $U = V \subseteqq H$

Throughout this section let $\mathfrak{A} = A \cup B|H$ be an amalgam and $\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V)$ a factor amalgam, where $U = V = N \subseteqq H$. Thus,

$$\mathfrak{F} = A/N \cup B/N|H/N.$$

Since the transversals S and T are now arbitrary transversals of H, take $K = L = \{1\}$, $P = P(\mathfrak{A}; S, T)$ and $P' = P(\mathfrak{F}; S', T')$.

Each $a \in A$ induces an automorphism a'' of N given by $n^{a''} = n^a$ $(n \in N)$, because N is normal in A. Suppose G is a group generated by $\mathfrak{A}$, and let

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

$A^* \cup B^*$ be the image of $\mathfrak{A}$ in G. Since N^* is again normal in G, there is an epimorphism of G onto the subgroup P'' of $\mathrm{Aut}(N)$ generated by the automorphisms induced on N by A and B. If $x = a^*b^* \cdots \in G$, then the induced automorphism, x'', is defined by $n^{x''} = n^{ab\cdots}$ $(n \in N)$, and the homomorphism from G onto P'' is given by $x \to x''$.

THEOREM 4.1. *Let $\mathfrak{A}$, $\mathfrak{F}$ and P'' be as above, where $N = U = V$. Let E denote the semi-direct product $P''N$. Then the permutational product $P = P(\mathfrak{A}; S, T)$ can be embedded in*

$$EWr(P'; W'),$$

where $P' = P(\mathfrak{F}; S', T')$ and $W' = S' \times T' \times H/N$.

PROOF. Let H' be a transversal of N in H and write $Y = S \times T \times H' \subseteqq W$. If

$$d' = (sN, tN, h'N) \in W',$$

write

$$[d'] = (s, t, h') \in Y.$$

Thus, if

$$d = (s, t, h) \in W = S \times T \times H,$$

then there exist unique elements $[d'] = (s, t, h') \in Y$ and $n \in N$, given by $h = h'n$, such that

$$d = [d']\rho(n). \tag{8}$$

It follows from (7) that if $x \in P$, then

$$[d']xg = [d']gx' = d'x'.$$

Thus, by (8), for each $x \in P$ and $d' \in W'$, there exists a unique $n_x(d') \in N$ such that

$$[d']x = [d'x']\rho(n_x(d')). \tag{9}$$

Let $n_x \in N^{W'}$ be defined by (9). Define g_x in $E^{W'}$ and h_x in $EWr(P'; W')$ by

$$g_x(d') = x''n_x(d')^{-1}, \ (d' \in W'), \text{ and } h_x = g_x x'. \tag{10}$$

Then

$$x \to h_x \tag{11}$$

defines the required embedding. To prove (11) defines a homomorphism, we proceed as follows. Let $x, y \in P$. Then

$$h_{xy} = g_{xy}x'y' \quad \text{and} \quad h_x h_y = g_x g_y^{(x')^{-1}} x'y'.$$

Thus

$$g_{xy}(d') = g_x(d')g_y^{(x')^{-1}}(d') \qquad (d' \in W') \tag{12}$$

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

must be verified. Note that if, say, $a \in A$, then $\rho(n)^{\rho(a)} = \rho(n^a)$; hence, if $y \in P$ and $n \in N$, then $\rho(n)^y = \rho(n^{y''})$. Using this remark and applying (9) repeatedly, it follows that

$$\begin{aligned}[][d'x'y']\rho(n_{xy}(d')) &= [d']xy \\ &= [d'x']\rho(n_x(d'))_y \\ &= [d'x']y\rho(n_x(d'))^y \\ &= [d'x'y']\rho(n_y(d'x'))\rho(n_x(d')^{y''}) \\ &= [d'x'y']\rho(n_y(d'x')n_x(d')^{y''}).\end{aligned}$$

Thus, for all $d' \in W'$,

$$n_{xy}(d') = n_y(d'x')n_x(d')^{y''},$$

so

$$[x''n_x(d')^{-1}][y''n_y(d'x')^{-1}] = (x''y'')n_{xy}(d')^{-1}.$$

Equation (12) follows immediately, proving that (11) defines a homomorphism. To show that (11) defines a monomorphism suppose $h_x = g_x x' = 1$, so $x' = 1$ and $g_x = 1$. Since $g_x = 1$, it follows that $n_x(d') = 1$, $(d' \in W')$, and $x'' = 1$, that is, x centralizes N. By (8) every element of W is of the form

$$[d']\rho(n) \quad (d' \in W', n \in N),$$

and

$$\begin{aligned}[][d']\rho(n)x &= [d']x\rho(n) \\ &= [d'x']\rho(n) \qquad (\text{by (9), since } n_x(d') = 1) \\ &= [d']\rho(n).\end{aligned}$$

Thus x is the identity permutation, completing the proof of the theorem.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups let $\mathfrak{X}\mathfrak{Y}$ be the class of all groups which are an extension of a group in $\mathfrak{X}$ by a group in $\mathfrak{Y}$.

If N, P'', and P' generate the varieties $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B}_3$, respectively, then Theorem 4.1 immediately shows that $P \in (\mathfrak{B}_1\mathfrak{B}_2)\mathfrak{B}_3$. More, in fact, can be shown.

COROLLARY 4.2. *Suppose N generates the variety $\mathfrak{B}_1$ and that P' and P'' in Theorem* 4.1 *together generate the variety $\mathfrak{B}$. Then $P \in \mathfrak{B}_1\mathfrak{B}$.*

PROOF. Let $\mathfrak{B}(P)$ be the verbal subgroup of P corresponding to $\mathfrak{B}$. Let v be a generator of $\mathfrak{B}(P)$, say $v = v(x_1, x_2, \cdots, x_n)$, $(x_i \in P)$, where $v(X_1, X_2, \cdots, X_n) = 1$ is a law of $\mathfrak{B}$. Since $P' \in \mathfrak{B}$, $\mathfrak{B}(P)$ is in the kernel of the natural homomorphism $f : P \to P'$. Thus, by (11), $v \to h_v = g_v$. But

$$g_v(d') = v''n_v(d')^{-1}, \ (d' \in W'),$$

and $P'' \in \mathfrak{B}$, so $v'' = 1$. Therefore $h_v \in N^{W'} \in \mathfrak{B}_1$; that is, $\mathfrak{B}(P) \in \mathfrak{B}_1$. It is clear that $P/\mathfrak{B}(P) \in \mathfrak{B}$ proving the corollary.

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

If $H = N$, it follows by Lemma 2.1 that $P' = A/H \times B/H$. Thus, if $H = N$, and A, B and P'' are all solvable, or finite p-groups, respectively, then $\mathfrak{A}$ can be embedded in a solvable group (Wiegold [13]) or a finite p-group (Higman [5]), respectively. We note that when $H = N$, the corollary shows the solvable length $l(P)$ of the embedding group P to be at most $l(H) + \max(l(P''), l(A/H \times B/H))$, improving the bound given by Wiegold [13].

COROLLARY 4.3. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam such that A and B are split-extensions of N, say $A = CN$, $B = DN$, and $H = (C \cap D)N$. If $S \subseteqq C$ and $T \subseteqq D$, then $P = P(\mathfrak{A}; S, T)$ is a split-extension of $\rho(N)$ by the subgroup of P generated by $\rho(C)$ and $\rho(D)$.*

PROOF. In the proof of Theorem 4.1, let $H' = (C \cap D)$. The transversals S and T are here chosen from the *subgroups* C and D, so by (9), if $c \in C$, $e \in D$ and $d' \in Y$, then $[d']\rho(c) = [d'c']$ and $[d']\rho(e) = [d'e']$. Therefore, if $x \in \langle \rho(C), \rho(D) \rangle$, then $n_x = 1$. If $x = \rho(n) \in \rho(N)$, then $n_x(d') = n$, $(d' \in W')$; hence, $\rho(N) \cap \langle \rho(C), \rho(D) \rangle = 1$. The corollary immediately follows.

