INTEGRAL REPRESENTATIONS OF THE DIRECT PRODUCT OF GROUPS

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1. Introduction. Let G be a finite group and R a Dedekind domain with quotient field K. We denote by RG the group ring of formal linear combinations of elements of G with coefficients in R. By an RG-module we understand a unital left RG-module which is finitely generated and torsion-free as R-module. In particular, if R is a principal ideal domain this is equivalent to considering representations of G by matrices with entries in R. Given a prime ideal P in R we let

$$R_P = \{a/b; a, b \in R, b \notin P\}.$$

If M is an RG-module we write $R_P M = R_P \otimes_R M$.

In this paper we consider groups G which are the direct product of groups, $G = G_1 \times G_2$. Given RG_i-modules M_i, $i = 1, 2$, the outer tensor product of M_1 and M_2, denoted $M_1 \# M_2$, is defined as the RG-module obtained by defining the action of any $(g_1, g_2) \in G$ on an element $m_1 \otimes m_2$ of the R-module $M_1 \otimes_R M_2$, by $(g_1, g_2)(m_1 \otimes m_2) = g_1 m_1 \otimes g_2 m_2$. We say that an RG-module M can be "expressed as a tensor product" when there exist RG_i-modules M_i, $i = 1, 2$, such that $M \cong M_1 \# M_2$. It is known that if K is a splitting field for G, that is if every irreducible KG-module remains irreducible under any extension of K, then every irreducible KG-module can be expressed as a tensor product. This does not hold in general for RG-modules. We shall prove that if K is a splitting field for G and if G_1 and G_2 have relatively prime orders, then for any prime ideal P of R, every indecomposable $R_P G$-module can be expressed as a tensor product. Furthermore, under the same hypothesis every irreducible RG-module can be expressed as a tensor product. We show by an example that the condition on the orders of G_1 and G_2 cannot be dropped in these theorems. We also give an expression for Ext^1 of two outer tensor products.

2. Indecomposable modules.

THEOREM 1. *Let G_1, G_2 be arbitrary groups, P a prime ideal of R relatively prime to the order of G_1, K a splitting field for G_1. Then every indecomposable $R_P G$-module is the outer tensor product of an irreducible $R_P G_1$-module and an indecomposable $R_P G_2$-module.*

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Proof. Let M be an indecomposable R_PG-module. We can consider M as an R_PG_1-module by defining for any $g_1 \in G_1$, $m \in M$, $g_1m = (g_1, 1)m$. Let M_{G_1} denote this module. The first step will be to show that all the irreducible R_PG_1-submodules of M_{G_1} are isomorphic.

Let M_1 be any non-zero irreducible submodule of M_{G_1}, and let M_1' be the sum of all submodules of M_{G_1} which are isomorphic to M_1. Because P is prime to the order of G_1, either $M_{G_1} \cong M_1'$ or M_1' is isomorphic to a direct summand of M_{G_1} (see **4**). Repeating the argument we can get $M_{G_1} \cong M_1' \dotplus \ldots \dotplus M_t'$, where M_i' is a sum of irreducible submodules of M_{G_1} isomorphic to M_i, $1 \leqslant i \leqslant t$, and such that M_i is not isomorphic to M_j for $i \neq j$. Since the M_i are irreducible, for all i we have $KM_i' \cong KM_i^{(1)} \dotplus \ldots \dotplus KM_i^{(s)}$ (s depending on i), where $M_i^{(j)} \cong M_i$ for all j. This implies that $M_i' \cong M_i^{(1)} \dotplus \ldots \dotplus M_i^{(s)}$ (see **4**). Since for $i \neq j$, $\mathrm{Hom}_{G_1}(M_i', M_j') = 0$, it follows that $(1, g_2)\, M_i' \subset M_i'$ for all $g_2 \in G_2$, and all i, $1 \leqslant i \leqslant t$. So M_i' is a G_2-module, therefore a G-module. Since we assumed that M was indecomposable, this implies that $M_{G_1} \cong M_1' \cong M_1 \dotplus \ldots \dotplus M_1$.

We can then find an R_P-submodule M_2 of M such that $M_{G_1} \cong M_1 \otimes_{R_P} M_2$, and such that for $g_1 \in G_1$, $m_i \in M_i$, $i = 1, 2$, $g_1(m_1 \otimes m_2) = g_1m_1 \otimes m_2$.

Let $\otimes$ denote $\otimes_{R_P}$ and $\mathrm{Hom}_G(\ ,\ )$ denote $\mathrm{Hom}_{R_PG}(\ ,\ )$.

