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A FARTHEST-POINT CHARACTERISATION OF THE RELATIVE CHEBYSHEV CENTRE

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We characterise the relative Chebyshev centre of a compact subset F of a real Banach space in terms of the Gateaux derivative of the distance to farthest points. We present a relative-Chebyshev-centre characterisation of Hilbert space. In Hilbert space we show that the relative Chebyshev centre is in the closed convex hull of the metric projection of F, and we estimate the relative Chebyshev radius of F.

Suppose X is a real Banach space and $K \subset X$ is closed and convex. If F is a compact subset of X, let

$$r_K(F) := \inf_{u \in K} \sup_{f \in F} \|u - f\|$$

and

$$\mathcal{Z}_K(F) := \{w \in K : \sup_{f \in F} \|w - f\| = r_K(F)\}.$$

We call $r_K(F)$ (respectively, $\mathcal{Z}_K(F)$) the *K-relative Chebyshev radius* (respectively, *centre*) of F. The X-relative Chebyshev centre is called the *Chebyshev centre* of F and is denoted by $\mathcal{Z}(F)$. The definition of the Chebyshev radius and centre, and the initial study thereof, are due to Garkavi [**4, 5**]. If F consists of a single point f, we shall denote $\mathcal{Z}_K(F)$ by $\mathcal{P}_K(f)$. In this case $\mathcal{P}_K$ is called the *metric projection* of X onto (the power set of) K and each element of $\mathcal{P}_K(f)$ is called a *best approximation* to f from K. If $A \subset X$, we shall denote by $\mathcal{P}_K(A)$ the set $\bigcup_{f \in A} \mathcal{P}_K(f)$. Given $u \in K$, let $\mathcal{Q}_F(u)$ consist of all points in F farthest from u, that is,

$$\mathcal{Q}_F(u) := \{f \in F : \|f - u\| = \sup_{g \in F} \|g - u\|\}.$$

Given $f, g \in X$ define

$$\psi_{f,g}(t) := \frac{\|f + tg\| - \|f\|}{t}$$

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It is shown in [9] that the function $\psi_{f,g}$ is nondecreasing and bounded below on $(0,\infty)$. The (one-sided) *Gateaux derivative* of the norm at f in the direction g is defined by

$$\tau_+(f,g) := \lim_{t\downarrow 0} \psi_{f,g}(t).$$

Our principal result combines characterisations found in the work of Pinkus [9] and Amir and Ziegler [1]. Theorem 1.6 in [9] states that $f^* \in \mathcal{P}_K(f)$ if and only if $\tau_+(f-f^*, f^*-g) \geqslant 0$ for every $g \in K$. Lemma 2.1 in [1] implies that if K is a linear subspace, w is in the K-relative Chebyshev centre of $\{f_1, f_2\}$ and f_1 is farther from w than is f_2 (that is, $\|f_1 - w\| > \|f_2 - w\|$), then w is a best approximation to f_1 from K. The inspiration for the present paper was the realisation that the conclusion of the last sentence is equivalent to the following statement: w is in the K-relative Chebyshev centre of the set of vectors in F which are farthest from w. The following lemma develops this insight in the case where F is not necessarily finite. It also builds on previous work incorporating Pinkus' derivative characterisation into the study of the relative Chebyshev centre, including that of [6] (which combined the characterisations in [9] and [1] in the case where X is smooth and K is a subspace of X) and [7] (where the requirements that X be smooth and K be linear were dropped).