COROLLARY 4.4. *Using the notation of Corollary 4.3., suppose C and D are direct complements of N in A and B, respectively. Then*

$$P(\mathfrak{A}; S, T) \cong P(\mathfrak{F}; S', T') \times \rho(N).$$

PROOF. It is clear that $P(\mathfrak{A}; S, T) = \langle \rho(C), \rho(D) \rangle \times \rho(N)$, since $\rho(C)$ and $\rho(D)$ permute elementwise with $\rho(N)$. If $x \in \langle \rho(C), \rho(D) \rangle$ such that $x' = 1$, then from the proof of Corollary 4.3, it follows that $[d']x = [d']$, $(d' \in W')$. But, if $d \in W$, then from (8),

$$\begin{aligned} dx &= [d']\rho(n)x \\ &= [d']x\rho(n) \\ &= [d']\rho(n) \\ &= d. \end{aligned}$$

Thus $x \to x'$ is an isomorphism from $\langle \rho(C), \rho(D) \rangle$ onto $P(\mathfrak{F}; S', T')$, as required.

Any amalgam $\mathfrak{A}$ of two finite groups can be embedded in a finite group, because any permutational product on $\mathfrak{A}$ is finite. A related question is: if H has finite index in both A and B, can $A \cup B|H$ be embedded in a group G such that H has finite index in G? B. H. Neumann showed by example that such an embedding group G may not exist [9]. In this example $H = N$ is normal in both constituents, the constituents are both split-extensions of H, and H is abelian (cf. Example 4.12). In this example P'' is not finite and this is a requirement for the proper kind of embedding group G to exist, for here $H \subseteqq C_G(H)$ and $P'' \subseteqq \mathrm{Aut}_G(H) \cong G/C_G(H)$, which would have to be finite. We partially answer this question with

COROLLARY 4.5. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam such that A and B are split-extensions of N, say $A = CN$, $B = DN$, and $H = (C \cap D)N$. Suppose P'' is finite and N has finite index in both A and B. If $S \subseteqq C$ and $T \subseteqq D$, then N has finite index in $P(\mathfrak{A}; S, T)$.*

PROOF. Here the hypotheses of Corollary 4.3 are satisfied, so $P(\mathfrak{A}; S, T)$ is a split-extension of $\rho(N)$. Furthermore, $P' = P(\mathfrak{F}; S', T')$ is finite since A/N and B/N are finite. It must therefore be shown that the index of $\rho(N)$ in the kernel of the natural homomorphism, $f : P \to P'$, is finite. Note that $\ker f \supseteqq \rho(N)$, so

$$\ker f = (\ker f \cap \langle \rho(C), \rho(D) \rangle)\rho(N).$$

Now $n_x = 1$ when $x \in \langle \rho(C), \rho(D) \rangle$, so if $x \in \langle \rho(C), \rho(D) \rangle \cap \ker f = F$, then $h_x = g_x \cdot 1$ and $g_x(d') = x'' \cdot 1$. Therefore,

$$[F : 1] = [\ker f : \rho(N)] \leqq [P'' : 1]$$

which is finite, completing the proof.

The assumptions of Corollary 4.5 are not necessary to embed $A \cup B|H$ in a group G such that H has finite index in G; it is, however, necessary that H contain a subgroup N which is normal and of finite index in both A and B. For a result on this embedding problem with different assumptions see Corollary 4.10.

I wish to thank Professor B. H. Neumann for refering me to his paper [9], thus suggesting the following results on periodic groups.

Refering to the notation of the proof of Theorem 4.1, any elements $a \in A$ and $b \in B$ can be written uniquely in the form

$$a = shn, \ b = th_1n_1 \quad (s \in S, t \in T, h, h_1 \in H', n, n_1 \in N).$$

Write $a^\gamma = n$, $b^\gamma = n_1$,

$$Q(a) = \{(sha)^\gamma | s \in S, h \in H'\}$$

and

$$R(b) = \{(thb)^\gamma | t \in T, h \in H'\}.$$

HYPOTHESIS 4.6. *For each $a \in A$ and $b \in B$ the sets $Q(a)$ and $R(b)$ are finite.*

COROLLARY 4.7. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of periodic groups. Then $P = P(\mathfrak{A}; S, T)$ is periodic, if any one of the following is true.*

(i) *Hypothesis 4.6 holds and both $P' = P(\mathfrak{F}; S', T')$ and P'' are periodic.*

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

(ii) *The group N has finite exponent and both P' and P'' are periodic.*
(iii) *The group N is finite and P' is periodic.*
(iv) *The index of N is finite in both A and B and P'' is periodic.*

PROOF. Hypothesis 4.6 merely states that if $x \in P$, then the function n_x (see (8) and (9)) has only a finite number of distinct values, $n_1, n_2, \cdots, n_k \in N$. If m is the least common multiple of the orders of the n_i, $i = 1, \cdots, k$, then $n_x^m = 1$. Thus, if (i) holds, h_x (and hence x) has finite order for all $x \in P$.

If (ii) holds, there is an integer m, $m =$ exponent of N, such that $n_x^m = 1$. Thus h_x has finite order.

If (iii) holds, then both N and P'' are finite, so the result follows from (ii).

If (iv) holds, then $E = P''N$ is periodic and P' and W' are finite. Thus $EWr(P'; W')$ is periodic.

We also note that if $\mathfrak{A}$ is an amalgam of groups of finite exponent and P' and P'' have finite exponent, then $EWr(P'; W')$ has finite exponent.

COROLLARY 4.8. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of locally finite groups A and B and suppose P'' is periodic. If N has countable index in both A and B, and finite index in H, then $P = P(\mathfrak{A}; S, T)$ is periodic. If, in addition, P'' is locally finite, then P will be locally finite.*

If the index of N in H is not finite, but $[A : N]$ and $[B : N]$ are countable, P' is periodic (locally finite) and P'' is periodic (locally finite), then P will be periodic (locally finite).

PROOF. It is implicit in Lemma 8.3 [9] that Hypothesis 4.6 holds. The first statement will follow from Corollary 4.7 if P' is periodic. If H/N is finite, then P' will be locally finite (Theorem 5.2 [9]), proving the first statement.

To prove the second statement it suffices to prove that the kernel of the natural homomorphism $f : P \to P'$ is locally finite (p. 153 [6]). Let $x_1, x_2, \cdots, x_k \in \ker f$. Then $h_{x_i} = g_{x_i}$, $i = 1, \cdots, k$. Since Hypothesis 4.6 holds all $Q(a)$ and $R(b)$ are finite, so each n_{x_i} can assume only a finite number of distinct values $n_{x_i}(d')$, $(d' \in W')$. (The values $n_{x_i}(d')$ are given, as in (8), by $(sh'x_i)^\gamma$.) Thus the h_{x_i} can assume only a finite number of distinct values in $E = P''N$ which is locally finite, so these values generate a finite subgroup E^* of E. Thus each $h_{x_i} \in (E^*)^{W'}$, which is locally finite (Lemma 5.4 [9]); hence the h_{x_i} generate a finite group, which was to be shown.

The last statement follows immediately from the fact that Hypothesis 4.6 holds.

