ON NON-ORIENTABLE CLOSED SURFACES IN EUCLIDEAN SPACES

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Let us begin with a simple result.

PROPOSITION. *Let X be a non-orientable closed surface differentiably imbedded into the euclidean 4-space R^4. Then there is a line in R^4 which intersects X at more than two points.*

Proof. Let c be any point of R^4 and let r be the smallest number such that X is contained in the closed 4-spheroid W of centre c and radius r. Clearly the boundary 3-sphere S of W intersects X.

Let a be a point of $S \cap X$. Then there is a function $f: X \to S$ defined as follows:

(i) $f(a) = a$.

(ii) Whenever $x \in X - \{a\}$, the line determined by a and x intersects S at a and another point. The second point of intersection is taken to be $f(x)$.

Obviously f is continuous at every point of $X - \{a\}$. Since X is differentiably imbedded into R^4, every line which is tangent to X at a is also tangent to S at a. It follows that $f(x) \to a = f(a)$ as $x \to a$. Hence f is also continuous at a.

Now we assert that there is a line in R^4 which contains a and intersects X at more than two points. If the assertion is false, then the map f constructed above is one-to-one so that it is a homeomorphism of X into S. But it is well known that such a homeomorphism into does not exist. The contradiction proves the assertion and thus the proof is completed.

The purpose of the present paper is to establish a more general result. In fact, we shall prove

THEOREM. *Let X be a non-orientable closed surface topologically imbedded into the euclidean n-space R^n. Then there is an $(n-3)$-plane in R^n which intersects X at more than $n-2$ points.*

Notice that since any non-orientable closed surface cannot be topologically imbedded into R^3, the integer n in the theorem must be >3.

The statement of the theorem can be rephrased: "Any non-orientable closed surface in the euclidean n-space cannot be $(n-3)$-independent in the sense of Borsuk **(1)**." Hence the theorem confirms a special case of the following conjecture of Professor A. M. Gleason: *Any compact subset of the euclidean*

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n-space which is k-independent in the sense of Borsuk can be topologically imbedded into an $(n-k)$-sphere. The author learned the conjecture from Dr. R. R. Phelps and a study on the conjecture in a forthcoming paper **(2)** induced the author to prepare the present paper.

To prove the theorem we assume that the theorem is false, and then establish a contradiction by computing the linking number of certain integral cycles. The proof will be given after the following four lemmas.

LEMMA 1. *Let X be a non-orientable closed surface. Let E be a closed 2-cell in X, let α be the boundary of E and let $Y = X - (E - \alpha)$. Let β be a simple closed curve in $Y - \alpha = X - E$ such that the local orientation of X at a point x of β is reversed when x moves along β once. Let λ be a homeomorphism of Y into a 3-sphere S. Let S, $\lambda(\alpha)$, $\lambda(\beta)$ be oriented and let a, b be the integral fundamental cycles on $\lambda(\alpha)$, $\lambda(\beta)$ respectively. Then the linking number of a and b is odd.*

Proof. Let k be the first mod 2 Betti number of X. Then we may regard X as a decomposition space of the unit circular disk D: $|z| \leqq 1$ in the complex plane obtained as follows. Let

$$A_r = \exp ir\pi/k, \qquad r = 1, \ldots, 2k, 2k+1;$$
$$A_rA_{r+1} = \{\exp i(r\pi/k + \theta) | 0 \leqq \theta \leqq \pi/k\}, \qquad r = 1, \ldots, 2k.$$

Then X is obtained from D by identifying $A_{2s-1}A_{2s}$ with $A_{2s}A_{2s+1}$, $s = 1, \ldots, k$.

Let ρ be the projection of D onto X. Let E' be the closed 2-cell $|z| \leqq 1/2$. Without loss of generality we may assume

$$E = \rho(E').$$

For every $s = 1, \ldots, k$,

$$\beta_s = \rho(A_{2s-1}A_{2s})$$

is a simple closed curve in $Y - \alpha$ and the local orientation of X at a point x of β_s is reversed when x moves along β_s once. β_s, when oriented, may be regarded as a closed path of basic point $p = \rho(A_1)$. It is easily seen that the fundamental group of $Y - \alpha$ is generated by the homotopy classes containing $\beta_1, \ldots, \beta_s$ respectively.

Let q be a point of β and let γ be a path in $Y - \alpha$ from p to q. Since β, when oriented, may be regarded as a closed path of basic point q, $\gamma\beta\gamma^{-1}$ is a closed path of basic point p. Therefore there are integers

$$k_1, \ldots, k_j \in \{1, \ldots, k\},$$

not necessarily distinct, such that $\gamma\beta\gamma^{-1}$ is homotopic to $\beta_{k_1} \ldots \beta_{k_j}$ in $Y - \alpha$. Since every one of $\gamma\beta\gamma^{-1}$, $\beta_1, \ldots, \beta_k$ has the property that the local orientation of X at a point x of the curve is reversed when x moves along the curve once, it follows that j is odd.