Lemma 1. *Suppose X is a real Banach space, $K \subset X$ is closed and convex, $F \subset X$ is compact, and $w \in K$. The vector w is an element of $\mathcal{Z}_K(F)$ if and only if for every $h \in K$ there is an $f_h = f_h(w) \in \mathcal{Q}_F(w)$ such that*

$$\tau_+(f_h - w, w - h) \geqslant 0.$$

Proof: Suppose $w \in \mathcal{Z}_K(F)$ and $h \in K$. Then for every $t \in (0,1)$ and $g \in \mathcal{Q}_F(w)$,

$$\|g - w\| = \sup_{f\in F} \|f - w\| \leqslant \sup_{f\in F} \|f - ((1-t)w + th)\|$$

so, by the compactness of F and the continuity of the norm, for each natural number k there exists an $f_h^k \in F$ such that

$$\|f_h^k - w\| \leqslant \|g - w\| \leqslant \left\| f_h^k - w - \frac{1}{k}(h - w)\right\|. \tag{1}$$

We may suppose without loss of generality that there exists an $f_h \in F$ such that $f_h^k \to f_h$ as $k \to \infty$. Since $g \in \mathcal{Q}_F(w)$, it must be that $f_h \in \mathcal{Q}_F(w)$ also.

Fix $t \in (0,1)$. Let $\varepsilon > 0$ be given. Choose a natural number N such that for every $k > N$, $1/k < t$ and $\|f_h^k - f_h\| < \varepsilon$. Then

$$\begin{aligned}\|f_h - w - t(h-w)\| &\geqslant \left\|f_h^k - w - t(h-w)\right\| - \varepsilon \\ &\geqslant \left\|f_h^k - w - \frac{1}{k}(h-w)\right\| - \varepsilon \\ &\geqslant \left\|f_h^k - w\right\| - \varepsilon \\ &\geqslant \|f_h - w\| - 2\varepsilon,\end{aligned}$$

where the second inequality follows from (1) and the convexity of the norm and the third from (1). Since ε was arbitrary,

$$\|f_h - w + t(w-h)\| \geqslant \|f_h - w\|. \tag{2}$$

Since t was arbitrary, (2) implies that $\tau_+(f_h - w, w-h) \geqslant 0$.

Conversely, suppose $h \in K$ and $f_h \in \mathcal{Q}_F(w)$ is chosen so that $\tau_+(f_h - w, w-h) \geqslant 0$. Since $\psi_{f_h - w, w-h}$ is nondecreasing on $(0, \infty)$, we may let $t = 1$ and obtain

$$\|f_h - h\| - \|f_h - w\| \geqslant \tau_+(f_h - w, w-h) \geqslant 0,$$

whence $\|f_h - w\| \leqslant \|f_h - h\|$. By the definition of $\mathcal{Q}_F(w)$,

$$\sup_{z \in F} \|z - w\| = \|f_h - w\| \leqslant \|f_h - h\| \leqslant \sup_{z \in F} \|z - h\|.$$

Thus, w is an element of $\mathcal{Z}_K(F)$. ∎

By Lemma 1 and Theorem 1.6 in [**9**] we have the following Corollary.

COROLLARY 2. *In the context of Theorem 1, if $\mathcal{Q}_F(w)$ consists of a single point g, then $w \in \mathcal{Z}_K(F)$ if and only if $w \in \mathcal{P}_K(g)$.*

Our principal theorem is an immediate consequence of Lemma 1.

THEOREM 3. *Suppose X is a real Banach space, $K \subset X$ is closed and convex, and $F \subset X$ is compact. Then*

$$\mathcal{Z}_K(F) = \{w \in K : w \in \mathcal{Z}_K(\mathcal{Q}_F(w))\}$$

An even sharper statement can be made by replacing $\mathcal{Q}_F(w)$ by $G := \{f_h(w) : h \in K\}$. In certain circumstances, it is possible to get by with a small subset of G. Perhaps an algorithm for the calculation of w can be devised using this idea. We also note that for every $w \in K$, if $g \in \mathcal{Q}_F(w)$ then g is an extreme point of the convex hull of F. If, for example, $X = \mathbf{R}^n$ and F is a polyhedral convex subset of X then $\{f_h(w) : w, h \in K\}$ is a finite set.

It is natural to ask about the relationship between the K-relative Chebyshev centre of F, $\mathcal{Z}_K(F)$, and the metric projection of F onto K, $\mathcal{P}_K(F)$. An investigation of this question leads, surprisingly, to a characterisation of Hilbert space. We now present our results along this line, beginning with a Hilbert-space version of Lemma 1.