COROLLARY 4.9. *Let $A \cup B|H = \mathfrak{A}$ be an amalgam of periodic groups* *(locally finite groups, groups of finite exponent) such that A and B are split-extensions, say $A = CN$, $B = DN$, and $H = (C \cap D)N$. If $S \subseteqq C$, $T \subseteqq D$, and P'' and $P(\mathfrak{F}; S', T')$ are both periodic (locally finite, of finite exponent), then $P(\mathfrak{A}; S, T)$ is periodic (locally finite, of finite exponent).*

Proof. Here $P(\mathfrak{A}; S, T)/\ker f$ and $\ker f/\rho(N)$ are both periodic (locally finite, of finite exponent) by Corollary 4.3 proving this corollary.

Corollary 4.10. *Let $A \cup B|H = \mathfrak{A}$ be an amalgam of locally finite groups A and B containing a normal subgroup N of both A and B. If N has finite index in both A and B, and P'' is locally finite, then there exists a group P embedding $\mathfrak{A}$ such that N has finite index in P.*

Proof. By Corollary 4.8, there exists a locally finite permutational product P on $\mathfrak{A}$. Choose a transversal of $\rho(N)$ in P from the subgroup of P generated by $\rho(S)$, $\rho(T)$ and $\rho(H')$. These are finite sets, hence the transversal of $\rho(N)$ is finite, as required.

Example 4.11. We now show that the hypothesis, 'P'' is locally finite', cannot be weakened to, 'P'' is periodic', in Corollary 4.10.

Let G be an infinite periodic p-group which is generated by two elements a and b. Such groups were shown to exist by Golod and Šafarevič [14]. Let $\phi : G \to \mathrm{Aut}(N)$ be a faithful representation of G where N is an abelian group; for example, let N be the direct sum of $|G|$ copies of the group of order 2, Z_2, and consider the right regular representation of G over the vector space N (over Z_2). Form the semi-direct products $A = \langle a\phi \rangle N$ and $B = \langle b\phi \rangle N$.

Since a and b have finite order in G, N has finite prime power index in both A and B. If R is a group generated by $A \cup B|N = \mathfrak{A}$, then, since $N \subseteqq C_R(N)$, there is a homomorphism from R/N onto $R/C_R(N) \cong G\phi = P''$ which is infinite and periodic. Thus no group R embeds the amalgam $\mathfrak{A}$ such that N has finite index in R.

Example 4.12. A similar argument shows that certain amalgams are embeddable only in their associated generalized free product.

Lemma 4.13. *Let $A = CN$ and $B = DN$ be split-extensions of the group N by C and D, respectively. Suppose $\mathfrak{A} = A \cup B|H$ is an amalgam such that $H = (C \cap D)N$. Then G is the generalized free product on $\mathfrak{A}$ if and only if G is a split-extension of N by the generalized free product on C and D with $E = C \cap D$ amalgamated.*

Proof. This follows immediately from the normal form for elements of a generalized free product (see [7]).

Let G be a generalized free product on $C \cup D|E$ and let $\phi : G \to \mathrm{Aut}(N)$ be a faithful representation of G with N abelian. Form the amalgam

$\mathfrak{A} = C\phi N \cup D\phi N | E\phi N$. Then as before, any group R generated by $\mathfrak{A}$ maps homomorphically onto $G\phi$, say $\psi : R \to R/C_R(N) \cong G\phi$, such that $C\phi$ and $D\phi$ in R map to $C\phi$ and $D\phi$ in $G\phi$, respectively. By the universal mapping property of generalized free products $G\phi$ maps onto the subgroup F of R generated by $C\phi$ and $D\phi$, say $\theta : G\phi \to R$. Since $\theta\psi|_F$ is the identity on $C\phi$ and $D\phi$, it must be the identity on $G\phi$. Similarly $\psi|_F\theta$ is the identity on F, so θ and $\psi|_F$ are isomorphisms, $R = FN$ and $F \cap N = 1$. By Lemma 4.13, R is a generalized free product. Finally, note that R must be unique (within isomorphism) since the automorphisms induced by F on N are always $G\phi$. (This example was suggested by a special case given in [16].)

5. N central in A

Recall (see Section 3) that in the general case, if $\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V)$ is a factor amalgam, then $N = U \cap H = V \cap H$ and $W = SK \times TL \times H$. Let H' be a set of coset representatives of N in H and

$$d = (sk, tl, h'n) \in W,$$

where $h' \in H'$ and $n \in N$. Since SK and TL are sets of coset representatives of H, if we set $a = skh'n$ and $b = tlh'n$, then d is uniquely determined by a and bH, and also by aH and b. Write $d = [a, bH) = (aH, b]$. If $a_1 \in A$ and $b_1 \in B$, then

$$[a, bH)\rho(a_1) = [aa_1, bH) \quad \text{and} \quad (aH, b]\rho(b_1) = (aH, bb_1].$$

Define permutations on W, $\lambda_1(a_1)$, $(a_1 \in A)$, and $\lambda_2(b_1)$, $(b_1 \in B)$, by

$$[a, bH)\lambda_1(a_1) = [a_1 a, bH) \quad \text{and} \quad (aH, b]\lambda_2(b_1) = (aH, b_1 b].$$

Evidently

$$\rho(a_1)\lambda_1(a_2) = \lambda_1(a_2)\rho(a_1), \quad (a_1, a_2 \in A) \tag{13}$$

and

$$\rho(b_1)\lambda_2(b_2) = \lambda_2(b_2)\rho(b_1), \quad (b_1, b_2 \in B). \tag{14}$$

LEMMA 5.1. *Let $\mathfrak{A} = A \cup B|H$. Using the preceeding notation, if N is contained in the centre $Z(A)$ of A, and $v \in V$, then $\lambda_2(v)$ commutes with every element of $P = P(\mathfrak{A}; SK, TL)$.*

PROOF. By (14) we need only verify

$$\rho(a)\lambda_2(v) = \lambda_2(v)\rho(a), \quad (a \in A).$$

We first introduce some notation; suppose H is a subgroup of A and S is a transversal of H in A. If $g = sh \in A$, $s \in S$, $h \in H$, write $g^\sigma = s$ and $g^{-\sigma+1} = h$. Now let

$$d = (sk, tl, h'n) \in W,$$

so

$$d = [a, bH) = (aH, b],$$

where

$$a = skh'n \quad \text{and} \quad b = tlh'n.$$

Then

$$\begin{aligned}[a, bH)\rho(a_1)\lambda_2(v) &= [aa_1, bH)\lambda_2(v)\\ &= (aa_1H, vb'],\end{aligned}$$

where

$$b' = tl(aa_1)^{-\sigma+1}.$$

Thus,

(15) $$(aa_1H, vb'] = \big((aa_1)^\sigma, (vb')^\sigma, (vb')^{-\sigma+1}\big).$$

Similarly,

(16) $$\begin{aligned}(aH, b]\lambda_2(v)\rho(a_1) &= [a'a_1, vbH)\\ &= \big((a'a)^\sigma, (vb)^\sigma, (a'a_1)^{-\sigma+1}\big),\end{aligned}$$

where

$$a' = sk(vb)^{-\sigma+1}.$$

To complete the proof of this lemma, the expressions in (15) and (16) must be shown to be equal. First note that $(vb')^\sigma = (vb)^\sigma$, for

$$\begin{aligned}(vb')^\sigma &= \big(vtl(aa_1)^{-\sigma+1}\big)^\sigma\\ &= (vtl)^\sigma\\ &= \big(vtl(h'n)\big)^\sigma\\ &= (vb)^\sigma.\end{aligned}$$