Let

$$a, b, b_1, \ldots, b_k$$

be the integral fundamental cycles on $\lambda(\alpha), \lambda(\beta), \lambda(\beta_1), \ldots, \lambda(\beta_k)$ respectively. Then b and $b_{k_1} + \ldots + b_{k_j}$, as singular cycles in $\lambda(Y - \alpha)$ are homologous. Hence we remain to prove that the linking number of a and b_s, for every $s = 1, \ldots, k$, is not congruent to 0 mod 2.

Let $0 < \delta < (\sin \pi/2k)/2$ and let $r = 1, \ldots, 2k$. Denote by $0A_r$ the radius of D of terminal point A_r, let K_r be the circle of centre A_r and radius δ and let B_r, C_r, D_r be the respective points of intersection of K_r with $A_{r-1}A_r$, $0A_r$, A_rA_{r+1}, where $A_0 = A_{2k}$ and $A_{2k+1} = A_1$. Let $B_rC_rD_r$ be the arc of K_r of endpoints B_r, D_r containing C_r and let B_rC_r, C_rD_r be the subarcs of $B_rC_rD_r$ of endpoints B_r, C_r and C_r, D_r respectively. The circle K' of centre 0 and radius $1 - \delta$ clearly contains $C_1, \ldots, C_{2k}$. Let $C_{r-1}C_rC_{r+1}$ be the arc of K' of endpoints C_{r-1}, C_{r+1} containing C_r and let $C_{r-1}C_r$ be its subarc of endpoints C_{r-1}, C_r.

Fix an integer t, $t = 1, \ldots, k$. Clearly ρ maps the union of the arcs

$$B_{2t-1}C_{2t-1},\ C_{2t-1}C_{2t}C_{2t+1},\ C_{2t+1}D_{2t+1};$$
$$B_rC_rD_r,\ r = 1, \ldots, 2t-2, 2t+2, \ldots, 2k,$$

into a simple closed curve γ_t in $Y - \alpha$ which is the boundary of a Möbius band M_t containing β_t in its interior, where $C_{2k+1}D_{2k+1} = C_1D_1$. Let $\lambda(\gamma_t)$ be oriented and let c_t be the integral fundamental cycle on $\lambda(\gamma_t)$. A direct observation yields that the linking number of c_t and b_t is not congruent to 0 mod 2.

For every $s = 1, \ldots, k$, ρ maps the union of the arcs

$$B_{2s}C_{2s},\ C_{2s}C_{2s+1},\ B_{2s+1}C_{2s+1}$$

into a simple closed curve β_s', where $B_{2k+1}C_{2k+1} = B_1C_1$. Let β_s' be oriented and let b_s' be the integral fundamental cycle on $\lambda(\beta_s')$. Then a is homologous to

$$c_t + 2(b_1' + \ldots + b_{t-1}' + b_{t+1}' + \ldots + b_k') \equiv c_t \mod 2$$

in $\lambda(Y - (M_t - \gamma_t))$. Hence the linking number of a and b_t is not congruent to 0 mod 2. The proof of Lemma 1 is thus completed.

LEMMA 2. *Let Y, α, β be as in Lemma 1 and let Y be topologically imbedded into the euclidean n-space R^n. Let $x_1, \ldots, x_{n-3}$ be $n-3$ points of R^n such that every $(n-3)$-plane containing these $n-3$ points intersects Y at no more than one point. Let P be the $(n-4)$-plane determined by $x_1, \ldots, x_{n-3}$, let Px, for every $x \in Y$, denote the half $(n-3)$-plane of boundary P containing x, and let*

$$P\alpha = \bigcup_{x \in \alpha} Px.$$

Then in the one-point-compactification $R^n \cup \{\infty\}$ of R^n, $P\alpha \cup \{\infty\}$ is homeomorphic to an $(n-2)$-sphere and is contained in the complement of β. Moreover, if we orient $R^n \cup \{\infty\}$, $P\alpha \cup \{\infty\}$ and β, then the linking number of the integral fundamental cycle on $P\alpha \cup \{\infty\}$ and that on β is odd.