Lemma 4. *If H is a real Hilbert space, V is a closed subspace of X, $F \subset H$ is compact, and $w \in V$, then the following are equivalent.*

(i) $\mathcal{Z}_V(F) = \{w\}$.

(ii) *For every $h \in V \setminus \{w\}$ there is a vector $f_h \in \mathcal{Q}_F(w)$ such that*

$$\langle f_h - w, w - h \rangle \geqslant 0.$$

(iii) *For every $h \in V \setminus \{w\}$ there is a vector $f_h \in \mathcal{Q}_F(w)$ such that*

$$\langle \mathcal{P}_V(f_h) - w, w - h \rangle \geqslant 0.$$

Proof: Since H is a real Hilbert space, for any $f \in H$ and $0 \neq g \in H$,

$$\tau_+(f,g) = \langle f,g \rangle / \|f\| \tag{1}$$

and, given $f^* \in V$,

$$\mathcal{P}_V(f) = \{f^*\} \quad \Leftrightarrow \quad (f - f^*) \perp V. \tag{2}$$

Suppose $\mathcal{Z}_V(F) = \{w\}$ and $h \in V \setminus \{w\}$. By Theorem 1, there exists a vector $f_h \in \mathcal{Q}_F(w)$ such that $\tau_+(f_h - w, w - h) \geqslant 0$ so (1) implies that the inequality in (ii) holds. By (2) $\langle f_h - \mathcal{P}_V(f_h), w - h \rangle = 0$. Subtracting this equality from the last inequality, we have the inequality in (iii).

To prove the converse, reverse the above procedure. ∎

The following example shows that Lemma 4 does not hold in general if V is replaced by a closed convex set K. Let X be the Euclidean space $\ell_2(2)$, $K = \{(x_1, x_2) : x_2 \leqslant -|x_1|\}$, $F = \{(-3,1),(3,1)\}$, and $h = (0,-1)$. Then $\mathcal{Z}(F) = \{w\}$, where $w = (0,0)$. However, for each $f \in F$,

$$\langle \mathcal{P}(f) - w, w - h \rangle = -1.$$

Klee [8] showed that a normed linear space X of dimension greater than two is an inner product space if and only if for every compact convex subset K of X,

$$\mathcal{Z}(K) \cap K \neq \emptyset.$$

Klee's theorem was strengthened by Garkavi [5] in the case where X is a Banach space: A Banach space X is a Hilbert space if and only if for every bounded set A, $[\mathcal{Z}(A)] \cap [\overline{\mathrm{co}}A] \neq \emptyset$, where $\overline{\mathrm{co}}A$ denotes the closed convex hull of A. The following theorem is an extension of Corollary 2.7 in [1]. It gives a characterisation, via the *relative* Chebyshev centre, of Hilbert space. Let $\mathcal{V}$, $\mathcal{V}_2$ and $\mathcal{F}$ denote, respectively, the family of all closed subspaces, all closed subspaces of dimension 2, and all compact subsets of X, and given $u, v \in X$, let $[u,v]$ denote the closed interval $\{u + t(v-u) : t \in [0,1]\}$.

THEOREM 5. *If X is a real Banach space, then the following are equivalent.*

(i) *X is a Hilbert space*

(ii) *For every $V \in \mathcal{V}$ and $F \in \mathcal{F}$, there is a $w \in V$ such that*

$$[\mathcal{Z}_V(F)] \cap [\overline{\text{co}}(\mathcal{P}_V(\mathcal{Q}_F(w)))] \neq \emptyset.$$