Note also that

$$(vtl)^{-\sigma+1} = (tlv^{tl})^{-\sigma+1} = (lv^{tl})^{-\sigma+1} \in V,$$

so $(vtl)^{-\sigma+1} \in V \cap H = N$, that is, $(vtl)^{-\sigma+1}$ is in the centre of A. Furthermore

$$(aa_1)^\sigma = (skh'na_1)^\sigma = (skh'a_1)^\sigma \qquad (n \in Z(A))$$

and

$$(a'a_1)^\sigma = (sk(vb)^{-\sigma+1}a_1)^\sigma.$$

Thus $(aa_1)^\sigma$ will equal $(a'a_1)^\sigma$, if

$$h'a_1H = (vb)^{-\sigma+1}a_1H.$$

But

$$\begin{aligned}(vb)^{-\sigma+1}a_1H &= (vtlh'n)^{-\sigma+1}a_1H\\ &= (vtl)^{-\sigma+1}(h'n)a_1H\\ &= h'a_1H \qquad \big((vtl)^{-\sigma+1}, n \in Z(A)\big),\end{aligned}$$

so

$$(aa_1)^\sigma = (a'a_1)^\sigma.$$

Finally,

$$\begin{aligned}(vb')^{-\sigma+1} &= \bigl(vtl(aa_1)^{-\sigma+1}\bigr)^{-\sigma+1}\\ &= (vtl)^{-\sigma+1}(aa_1)^{-\sigma+1}\\ &= (vtl)^{-\sigma+1}(skh'na_1)^{-\sigma+1}\\ &= \bigl(sk(vtl)^{-\sigma+1}(h'n)a_1\bigr)^{-\sigma+1}\\ &= \bigl(sk(vb)^{-\sigma+1}a_1\bigr)^{-\sigma+1}\\ &= (a'a_1)^{-\sigma+1}.\end{aligned}$$

This completes the proof of the lemma.

THEOREM 5.2. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam and $\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V)$ a factor amalgam, where V is a normal subgroup of B such that $V \cap H = N$ is in the centre of A and $U = N$. Then the permutational product $P = P(\mathfrak{A}; S, TL)$ can be embedded in $VWr(P'; W')$, where $P' = P(\mathfrak{F}; S', T')$ and $W' = S' \times T' \times H/N$.*

PROOF. Let H', Y and $[d']$, $(d' \in W')$, be as in Theorem 4.1. If

$$d = (s, tl, h'n) \in W,$$

define $v \in V$ by $ln = v^t$. Then

$$d = [d']\lambda_2(v), \tag{17}$$

because

$$tlh'n = vth' \qquad (n \in Z(A))$$

and

$$d = (sH, vth'] = (sH, th']\lambda_2(v)$$

as required.

Thus, for each $x \in P$ and $d' \in W'$, there exists a unique $v_x(d') \in V$ such that

$$[d']x = [d'x']\lambda_2(v_x(d')). \tag{18}$$

Define $v_x \in V^{W'}$ by (18) and let $r_x = v_x x'$.

LEMMA 5.3. *The function given by*

$$x \to r_x, \qquad (x \in P), \tag{19}$$

is a monomorphism from P into $VWr(P'; W')$.

PROOF. Note that by the definition of λ_2,

$$\lambda_2(b_1b_2) = \lambda_2(b_2)\lambda_2(b_1), \qquad (b_1, b_2 \in B).$$

Let $x, y \in P$. By (18) and Lemma 5.1, if $d' \in W'$, then

$$
\begin{aligned}
[d'x'y']\lambda_2(v_{xy}(d')) &= [d']xy \\
&= [d'x']\lambda_2(v_x(d'))y \\
&= [d'x'y']\lambda_2(v_y(d'x'))\lambda_2(v_x(d')).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
v_{xy}(d')x'y' &= v_x(d')v_y(d'x')x'y' \\
&= v_x(d')x'v_y(d')y',
\end{aligned}
$$

proving (19) defines a homomorphism.

If $r_x = v_x x' = 1$, then $v_x(d') = 1$, $(d' \in W')$, and $x' = 1$. Let $d = [d']\lambda_2(v) \in W$ and suppose $x \to 1$. Then

$$
\begin{aligned}
dx &= [d']x\lambda_2(v) \\
&= [d']\lambda_2(v_x(d'))\lambda_2(v) \\
&= d,
\end{aligned}
$$

because $\lambda_2(1) = 1$. Thus (19) defines a monomorphism, which was to be shown.

Several simple results follow immediately. Using the notation of Theorem 5.2, we note (i) if V and P' are in the varieties $\mathfrak{V}$ and $\mathfrak{V}_1$, respectively, then $P \in \mathfrak{V}\mathfrak{V}_1$; (ii) if V and P' are both locally finite (of finite exponent), then P is locally finite (of finite exponent); (iii) if V and P' are periodic and W' is finite, then P is periodic.

Since the case $N = H$ has been discussed in several other papers, we only give two corollaries here. The first slightly improves a bound on $l(P)$ given by Neumann [8] and the second concerns the splitting of the permutational product. We refer the interested reader to the papers of Bryce [17], Majeed [16] and, especially, Neumann [8], [9] and Wiegold [11], [12], for further results concerning this case.

Corollary 5.4. (Neumann [8]). *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of solvable groups of solvable lengths $l(A) = n$ and $l(B) = m$, respectively, and let B' be the derived group of B. If H is central in A, then $\mathfrak{A}$ can be embedded in a solvable permutational product P such that*

$$
l(P) \leqq l(A/B' \cap H) + l(B') \leqq n+m-1.
$$

Proof. Set $V = B'$ and $U = B' \cap H$. Then H/U is central both in A/U and in the abelian group B/B'. Let $P = P(\mathfrak{A}; S, TL)$. Then P' is the generalized direct product on $A/U \cup B/B'|H/U$ which is solvable of length $l(A/B' \cap H)$, so the result follows from Theorem 5.2.

Corollary 5.5. *In Theorem 5.2 suppose A and B are split-extensions $A = CN$ and $B = DV$, respectively, where $H = (C \cap D)N$. If $S \subseteqq C$ and $T \subseteqq D$, then $P = P(\mathfrak{A}; S, TL)$ is a split-extension of the kernel of the natural homomorphism $f: P \to P'$ by $E = \langle \rho(C), \rho(D) \rangle$.*

PROOF. Let $H' = C \cap D$. As in Corollary 4.3, if $x \in E$, then $[d']x = [d'x']$, $(d' \in W')$. Suppose now that $x \in \ker f$ and $[d']x = [d'x'] = [d']$. If $d \in W$, then $d = [d']\lambda_2(v)$ and

$$\begin{aligned} dx &= [d']\lambda_2(v)x \\ &= [d']x\lambda_2(v) && \text{(Lemma 5.1)} \\ &= d. \end{aligned}$$

Thus $E \cap \ker f = 1$ as required.

6. N central in both A and B

THEOREM 6.1. *Let $A \cup B|H$ be an amalgam and $\mathfrak{F}(\mathfrak{A}; U, V)$ a factor amalgam such that $N = U \cap H = V \cap H$ is a subgroup of the centres of both A and B. Then $P = P(\mathfrak{A}; SK, TL)$ can be embedded in*

$$DWr(P'; W'),$$

where $P' = P(\mathfrak{F}; S', T')$, $W' = S' \times T' \times H/N$, and D is the generalized direct product of U and V with N amalgamated.

PROOF. We shall pattern this proof after that of Theorem 5.2. Let W, H', Y, $\lambda_1(u)$, $(u \in U)$, and $\lambda_2(v)$, $(v \in V)$, be as in Theorem 5.2. Write $\lambda_1(U) = \{\lambda_1(u) | u \in U\}$.