Make $M_1 \otimes M_2$ into an R_PG-module as follows: for any $g \in G$, $m_i \in M_i$, $i = 1, 2$, choose $g(m_1 \otimes m_2)$ to be the image of gm under the R_P-isomorphism $M \cong M_1 \otimes M_2$, where m corresponds to $m_1 \otimes m_2$ in this isomorphism. We then have, for any $g_2 \in G_2$,

$$(1, g_2) \in \mathrm{Hom}_{G_1 \times 1}(M_1 \otimes M_2, M_1 \otimes M_2) \cong \mathrm{Hom}_{G_1}(M_1, M_1) \otimes \mathrm{Hom}_{R_P}(M_2, M_2).$$

Furthermore, since K is a splitting field for G_1, by Schur's lemma, $\mathrm{Hom}_{G_1}(M_1, M_1) \cong R_P$; therefore

$$\mathrm{Hom}_{G_1 \times 1}(M_1 \otimes M_2, M_1 \otimes M_2) \cong 1 \otimes \mathrm{Hom}_{R_P}(M_2, M_2).$$

Thus for $g_2 \in G_2$, $m_2 \in M_2$, we can define g_2m_2 by the formula

$$(1, g_2)(m_1 \otimes m_2) = m_1 \otimes g_2m_2.$$

It is easily verified that with this multiplication M_2 becomes an R_PG_2-module. Finally $(g_1, g_2)(m_1 \otimes m_2) = g_1m_1 \otimes g_2m_2$, so we have an isomorphism of R_PG-modules,

$$M \cong M_1 \# M_2.$$

Since M is indecomposable it follows that M_2 must be indecomposable.

We observe that M_1 is uniquely determined by M because, since P is prime to the order of G_1, the Krull-Schmidt theorem holds for R_PG_1-modules (see **3**); thus the components of M_{G_1} are uniquely determined. M_2 is isomorphic to all the indecomposable components of M_{G_2}; therefore when the Krull-Schmidt theorem holds for R_PG_2-modules, M_2 will also be uniquely determined.

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It follows from Theorem 1 that when the orders of G_1 and G_2 are relatively prime, and K is a splitting field for G, then for all prime ideals P of R, every indecomposable R_PG-module can be expressed as a tensor product in a unique way.

3. Irreducible modules.

THEOREM 2. *If G_1 and G_2 have relatively prime orders, and K is a splitting field for G, then every irreducible RG-module M can be expressed as a tensor product.*

Proof. By Theorem 1, for every prime ideal P in R which divides the order of G, there are R_PG_i-modules, M_{iP}, $i = 1, 2$, such that

$$R_PM \cong M_{1P} \# M_{2P}.$$

Then

$$KM \cong KM_{1P} \# KM_{2P}.$$

Therefore, since every irreducible KG-module is uniquely expressible as an outer tensor product, it follows that the modules M_{iP}, for the different primes P, are all K-isomorphic, for $i = 1, 2$. From a result of Maranda **(4)** it follows that there exist RG_i-modules M_i, $i = 1, 2$, such that $R_PM_i \cong M_{iP}$ for all P. Thus

$$R_PM \cong R_P(M_1 \# M_2).$$

This implies that for some ideal A in R

$$M \cong A(M_1 \# M_2) = AM_1 \# M_2.$$

If M_i, M_i' are irreducible RG_i-modules, $i = 1, 2$, such that

$$M_1 \# M_2 \cong M_1' \# M_2',$$

then for all P

$$R_PM_1 \# R_PM_2 \cong R_PM_1' \# R_PM_2'.$$

Since the Krull-Schmidt theorem holds for R_PG-modules, it follows that $R_PM_i \cong R_PM_i'$, $i = 1, 2$. Then there are ideals A_i in R such that $M_i' \cong A_iM_i$, $i = 1, 2$. Therefore

$$A_1A_2(M_1 \# M_2) \cong M_1 \# M_2.$$

It is shown in **(4)** that this implies the existence of $\alpha \in K$ such that $A_1A_2 = \alpha R$; from this we conclude that

$$M_1' \cong A_1M_1, \qquad M_2' \cong A_1^{-1}M_2.$$

If the condition that the orders of G_1 and G_2 be relatively prime is dropped from Theorems 1 and 2, they do not hold in general. The following example will prove this assertion.

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Let $G_1 = G_2 = D$, the dihedral group of order 8, defined by generators a, b, and relations

$$a^4 = b^2 = 1, \qquad bab = a^{-1}.$$

The irreducible representations of D over the rationals Q consist of four one-dimensional representations and a two-dimensional representation X, which can be defined by the matrices

$$X(a) = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \qquad X(b) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

It is easily seen that X remains irreducible under any extension of Q, and this shows that Q is a splitting field for D.