Proof. We first note that since, by hypothesis, every $(n-3)$-plane containing $x_1, \ldots, x_{n-3}$ intersects Y at no more than one point, $x_1, \ldots, x_{n-3}$ are distinct

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and cannot be contained in the same $(n-5)$-plane so that they determine a unique $(n-4)$-plane P. Moreover, it follows from the hypothesis that $P \cap Y = \phi$ so that P and any point x of Y determine a unique $(n-3)$-plane. Hence Px for $x \in Y$ and then $P\alpha$ are well-defined. For any two distinct points x and x' of α, $Px \neq Px'$ and so $Px \cap Px' = P$. We infer that $P\alpha$ is homeomorphic to an $(n-2)$-plane.

Let Q be a 4-plane orthogonal to P and let S be a 3-sphere in Q with $P \cap Q$ as its centre. Then for every $x \in Y$, $Px \cap S$ contains exactly one point. The function $\lambda: Y \to S$ mapping every $x \in Y$ into the point in $Px \cap S$ is clearly continuous and one-to-one so that it is a homeomorphism into.

Let $\lambda(\alpha)$, $\lambda(\beta)$, S be oriented and let a, b be the respective integral fundamental cycles on $\lambda(\alpha)$, $\lambda(\beta)$. Then, by Lemma 1, the linking number of a and b is odd.

Let $R^n \cup \{\infty\}$ be the one-point-compactification of R^n. Then topologically $R^n \cup \{\infty\}$ is an n-sphere, $P \cup \{\infty\}$ is an $(n-4)$-sphere and $P\alpha \cup \{\infty\}$ is an $(n-2)$-sphere. Let $R^n \cup \{\infty\}$ and $P \cup \{\infty\}$ be oriented such that the integral fundamental cycle on $P \cup \{\infty\}$ and that on S have 1 as their linking number. Let α and β be oriented such that the homeomorphisms $\lambda_\alpha: \alpha \to \lambda(\alpha)$ and $\lambda_\beta: \beta \to \lambda(\beta)$ defined by λ are orientation-preserving, and let $P\alpha \cup \{\infty\}$ be oriented such that the integral fundamental cycle on $P \cup \{\infty\}$ and that on α have 1 as their linking number. Then the linking number of the integral fundamental cycle on $P\alpha \cup \{\infty\}$ and that on β is equal to the linking number of a and b and hence is odd. This completes the proof of Lemma 2.

LEMMA 3. *Let α and α' be simple closed curves and let K and K' be triangulations on α and α' respectively. Let $\phi: \alpha' \to \alpha'$ be a simplicial involution without fixed point and let $\mu: \alpha \to \alpha'$ be a simplicial map of degree* 1 *such that whenever σ is a* 1-*simplex of K, $\mu(\sigma)$ is a* 1-*simplex of K'. Let*

$$I = \{(x, y) \in \alpha \times \alpha \mid \mu(x) = \phi\mu(y)\}$$

and let $p: I \to \alpha$ be given by

$$p(x, y) = x, \qquad (x, y) \in I.$$

Then there is a map $\nu: \alpha \to I$ such that $p\nu$ is homotopic to the identity map.

Proof. Since μ is of degree 1, we may let 1-simplexes of K and those of K' be oriented such that (i) the sum of the oriented 1-simplexes of K is an integral fundamental cycle c of α, (ii) the sum of the oriented 1-simplexes of K' is an integral fundamental cycle c' of α' and (iii) $\mu(c) = c'$. Then for every oriented 1-simplex σ' of K' the number of those oriented 1-simplexes σ of K with $\mu(\sigma) = \sigma'$ (that is, μ maps σ onto σ' with orientation preserved) is exactly one larger than the number of those σ with $\mu(\sigma) = -\sigma'$ (that is, μ maps σ onto σ' with orientation reversed).

Let σ' be an oriented 1-simplex of K' and let σ_1 and σ_2 be oriented 1-simplexes of K such that

$$\mu(\sigma_1) = \epsilon_1\sigma', \qquad \mu(\sigma_2) = \epsilon_2\phi(\sigma'),$$

where $\epsilon_1, \epsilon_2 = 1$ or -1. Let u', v' be vertices of σ' and let u_1, v_1 and u_2, v_2 be respective vertices of σ_1 and σ_2 such that

$$\mu(u_1) = u', \qquad \mu(v_1) = v';$$
$$\mu(u_2) = \phi(u'), \qquad \mu(v_2) = \phi(v').$$

Then the diagonal of $\sigma_1 \times \sigma_2$ joining (u_1, u_2) and (v_1, v_2), which we denote by $\sigma_1 \triangle \sigma_2$, is in I.