LEMMA 6.2. *The group of permutations $\Lambda = \langle \lambda_1(U), \lambda_2(V) \rangle$ is isomorphic to the generalized direct product, D, of U and V with N amalgamated, when $N \subseteqq Z(A) \cap Z(B)$.*

PROOF. The function $\lambda_1(u) \to u^{-1}$ is an isomorphism from $\lambda_1(U)$ onto U; similarly, $\lambda_2(V) \cong V$. Thus it remains to be verified that

$$\lambda_1(u)\lambda_2(v) = \lambda_2(v)\lambda_1(u), \quad (u \in U, v \in V), \tag{20}$$

and

$$\lambda_1(U) \cap \lambda_2(V) = \lambda_1(N) = \lambda_2(N). \tag{21}$$

We sketch the proof using the notation of Lemma 5.1. Let $d = (sk, tl, h'n) \in W$. Then

$$d\lambda_1(u)\lambda_2(v) = \big((ua)^\sigma H, vtl(ua)^{-\sigma+1}\big] = d',$$

where $a = skh'n$, and

$$d\lambda_2(v)\lambda_1(u) = \big[usk(vb)^{-\sigma+1}, (vb)^\sigma H\big) = d'',$$

where $b = tlh'n$. Then

$$\begin{aligned} (ua)^\sigma &= (usk)^\sigma = \big(usk(vb)^{-\sigma+1}\big)^\sigma, \\ (vb)^\sigma &= (vtl)^\sigma = \big(vtl(ua)^{-\sigma+1}\big)^\sigma, \end{aligned}$$

and
$$\big(vtl(ua)^{-\sigma+1}\big)^{-\sigma+1} = (v^t l)^{-\sigma+1}(u^s k)^{-\sigma+1}h'n$$
$$= \big(usk(vb)^{-\sigma+1}\big)^{-\sigma+1}$$

since $(v^t l)^{-\sigma+1} \in N$, which is in the centre of A. Thus $d' = d''$ proving (20). To prove (21), note first that

$$\lambda_1(n) = \lambda_2(n), \qquad (n \in N).$$

Now let $x = \lambda_1(u) = \lambda_2(v)$, $u \in U$, $v \in V$. If d is as before,

$$dx = d\lambda_1(u) = \big((usk)^\sigma,\ tl,\ (usk)^{-\sigma+1}h'n\big)$$

and

$$dx = d\lambda_2(v) = (sk,\ (vtl)^\sigma,\ (vtl)^{-\sigma+1}h'n).$$

Comparing first co-ordinates of the expressions for dx, $sk(usk)^{-\sigma+1} = usk$, so

$$(usk)^{-\sigma+1} = u^{sk} \in H \cap U = N.$$

Thus,

$$(usk)^{-\sigma+1} = u.$$

Similarly,

$$(vtl)^{-\sigma+1} = v \in N.$$

Finally, comparing third co-ordinates, $u = v \in N$, completing the proof of the lemma.

If $d = (sk, tl, h'n) \in W$, let $u = k^{s^{-1}}n$, $v = l^{t^{-1}}$, and $\lambda^* = \lambda_1(u)\lambda_2(v) \in \Lambda$. Then

$$d = [d']\lambda^*. \tag{22}$$

If $\lambda_1^* = \lambda_1(u_1)\lambda_2(v_1) \in \Lambda$ and $d = [d']\lambda_1^*$, then a routine calculation using the methods of Lemma 6.2 shows that

$$u_1 = u[(v_1^t)^{-\sigma+1}]^{-1} \text{ and } v_1 = v[(v_1^t)^{-\sigma+1}];$$

that is, λ^* is a uniquely determined element of Λ.

Let $x \in P$ and $d \in W$. By (22) define $e_x \in D^{W'}$ by

$$e_x(d') = u_x(d')v_x(d') \in P,$$

where

$$[d']x = [d'x']\lambda_x^*(d')$$

and

$$\lambda_x^*(d') = \lambda_1\big(u_x(d')\big)\lambda_2\big(v_x(d')\big) \in \Lambda.$$

By Lemma 6.2 $e_x(d')$ is a uniquely determined element of D. Finally, set $r_x = e_x x'$.

LEMMA 6.3. *The function defined by*

$$x \to r_x, \ (x \in P), \tag{23}$$

is a monomorphism from P into $DWr(P'; W')$.

PROOF. The proof that (23) defines a monomorphism follows that of Lemma 5.3, so the details are omitted.

This completes the proof of Theorem 6.1.

Several simple results again follow from Theorem 6.1, but only one will be stated (without proof).

COROLLARY 6.4. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam where A and B are split-extensions, $A = CU$, $B = DV$, N is in the centres of both A and B, and $H = (C \cap D)N$. Then $P = P(\mathfrak{A}; SK, TL)$ is a split-extension of the subgroup $E = \langle \rho(C), \rho(D) \rangle$ by the kernel of the natural homomorphism $f : P \to P' = P(\mathfrak{F}; S', T')$, whenever $S \subseteqq C$ and $T \subseteqq D$. If U and V are direct factors, so is ker f.*

The following result is presented as a simple example of how the results of sections 4 and 6 may be applied to embedding theorems.

THEOREM 6.5. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of solvable groups A and B such that H is normal in both A and B. Then $\mathfrak{A}$ can be embedded in a solvable group if and only if A and B have series of normal subgroups $U_0 = 1 \subseteqq U_1 \subseteqq \cdots \subseteqq U_n = A$ and $V_0 = 1 \subseteqq V_1 \subseteqq \cdots \subseteqq V_n = B$ such that U_{i+1}/U_i and V_{i+1}/V_i are abelian, $N_i = U_i \cap H = V_i \cap H$, and A/U_i and B/V_i induce a solvable group of automorphisms P_i'' on N_{i+1}/N_i for $i = 0, 1, 2, \cdots, n-1$.*

PROOF. One way is clear. Let A and B be solvable groups having series of normal subgroups of length n as in the statement of the theorem. We always assume the amalgam is proper, that is, $A \neq H$, $B \neq H$.

If A and B are abelian, then any permutational product on the amalgam will be abelian by Lemma 2.1, so suppose the result is true whenever A and B have normal series of length less than n. Thus the factor amalgam $\mathfrak{F} = A/U_1 \cup B/V_1|H/N_1$ generates a solvable permutational product $P^* = P(\mathfrak{F}; S^*, T^*)$ for some choice of S^* and T^*.

Let K_1 and L_1 be transversals of N_1 in U_1 and V_1, respectively. With respect to U_1 and V_1 choose transversals SK and TL as in (2)—(5) such that the image SU_1/U_1 of S is S^* and $TV_1/V_1 = T^*$. Let $\mathfrak{F}_1 = A/N_1 \cup B/N_1|H/N_1$. Then $P' = P(\mathfrak{F}_1; S'K', T'L')$ is solvable by Theorem 6.1, for it can be embedded in a solvable group of the form $DWrP^*$, where D is the direct product of U_1N_1/N_1 and V_1N_1/N_1. Finally, the group $P(\mathfrak{A}; SKK_1, TLL_1)$ is solvable since it can be embedded in a solvable group of the form $P_1''N_1WrP'$ by Theorem 4.1. The theorem follows by induction on n.

I thank Dr R. B. J. T. Allenby for the following (unpublished) result which suggested Theorem 6.7.

THEOREM 6.6. (Allenby). *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of metabelian groups such that H is normal in both A and B and P'' is abelian. If the derived*

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

subgroups of A and B are contained in H, then $\mathfrak{A}$ is embeddable in a metabelian group.

Proof. If P'' is abelian, then $A'B'$ is a subgroup of the centre $Z(H)$ of H. Let $N = Z(H)$. Since A/N and B/N are abelian groups P' is abelian, and the result follows by Corollary 4.2.