Consider also the representation Y defined by

$$Y(a) = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, \qquad Y(b) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

It can be verified that X and Y considered as representations over the 2-adic valuation ring of Q, Z_2, are not equivalent. Now let $W = VYV^{-1}$, where

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix};$$

then

$$W(a) = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}, \qquad W(b) = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}.$$

Finally consider the representation T of $D \times D$ defined by

$$T = \begin{bmatrix} I & \\ & U \end{bmatrix} X \otimes W \begin{bmatrix} I & \\ & U^{-1} \end{bmatrix},$$

where

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

and I is an identity matrix of suitable dimension. T is seen to have entries in the integers. Further, T is an irreducible representation because X and W are irreducible. Now T restricted to one of the factor groups D is of the form

$$\begin{bmatrix} X & \\ & Y \end{bmatrix},$$

where X and Y are irreducible and non-equivalent over Z_2. But since Q is a splitting field for D, the Krull-Schmidt theorem holds for Z_2D-modules. Thus T cannot be the tensor product of two representations of D over Z_2.

4. Extensions of tensor products. For the notions used in this section we refer the reader to **(1)**.

THEOREM 3. *Let G_1 and G_2 be arbitrary groups,* $\Gamma_i = RG_i$, $i = 1, 2$, $\Gamma = RG$. *If M_i are Γ_i-modules,* $i = 1, 2$, *then*

$$\operatorname{Ext}^1_\Gamma(M_1 \# M_2, M'_1 \# M'_2) \cong \operatorname{Hom}_{\Gamma_1}(M_1, M'_1) \otimes_R \operatorname{Ext}^1_{\Gamma_2}(M_2, M'_2) \dotplus \operatorname{Ext}^1_{\Gamma_1}(M_1, M'_1) \otimes_R \operatorname{Hom}_{\Gamma_2}(M_2, M'_2),$$

$$\operatorname{Hom}_\Gamma(M_1 \# M_2, M'_1 \# M'_2) \cong \operatorname{Hom}_{\Gamma_1}(M_1, M'_1) \otimes_R \operatorname{Hom}_{\Gamma_2}(M_2, M'_2).$$

Proof. Let $\otimes$ denote $\otimes_R$.

Suppose the complex X^i with derivation d_i is a projective resolution of M_i, $i = 1, 2$. All the modules of X^i can be assumed Γ_i-free, and finitely generated as Γ_i-modules. Consider now the complex $X^1 \# X^2$, where

$$(X^1 \# X^2)_m = \sum_{j+k=m} X^1_j \# X^2_k,$$

and where the derivation on $X_j{}^1 \# X_k{}^2$ is $d_1 \otimes 1 + (-1)^j \otimes d_2$. The modules of $X^1 \# X^2$ are Γ-free, and for $m > 0$,

$$H_m(X^1 \otimes X^2) = \operatorname{Tor}^m_R(M_1, M_2) = 0.$$

It follows that $X^1 \# X^2$ is a projective resolution of $M_1 \# M_2$.

Therefore

$$\operatorname{Ext}_\Gamma(M_1 \# M_2, M_1' \# M_2') \cong H[\operatorname{Hom}_\Gamma(X^1 \# X^2, M^{1\prime} \# M^{2\prime})].$$

Now observe that

$$\operatorname{Hom}_\Gamma(\Gamma, M_1' \# M_2') \cong M_1' \# M_2' \cong \operatorname{Hom}_{\Gamma_1}(\Gamma_1, M_1') \otimes \operatorname{Hom}_{\Gamma_2}(\Gamma_2, M_2').$$

Therefore since the modules of X^i are Γ_i-free and finitely generated, it follows that

$$\operatorname{Hom}_\Gamma(X^1 \# X^2, M_1' \# M_2') \cong \operatorname{Hom}_{\Gamma_1}(X^1, M_1') \otimes \operatorname{Hom}_{\Gamma_2}(X^2, M_2').$$

Next observe that $\operatorname{Hom}_{\Gamma_i}(\Gamma_i, M_i') \cong M_i'$ is torsion-free and R-projective, so the modules of $\operatorname{Hom}_{\Gamma_i}(X^i, M_i')$ are torsion-free, R-projective. Then from Künneth's theorem we get

$$H_m[\operatorname{Hom}_{\Gamma_1}(X^1, M'_1) \otimes \operatorname{Hom}_{\Gamma_2}(X^2, M'_2)] \cong \sum_{j+k=m} H_j[\operatorname{Hom}_{\Gamma_1}(X^1, M'_1)] \otimes H_k[\operatorname{Hom}_{\Gamma_2}(X^2, M'_2)] \dotplus \sum_{j+k=m-1} \operatorname{Tor}^R_1[H_j[\operatorname{Hom}_{\Gamma_1}(X^1, M'_1)], H_k[\operatorname{Hom}_{\Gamma_2}(X^2, M'_2)]].$$

From this, taking $m = 1$ and observing that $\operatorname{Tor}_1{}^R[\operatorname{Hom}_{\Gamma_1}(M_1, M_1'), \operatorname{Hom}_{\Gamma_2}(M_2, M_2')] = 0$, we obtain the formula for $\operatorname{Ext}_\Gamma{}^1$. Similarly, with $m = 0$, the formula for $\operatorname{Hom}_\Gamma$ follows.

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