(iii) *For every $V \in \mathcal{V}_2$ and $f \in X$, there is a $g \in \mathcal{P}_V(f)$ such that*

$$[\mathcal{Z}_V(\{0, f\})] \cap [0, g] \neq \emptyset.$$

PROOF: Suppose X is a real Hilbert space. Let w be the unique element of $\mathcal{Z}_V(F)$. Suppose w is not in $T := \overline{\text{co}}(\mathcal{P}_V(\mathcal{Q}_F(w)))$. Let v be the unique element of $\mathcal{P}_T(w)$. By Lemma 4, there exists an $f_v \in \mathcal{Q}_F(w)$ such that $\langle \mathcal{P}_V(f_v) - w, w - v \rangle \geqslant 0$. However, since $\mathcal{P}_V(f_v) \in T$ and v is the best approximation to w from T, the theorem of Pinkus mentioned above implies that $\langle w - v, v - \mathcal{P}_V(f_v) \rangle \geqslant 0$. By adding the last two inequalities, we obtain the contradiction $\langle v - w, w - v \rangle \geqslant 0$. Thus $w \in \overline{\text{co}}(\mathcal{P}_V(\mathcal{Q}_F(w)))$. This proves that (i) implies (ii).

Suppose that (iii) does not hold. Then there exists a subspace $V \in \mathcal{V}_2$ and an $f \in X$ such that for every $g \in \mathcal{P}_V(f)$ the set $\mathcal{Z}_V(\{0, f\})$ does not intersect the interval $[0, g]$. Let $F := \{0, f\}$ and let $A := \bigcup\{[0, g] : g \in \mathcal{P}_V(f)\}$. Then A is a closed convex set. Indeed, suppose $h_n \in A$ and $h_n \to h$. Then there exist $\lambda_n \in [0, 1]$ and $g_n \in \mathcal{P}_V(f)$ such that $h_n = \lambda_n g_n$. No generality is lost in assuming that there is a $\lambda \in [0, 1]$ such that $\lambda_n \to \lambda$. If $\lambda = 0$ then, since $\mathcal{P}_V(f)$ is bounded, it must be that $h_n \to 0 \in A$. If $\lambda \neq 0$ then, $g_n \to h/\lambda$. Since $\mathcal{P}_V(f)$ is closed, $h/\lambda \in \mathcal{P}_V(f)$. Since $h \in [0, h/\lambda]$, $h \in A$. Thus, A is closed. Since $\mathcal{P}_V(f)$ is convex, A is convex. Thus $\overline{\text{co}}(\mathcal{P}_V(F)) \subset A$, whence $[\mathcal{Z}_V(F)] \cap [\overline{\text{co}}(\mathcal{P}_V(F))] = \emptyset$. For every $z \in V$, $\mathcal{Q}_F(z) \subset F$ so $[\mathcal{Z}_V(F)] \cap [\overline{\text{co}}(\mathcal{P}_V(\mathcal{Q}_F(z)))] = \emptyset$. This proves that (ii) implies (iii).

That (i) and (iii) are equivalent was proven by Amir and Ziegler [**1**]. ∎

Combining the above-cited theorem of Garkavi and (ii) in Theorem 5, we have a characterisation of Hilbert space in terms of the relationship between the relative Chebyshev centre of a compact set F and the metric projection of F.

COROLLARY 6. *If X is a real Banach space, then the following are equivalent.*

(i) *X is a Hilbert space.*

(ii) *For every $V \in \mathcal{V}$ and $F \in \mathcal{F}$,*

$$[\mathcal{Z}_V(F)] \cap [\overline{\text{co}}(\mathcal{P}_V(F))] \neq \emptyset.$$

Theorem 5 also enables a characterisation, in Hilbert space, of $\mathcal{Z}_V(F)$.

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Theorem 7. *Suppose X is a real Hilbert space, V is a closed subspace of X, $F \subset X$ is compact, and $w \in V$. Then $\mathcal{Z}_V(F) = \{w\}$ if and only if $w \in \overline{\mathrm{co}}(\mathcal{P}_V(\mathcal{Q}_F(w)))$.*

Proof: The necessity of the condition was proven in Theorem 5.