Theorem 6.7. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of metabelian groups. If $N = (A' \cap H)(B' \cap H)$ is in the centres of both A and B, then $\mathfrak{A}$ can be embedded in a metabelian group.*

Proof. The abelian groups $U = A'N$ and $V = B'N$ have the same intersection N with H and are normal subgroups of A and B, respectively. By Theorem 6.1. $P(\mathfrak{A}; SK, TL)$ is a metabelian group, since A/U and B/V, and thus P', are abelian.

Example 6.8. James Wiegold [10] has given an example of an amalgam $A \cup B|H$ of the Pruefer group of type 2^∞ with the dihedral group of order 8 which is embeddable in no nilpotent group. In this example Wiegold lets

$$A = gp(a_1, a_2, \cdots, a_n, \cdots; a_1^2 = 1, a_{n+1}^2 = a_n, n = 1, 2, \cdots),$$
$$B = gp(c, d; c^4 = d^2 = (cd)^2 = 1) \text{ and}$$
$$H = gp(c), \text{ where } c = a_2.$$

This amalgam satisfies the conditions of Theorem 6.5 if we take $U_0 = V_0 = 1$, $V_1 = gp(c^2)$, $U_1 = V_1 \cap H$, $U_2 = A$, and $V_2 = B$. Hence the embedding group of Theorem 6.5 need not be nilpotent when A and B are.

Wiegold [11] has also given examples of cyclic groups of order 2^n amalgamated with certain generalized dihedral groups which are only embeddable in nilpotent groups of class $C_n \geqq f(n)$ where $f(n) \to \infty$ as $n \to \infty$. These examples together with Examples 6.8 give rise to the following remarks.

Remark 6.9. Let $\mathfrak{A} = A \cup B|H$ be an amalgam of groups which are each unions of ascending chains of subgroups, say $A = \bigcup_{i=1}^\infty A_i$, $B = \bigcup_{j=1}^\infty B_j$, where $A_i \subseteqq A_{i+1}$ and $B_j \subseteqq B_{j+1}$, for all positive integers i, j. Suppose for all i, j, $A_i \cap B_j = H$. Suppose also the amalgams $\mathfrak{A}_{ij} = A_i \cup B_j|H$ can be embedded in groups G_{ij} such that when $i_0 \leqq i$ and $j_0 \leqq j$, the subgroup of G_{ij} generated by $\mathfrak{A}_{i_0 j_0}$ is isomorphic to $G_{i_0 j_0}$ and the restriction of this isomorphism to $\mathfrak{A}_{i_0 j_0}$ is the identity. Then an ascending union of the G_{ij}, $G = \operatorname{Lim}_{\rightarrow} G_{ij}$, will embed $A \cup B|H$ (here $i_0 j_0 \leqq ij$ if and only if $i_0 \leqq i$ and $j_0 \leqq j$). Furthermore, G is solvable or nilpotent of length at most m if and only if each G_{ij} is solvable or nilpotent of length at most m. The following theorem shows such embedding groups G_{ij} can always be found.

Theorem 6.10. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam and $\mathfrak{A}_1 = A_1 \cup B_1|H$*

a subamalgam of $\mathfrak{A}$. Choose transversals K of H in A_1, S of A_1 in A, L of H in B_1 and T of B_1 in B. Then there is an isomorphism from $P_1 = P(\mathfrak{A}_1; K, L)$ onto the subgroup of $P = P(\mathfrak{A}; SK, TL)$ generated by $\rho(A_1)$ and $\rho(B_1)$ which acts as the identity when restricted to $\mathfrak{A}_1$.

PROOF. If $\rho(a_1) \in \rho(A_1) \subseteqq P$, then $(sk, tl, h)\rho(a_1) = (sk_1, tl, h_1)$, where $kha_1 = k_1h_1$ and this last equation is precisely the same equation arising by calculating $\rho(a_1)$ in P_1. If

$$x = \rho(a_1)\rho(b_2) \cdots \rho(a_n) \in \langle \rho(A_1), \rho(B_1) \rangle \subseteqq P,$$

then
$$x \to x' = \rho(a_1)\rho(b_2) \cdots \rho(a_n) \in P_1$$

is the required isomorphism.

Thus, in Remark 6.9, choose transversals S and T of H in A and B to be $S = \prod_{i=1}^{\infty} S_i$ and $T = \prod_{i=1}^{\infty} T_i$, where S_i is a transversal of A_{i-1} in A_i, $A_0 = H$, and T_i is similarly chosen. Then $P(\mathfrak{A}; S, T)$ is the ascending union of the subgroups $P(\mathfrak{A}_{ij}; S_i, T_j)$ by Theorem 6.10, proving the existence of the G_{ij} required in Remark 6.9. This remark also explains why the calculations in Lemma 8.3 [9] are natural and could be used to give a somewhat more direct proof of Corollary 4.8.

Finally, using the methods of Theorem 6.5 the following results of Higman [5] follow.

THEOREM 6.11. (Higman [5]) *Let $A = A \cup B|H$ be an amalgam of nilpotent groups. If there are central series*

$$U_0 = 1 \subseteqq U_1 \subseteqq \cdots \subseteqq U_n = A \text{ and } V_0 = 1 \subseteqq V_1 \subseteqq \cdots \subseteqq V_n = B$$

of A and B, respectively, such that $U_i \cap H = V_i \cap H$, $i = 0, 1, \cdots, n$, then $\mathfrak{A}$ can be embedded in a solvable group P.

THEOREM 6.12 (Higman [5]). *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of finite p-groups. The amalgam is embeddable in a finite p-group P if and only if there are chief series of A and B, say $\{U_i\}$ and $\{V_j\}$, respectively, such that $\{U_i \cap H\} = \{V_j \cap H\}$.*

7. Finite nilpotent groups

The results here can be derived from the foregoing results on permutational products, but the proofs here are based on the following very general result of Wiegold.

THEOREM 7.1. (Wiegold [11]) *Let $A_\alpha \cup B_\alpha|H_\alpha$ be an amalgam of two groups which is embeddable in a group of a variety $\mathfrak{B}$ for each α in some index set M. Then if $A = \times A_\alpha$, $B = \times B_\alpha$, $H = \times H_\alpha$ are the restricted direct products of the A_α, B_α, and H_α respectively, then the amalgam $A \cup B|H$ is also embeddable in some group of the variety $\mathfrak{B}$.*

Theorem 7.2. *Let A and B be finite nilpotent groups. The amalgam $\mathfrak{A} = A \cup B|H$ is embeddable in a finite nilpotent group G if and only if A and B have chief series (A_j) and (B_k), respectively, such that*

$$(A_j) \cap H = (B_k) \cap H.$$

Proof. Let $p_1, \cdots, p_r$ be the distinct primes dividing the orders of A and B. The groups A, B and H are the direct products of their Sylow p_i-subgroups, A_i, B_i and H_i, respectively. If (A_j) and (B_k) are chief series of A and B such that $(A_j) \cap H = (B_k) \cap H$, then, since A_i/A_{i+1} and B_r/B_{r+1} are cyclic of prime order, $(A_j) \cap A_i$ and $(B_k) \cap B_i$ are chief series of A_i and B_i, satisfying the conditions of Higman's Theorem (Theorem 6.12). Thus $A_i \cup B_i|H_i$ is embeddable in a p_i-group G_i for all $i = 1, \ldots, r$. Hence $\mathfrak{A}$ is embeddable in the direct product $\times_{i=1}^r G_i$ which is nilpotent.