It is easily seen that I is the union of these $\sigma_1 \triangle \sigma_2$ and a finite set. Therefore there is a natural triangulation on I with every $\sigma_1 \triangle \sigma_2$ as a 1-simplex. Let the 1-simplexes $\sigma_1 \triangle \sigma_2$ be oriented such that

$$p(\sigma_1 \triangle \sigma_2) = \sigma_1 \text{ or } -\sigma_1$$

according as $\mu(\sigma_2)$ has positive or negative orientation.

Whenever u_1, u_2 are vertices of K such that $(u_1, u_2) \in I$, the 1-simplexes $\sigma_1 \triangle \sigma_2$ having (u_1, u_2) as a vertex are either four or two or zero in number. A direct observation yields that the sum of all the oriented 1-simplexes $\sigma_1 \triangle \sigma_2$ is an integral cycle z in I. Since the union of these 1-simplexes is connected, there is a map $\nu: \alpha \to I$ such that $\nu(c)$ and z, as singular cycles in I, are homologous. Since $\mu p(z) = c'$, we have $p(z) = c$. It follows that $p\nu(c)$ and c, as singular cycles in α, are homologous. Hence $p\nu$ is homotopic to the identity map. This proves Lemma 3.

LEMMA 4. *In Lemma 3, if α is the boundary of a closed 2-cell E, $a \in E - \alpha$ and*

$$H = \{(x, y) \in E \times E | x \neq y\},$$

then there is a map $h: \alpha \times [0, 1] \to H$ such that for $x \in \alpha$,

$$h(x, 0) = (x, a), \qquad h(x, 1) \in I.$$

Proof. By Lemma 3, there is a map $\nu: \alpha \to I$ such that $p\nu$ is homotopic to the identity map. Let us consider E as the unit circular disk $|z| \leqq 1$ in the complex plane with $a = 0$ and let $q: I \to \alpha$ be the map given by

$$q(x, y) = y, \qquad (x, y) \in I.$$

Let $g: \alpha \times [0, 1] \to \alpha$ be a map such that for $x \in \alpha$,

$$g(x, 0) = x, \qquad g(x, 1) = p\nu(x).$$

Then the map $h: \alpha \times [0, 1] \to H$ given by

$$h(x, t) = (g(x, t), tq\nu(x)), \qquad (x, t) \in \alpha \times [0, 1],$$

is as desired. Hence Lemma 4 is proved.

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Proof of theorem. Suppose that our theorem is false. Then we may assume that X is a non-orientable closed surface topologically imbedded into the euclidean n-space R^n such that every $(n-3)$-plane in R^n intersects X at no more than $n-2$ points. We recall again that n must be >3.

Suppose first that $n > 4$. Let E be a closed 2-cell in X, let α be the boundary of E and let $Y = X - (E - \alpha)$. Let γ be a simple closed curve in $E - \alpha$ and let

$$x_1, x_2: [0, 1] \to \gamma$$

be maps such that

(i) $x_1(0) = x_2(1)$ and $x_1(1) = x_2(0)$ and

(ii) $x_1(t) \neq x_2(t)$ for all $t \in [0, 1]$.

Let $x_3, \ldots, x_{n-3}$ be any $n-5$ distinct points in $E - (\alpha \cup \gamma)$. Then for every $t \in [0, 1]$, $x_1(t)$, $x_2(t)$, $x_3, \ldots, x_{n-3}$ determine a unique $(n-4)$-plane P_t. As in Lemma 2, we have P_t and $P_t\alpha$ corresponding to P and $P\alpha$ respectively when $\{x_1(t), x_2(t), x_3, \ldots, x_{n-3}\}$ takes the place of $\{x_1, x_2, x_3, \ldots, x_{n-3}\}$.

Let $R^n \cup \{\infty\}$ be the one-point-compactification of R^n and assign an orientation to $R^n \cup \{\infty\}$. Let $P_t\alpha \cup \{\infty\}$ be oriented such that the orientation is continuous in t, that means, the map $h_t: P_0\alpha \cup \{\infty\} \to P_t\alpha \cup \{\infty\}$ such that for every $x \in \alpha$, h_t defines an affine transformation of P_0x into P_tx mapping $x_1(0), x_2(0), x_3, \ldots, x_{n-3}, x$ into $x_1(t), x_2(t), x_3, \ldots, x_{n-3}, x$ respectively, is orientation-preserving, $0 \leqq t \leqq 1$. Let β be an oriented simple closed curve in $X - E$ such that the local orientation of X at a point x of β is reversed when x moves along β once. By Lemma 2, the linking number l_t of the integral fundamental cycle on $P_t\alpha \cup \{\infty\}$ and that on β is an odd number. Since the orientation on $P_t\alpha \cup \{\infty\}$ is continuous in t, the number l_t is continuous in t so that it is independent of t. Hence

$$l_0 = l_1.$$

However, $P_0\alpha \cup \{\infty\}$ and $P_1\alpha \cup \{\infty\}$ are identical but have opposite orientations. It follows that

$$l_0 = -l_1.$$

Hence $l_0 = l_1 = 0$, contrary to the fact that it is an odd number.