We now prove the converse. Suppose that $w \in \overline{\mathrm{co}}(\mathcal{P}_V(\mathcal{Q}_F(w)))$. For notational convenience we let $U := \mathcal{P}_V(\mathcal{Q}_F(w))$, $S := \mathrm{co}(U)$ and $T := \overline{\mathrm{co}}(U)$. Note that since F is compact $\mathcal{Q}_F(w)$ must be compact also. Since $\mathcal{P}_V$ is continuous, U is compact so, by Theorem V.2.6 in [**3**], T is compact. Suppose $h \in V \setminus \{w\}$. Let $\alpha_0 := \sup\{\langle v, w-h\rangle : v \in T\}$. Since T is compact there exists a $v_0 \in T$ such that $\langle v_0, w-h\rangle = \alpha_0$. Since the closed convex hull is the closure of the convex hull, there exist $g^n \in S$ such that $g^n \to v_0$. Since each g^n is a convex combination of elements of U and every linear functional on a polyhedron attains its maximum at a vertex, for every natural number n there exists an $f_h^n \in \mathcal{Q}_F(w)$ such that $\langle \mathcal{P}_V(f_h^n), w-h\rangle \geqslant \langle g^n, w-h\rangle$. Since $\mathcal{Q}_F(w)$ is compact, $\{f_h^n\}$ has a subsequential limit f_h. Then $\langle \mathcal{P}_V(f_h), w-h\rangle = \alpha_0$. Since $w \in T$, $\langle w, w-h\rangle \leqslant \alpha_0$, so Lemma 4 implies that $\mathcal{Z}_V(F) = \{w\}$. ∎

If $V = X$, then $\mathcal{P}_V(f) = f$ for every $f \in X$, so we have the following corollary.

Corollary 8. *Suppose X is a real Hilbert space and $F \subset X$ is compact. Then, for $w \in X$, $\mathcal{Z}(F) = \{w\}$ if and only if $w \in \overline{\mathrm{co}}(\mathcal{Q}_F(w))$.*

The characterisation in the last theorem enables an estimate of the V-relative Chebyshev radius. If $A \subset X$ we shall denote by $\mathrm{diam}\,(A)$ the diameter of the set A, that is, $\mathrm{diam}\,(A) := \sup\{\|a-b\| : a, b \in A\}$.

Corollary 9. *Suppose X is a real Hilbert space, V is a closed subspace of X, $F \subset X$ is compact and $\mathcal{Z}_V(F) = \{w\}$. Then,*

$$r_V(F) \leqslant \inf_{f \in \mathcal{Q}_F(w)} r_V(\{f\}) + \mathrm{diam}\,(\mathcal{Q}_F(w)).$$

Proof: Given any $f \in \mathcal{Q}_F(w)$, the V-relative Chebyshev radius of F can be calculated by $r_V(F) = \|f - w\|$. Since $w \in \overline{\mathrm{co}}(\mathcal{P}_V(\mathcal{Q}_F(w)))$ it must be that for every $f \in \mathcal{Q}_F(w)$

$$\|w - \mathcal{P}_V(f)\| \leqslant \mathrm{diam}\,(\mathcal{P}_V(\mathcal{Q}_F(w))) \leqslant \mathrm{diam}\,(\mathcal{Q}_F(w)),$$

where the last inequality follows from the fact that the metric projection in Hilbert space is distance-reducing. Thus

$$\begin{aligned} r_V(F) &\leqslant \|f - \mathcal{P}_V(f)\| + \|\mathcal{P}_V(f) - w\| \\ &\leqslant \inf_{f \in \mathcal{Q}_F(w)} r_V(\{f\}) + \mathrm{diam}\,(\mathcal{Q}_F(w)). \end{aligned}$$ ∎

Our initial inspiration for Theorem 7 was the $\ell_2(n)$ version of Corollary 8 in the case where F is finite, apparently due to Pschenichny [**10**] and cited by Botkin and

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Turova-Botkina [**2**]. A finite algorithm for the calculation of the Chebyschev centre of a finite subset of $\ell_2(n)$ was presented in [**2**]. We conjecture that, thanks to Theorem 7, the algorithm in [**2**] can be generalised to the calculation of the V-relative Chebyshev centre of a finite subset of $\ell_2(n)$, where V is a subspace of $\ell_2(n)$.

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