Conversely, if $\mathfrak{A}$ is embeddable in the nilpotent group G, then any chief series of G, say (G_j), determines the necessary chief series in A and B, $(A_j) = A \cap (G_j)$ and $(B_j) = B \cap (G_j)$, completing the proof.

Corollary 7.3. *If A and B are finite nilpotent groups and H is cyclic, then $A \cup B|H$ is embeddable in a nilpotent group.*

Proof. As Higman points out [5], a cyclic p-group has one chief series, so the conditions of Theorem 7.2 must hold.

Theorem 7.4. *If H is a normal subgroup of both the finite nilpotent groups A and B, then $\mathfrak{A} = A \cup B|H$ is embeddable in a finite nilpotent group if and only if the group P'' of automorphisms induced by A and B on H is nilpotent.*

Proof. One way is clear. Again we assume A, B and H are direct products of their Sylow p_i-subgroups A_i, B_i, and H_i, for distinct primes p_i, $i = 1, \cdots, r$. The H_i are characteristic subgroups of H, so

$$\mathrm{Aut}\,(H) = \mathop{\times}_{i=1}^{r} \mathrm{Aut}\,(H_i).$$

Furthermore $A \cup B|H$ is embeddable in a nilpotent group if and only if $A_i \cup B_i|H_i$ is embeddable in a p_i-group, $i = 1, \cdots r$, by Theorem 7.1. We now show that A_i and B_i induce a p_i-group P_i'' on H_i, $i = 1, \cdots, r$, and the result will follow by Theorem 4.1. Let $\mathrm{Aut}_A\,(H)$ denote the automorphisms induced by A on H.

Now $\mathrm{Aut}_A\,(H) = \times_{i=1}^r \mathrm{Aut}_A\,(H_i)$ and if $x \in A$, then $x = ay$, where $a \in A_i$ and the order of y is prime to p_i. Thus y commutes with every element of A_i, and x induces the same automorphism as a. Hence, for each i, $\mathrm{Aut}_A\,(H_i) = \mathrm{Aut}_{A_i}\,(H_i)$, so $\mathrm{Aut}_A\,(H) = \times_{i=1}^r \mathrm{Aut}_{A_i}\,(H_i)$; similarly, $\mathrm{Aut}_B\,(H) = \times_{i=1}^r \mathrm{Aut}_{B_i}\,(H_i)$. Thus P'' is the direct product of the subgroups

P_i'' of Aut (H_i), because $\mathrm{Aut}_{A_i}(H_i) \subseteqq \mathrm{Aut}(H_i)$ and $\mathrm{Aut}_{B_i}(H_i) \subseteqq \mathrm{Aut}(H_i)$, $i = 1, \cdots, r$.

If P'' is a finite nilpotent group, then each P_i'' is a nilpotent group generated by the p_i-groups $\mathrm{Aut}_{A_i}(H_i)$ and $\mathrm{Aut}_{B_i}(H_i)$, so P_i'' is a p_i-group, $i = 1, \cdots, r$. Hence the amalgams $A_i \cup B_i | H_i$, $i = 1, \cdots, r$ are embeddable in a p_i-group as required.

8. The pull apart property

We now discuss a simple embedding result (Corollary 8.5) which does not seem to have appeared elsewhere. This section will also indicate uses to which amalgam embedding theorems can be put. Let $A\pi B|H$ denote the generalized free product associated with $A \cup B|H$.

Let Q be a group property. The group G is *residually* Q if and only if for each $g \in G$, $g \neq 1$, there is a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g has property Q.

THEOREM 8.1. (*see* Baumslag [2]). *Let L be a group property such that if G is an L-group, then every subgroup of G is an L-group. If the amalgam $A \cup B|H$ can be embedded in an L-group G, and $N \subseteqq H$, then $S = A\pi B|N$ is an extension of a free group by an L-group.*

DEFINITION 8.2. [4]. A property Q is a *root property*, if:

(1) if a group is Q, then so also is every subgroup.

(2) if G and H have Q, then so also has the direct product $G \times H$,

(3) if $G \geqq H \geqq K \geqq 1$ is a series of subgroups, each normal in its predecessor, and G/H, H/K are Q, then K contains a subgroup L, normal in G, such that G/L is Q.

Let us note that if Q is any property satisfying part (3) of the definition of root properties, and H is a normal subgroup of G such that G/H is Q and H is residually Q, then G is residually Q [4, Lemma 1.5], and if Q is a root property, then every free product of residually Q groups is itself residually Q if and only if every free group is residually Q [4].

Using these remarks together with Theorem 8.1, we easily have the following simple but useful, corollaries.

COROLLARY 8.3. *Suppose A and B are Q-groups, where Q is a root property such that every free group is residually Q. Suppose the amalgam $A \cup B|H$ can be embedded in a group having Q and $N \subseteqq H$. Then $A\pi B|N$ is residually Q.*

COROLLARY 8.4. *Let Q be a root property. Suppose every free group is residually a Q-group (residually a finite Q-group). If A and B are finite Q-groups, then $A\pi B|H$ is residually a Q-group (residually a finite Q-group)*

https://doi.org/10.1017/S1446788700006947 Published online by Cambridge University Press

if and only if the amalgam $A \cup B|H$ is embeddable in a Q-group (finite Q-group).

Corollary 8.5. *Let A and B be finite Q-groups, where Q is as in Corollary 8.4. If $A \cup B|H$ is embeddable in a Q-group (finite Q-group) and $N \subseteqq H$, then $A \cup B|N$ is embeddable in a Q-group (finite Q-group).*

Proof. Since $A \cup B|H$ is embeddable in a (finite) Q-group, $A\pi B|N$ is residually a (finite) Q-group by Corollary 8.3. But $A \cup B|N$ is embeddable in a (finite) Q-group by Corollary 8.4.

Intuitively, if $A \cup B|H$ is embeddable in a finite Q-group, then as A and B are 'pulled apart', so as to have a smaller common subgroup, the new amalgam of A and B is again embeddable in a finite Q-group.

The above theorem includes the cases when Q is 'solvable', 'finite' and 'finite p-group', but not 'finite nilpotent', which can be done as follows.

Let $\mathfrak{X}$ be a given abstract class of groups. (That is, all isomorphic copies of any group in $\mathfrak{X}$ are also in $\mathfrak{X}$.) We shall say that $\mathfrak{X}$ has the pull apart property, if, whenever $A \cup B|H$ is embeddable in a group in $\mathfrak{X}$, then $A \cup B|N$ is also embeddable in a group in $\mathfrak{X}$ for any subgroup $N \subseteqq H$.

Theorem 8.6. *The class of groups $\mathfrak{X}$ has the pull apart property if and only if for any group A in $\mathfrak{X}$ and any isomorphic copy A^* of A the amalgam $A \cup A^*|H$ is embeddable in a group in $\mathfrak{X}$, where H is any subgroup of A amalgamated with its image $H^* \subseteqq A^*$.*

Proof. Suppose $\mathfrak{X}$ has the pull apart property and let A be in $\mathfrak{X}$. Then we may consider A as embedding the amalgam $A \cup A^*|A$ and thus $A \cup A^*|H$ is embeddable in a group in $\mathfrak{X}$.

Conversely, suppose any amalgam $A \cup A^*|H$ is embeddable in a group in $\mathfrak{X}$ and that $A \cup B|H$ is embeddable in a group G in $\mathfrak{X}$. Let $N \subseteqq H$. If G^* is any isomorphic copy of G, then $G \cup G^*|N$ is embeddable in a group G^{**} in $\mathfrak{X}$ and since G^{**} embeds $A \cup B^*|N$ (where $N = N^*$) which is amalgam isomorphic to $A \cup B|N$, we are through.

Thus in view of Theorems 8.6 and 7.4, the pull apart property also holds for the class of finite nilpotent groups.

Suppose now that Q is a property such that every Q-group is finite. One might ask if the above 'pull apart property' holds for residually Q-groups, i.e., if $A \cup B|H$ is embeddable in a residually Q-group and $N \subseteqq H$, must $A \cup B|N$ be embeddable in a residually Q-group?

The following example shows this is not the case.

Let $gp(a)$ and $gp(b)$ be infinite cyclic groups. Consider the amalgam $gp(a) \cup gp(b)|a^p = b^q$, where the subgroup $\langle a^p \rangle$ generated by a^p is identified with the subgroup $\langle b^q \rangle$ by the isomorphism given by $a^p \leftrightarrow b^q$. Here p and q are distinct odd primes. The above amalgam can be embedded by the infinite cyclic group $G = gp(a, b, |a^p = b^q, ab = ba)$, in the natural way. (The group G is generated by $a^v b^u$, where u and v are integers such that $up+vq = 1$.)

Hence the above amalgam is embeddable in a residually finite p-group.

Now consider $\langle a^{pq}\rangle \subsetneqq \langle a^p\rangle \subseteqq gp(a)$. We shall show that the amalgam $\mathfrak{A} = gp(a) \cup gp(b)|a^{pq} = b^{q^2}$ is not embeddable in any residually finite p-group.

Suppose the contrary, that $\mathfrak{A}$ can be embedded in some group G which is residually a finite p-group, and recall that $[x, y] = x^{-1}y^{-1}xy$. Consider the element s of G, $s = [b^{2q}, a^{pq-2}]$.

Either $s = 1$ or $s \neq 1$.

Case 1. $s \neq 1$. Since G is residually a finite p-group, we can choose a normal subgroup N of index p^i in G, $i \geqq 1$, such that s is not an element of N. Since $(p, q) = 1$, we can choose integers u, v such that $1 = qu+p^{i+1}v$, so

$$q = q^2u+p^{i+1}qv. \tag{24}$$

Note that if θ is the natural homomorphism from S to S/N, then $\theta(b^{2q}) = b^{2uq^2}N$ by the above equation (24) and the fact that N has index p^i in G, so $\theta(b^{2q}) = a^{2pqu}N$.

Hence

$$\begin{aligned}\theta(s) &= [\theta(b^{2q}), \theta(a^{pq-2})]\\ &= [a^{2pqu}N, a^{pq-2}N] = N, \quad \text{so} \quad s \in N,\end{aligned}$$

contrary to hypothesis. Therefore, if G is residually a finite p-group, we must have case 2.

Case 2. $s = 1$. Now b^{2q} and a^{pq-2} commute, and $a^{pq-2} = b^{q^2} \cdot a^{-2}$, so b^{2q} and a^2 commute.

If we could show that $s = 1$ forces G to have an element x of order q or q^2, then G couldn't be residually a finite p-group, because any non-trivial image of x in a factor group of G would have order q or q^2. Now

$$(a^{2p}b^{-2q})^{q^2} = a^{2pq^2}b^{-2q^3} = b^{2q^3}b^{-2q^3} = 1,$$

because a^{2p} and b^{-2q} commute, and $a^{pq} = b^{q^2}$. The above remark shows that $a^{2p}b^{-2q}$ can't be of order q or q^2. Therefore $a^{2p} = b^{2q}$. But a^{2p} and b^{2q} are not in the amalgamated subgroup, because p and q are odd numbers, so if $a^{2p} = b^{2q}$ the amalgam $\mathfrak{A}$ is not embedded in G, a contradiction. Hence, $\mathfrak{A}$ is not embeddable in a group which is residually a finite p-group, which was to be shown.

For further results on the residual properties of generalized free products see [2] and [3].

Before closing we ask a question. Allenby [15] has shown that every permutational product is a *generalized regular product on* $\mathfrak{A}$, that is, if $\psi(S, T)$ is the natural homomorphism from $F = A\pi B|H$ onto $P(\mathfrak{A}; S, T)$ where $\mathfrak{A} = A \cup B|H$, and S and T are arbitrary transversals of H, then the kernel of $\psi(S, T)$ is in the *cartesian* subgroup C, $C = \langle [a, b] | a \in A, b \in B \rangle$, of F. For any given amalgam $\mathfrak{A}$, can S and T be chosen such that ker $\psi(S, T)$ is maximal with respect to being a normal subgroup of F contained in the cartesian C? In other words, is at least one permutational product on $\mathfrak{A}$ a minimal generalized regular product on $\mathfrak{A}$?

I thank Dr Allenby for sending me a copy of an unpublished result due to Graham Higman, which states that an amalgam $A \cup B|H$ can be embedded in a standard wreath product similar to that used in Theorem 4.1, when H is normal in both A and B. This in turn suggested a special case of Theorem 4.1 in an earlier version of this paper. Originally the rest of the results on permutational products were proven directly from Theorem 3.1. The present unified treatment, considering permutational products as subgroups of permutational wreath products is due to the referee, to whom we acknowledge our indebtedness and offer our thanks.

References

[1] R. B. J. T. Allenby, '*Generalized regular products of groups*', Ph. D. Thesis, University of Wales, 1966.

[2] G. Baumslag, 'On the Residual Finiteness of Generalized Free Products of Nilpotent Groups', *Trans. Amer. Math. Soc.* 106 (1963), 193—209.

[3] R. J. Gregorac, 'On the Residual Properties of Generalized Free Products', unpublished.

[4] K. Gruenberg, 'Residual Properties of Infinite Soluble Groups', *Proc. London Math. Soc.* (3) 7 (1957), 29—62.

[5] G. Higman, 'Amalgams of p-Groups', *J. Algebra*, 1, (1964), 301—305.

[6] A. Kurosh, *The Theory of Groups*, vol. II, (Chelsea Publishing Co., 1960).

[7] B. H. Neumann, 'An Essay on Free Products of Groups with Amalgamations', *Phil. Trans. Royal Soc. of London* (A) 246 (1954), 503—554.

[8] B. H. Neumann, 'Permutational Products of Groups', *J. Austral. Math. Soc.* 1 (1960) 299—310.

[9] B. H. Neumann, 'On Amalgams of Periodic Groups', *Proc. Roy. Soc.* (A) 255 (1960), 477—489.

[10] J. Wiegold, 'Nilpotent products of groups with amalgamations', *Publ. Math. Debrecen* 6 (1959), 131—168.

[11] J. Wiegold, 'Some Remarks on Generalized Products of Groups with Amalgamations', *Math. Zeitschr.* 75, (1961), 57—78.

[12] J. Wiegold, 'Embedding Group Amalgams in Wreath Products', *Math. Zeitschr.* 80 (1962), 148—153.

[13] J. Wiegold, 'Soluble Embeddings of Group Amalgams', *Publ. Math. Debrecen* 12 (1965), 227—230.

[14] E. S. Golod and I. R. Šafarevič, 'On the class field tower' [Russian]. *Izvestiya akad. Nauk. S.S.S.R.*, Ser. Mat. 28 (1964), 261—272.

[15] R. B. J. T. Allenby, 'Permutational Products and Regular Products of Groups', *Trans. Amer. Math. Soc.*, **136** (1969), 537—543.
[16] Abdul Majeed, *'Permutational Products of Groups and Embedding theory of Group Amalgams'*, thesis, Australian National University, Canberra, 1966.
[17] R. A. Bryce, *'Some Results concerning Group Amalgams and their Free Products'*, thesis University of Queensland, 1964.

Iowa State University
U.S.A.
and
The Australian National University
Canberra