Suppose now that $n = 4$. As in the proof of the proposition, there is a 4-spheroid W containing X such that the boundary 3-sphere S of W intersects X. Let $a \in S \cap X$ and let T be the 3-plane tangent to S at a. It is clear that whenever F is a closed subset of X not containing a there is a 3-plane T' which is parallel to T and separates a and F (that means, a and F are contained in different components of $R^n - T'$).

Let A be a closed 2-cell in X containing a in its interior. Then there is a 3-plane L which is parallel to T and separates a and the boundary of A. Let B be a closed 2-cell which contains a in its interior and is contained in $X - L$, let

D be the unit circular disk $|z| \leqq 1$ in the complex plane and let ξ be a homeomorphism of D onto B mapping 0 into a. Let α' be the boundary of D and let

$$\mu: B - \{a\} \to \alpha'$$

be the map defined by

$$\mu(x) = \xi^{-1}(x)/|\xi^{-1}(x)|, \qquad x \in B - \{a\}.$$

Let M be a 3-plane which is parallel to T and separates a and the boundary of B, and let F be the boundary of the component of $B - M$ containing a. Since

$$J = \{(x, y) \in F \times F \mid |\mu(x) - \mu(y)| \geqq 1\}$$

is compact and, for every $(x, y) \in J$, the line joining x and y does not intersect the closed 3-cell $L \cap W$, there is a number $\epsilon > 0$ such that whenever $(x', y') \in B \times B$ such that for some $(x, y) \in J$, $|\xi^{-1}(x') - \xi^{-1}(x)| < \epsilon$ and $|\xi^{-1}(y') - \xi^{-1}(y)| < \epsilon$, x' and y' are distinct and the line joining x' and y' does not intersect $L \cap W$.

Let ϵ' be a number such that $\epsilon \geqq \epsilon' > 0$ and that whenever $(x, y) \in F \times F$ such that for some $(x', y') \in B \times B$ with $|\xi^{-1}(x') - \xi^{-1}(x)| < \epsilon'$, $|\xi^{-1}(y') - \xi^{-1}(y)| < \epsilon'$ and $\mu(x') = -\mu(y')$, (x, y) belongs to J. Since F separates a and the boundary of B in X, there is a simple closed curve α in the ϵ'-neighbourhood of F which separates a and the boundary of B in X. It is clear that the curve α can be so chosen that there are triangulations K and K' of α and α' respectively such that (i) $\phi: \alpha' \to \alpha'$ given by $\phi(z) = -z$, $z \in \alpha'$, is simplicial, (ii) $\mu: \alpha \to \alpha'$ given by $\mu(x) = \xi^{-1}(x)/|\xi^{-1}(x)|$, $x \in \alpha$, is simplicial, and (iii) whenever σ is a 1-simplex of K, $\mu(\sigma)$ is a 1-simplex of K'.

Since α is a simple closed curve in the closed 2-cell B, there is a closed 2-cell E in B having α as its boundary and containing a in its interior. Let

$$H = \{(x, y) \in E \times E \mid x \neq y\},$$
$$I = \{(x, y) \in \alpha \times \alpha \mid \mu(x) = \phi\mu(y)\}.$$

It follows from Lemma 4 that there is a map

$$h: \alpha \times [0, 1] \to H$$

such that for $x \in \alpha$,

$$h(x, 0) = (x, a), \qquad h(x, 1) \in I.$$

Since E is a closed 2-cell containing a in its interior, there is a homeomorphism η of D onto E mapping 0 into a. Let C be the complex plane and let $p, q: H \to E$ be the maps given by

$$p(x, y) = x, \qquad q(x, y) = y, \qquad (x, y) \in H.$$

Now we define a map

$$\tau: C \times [0, 1] \to R^n$$

as follows: Whenever $r \geqq 0$, $z \in \alpha'$ and $t \in [0, 1]$,

$$\tau(rz, t) = \begin{cases} \eta((r/t)\eta^{-1}qh(\eta(z), t)) & \text{if } r < t; \\ qh(\eta(z), t) + (r - t)(ph(\eta(z), t) - qh(\eta(z), t)) & \text{if } r \geqq t. \end{cases}$$

For a fixed $z \in \alpha'$, τ maps the half-line $\{(rz, 0)|\ r \geqq 0\}$ into the half-line $a\eta(z)$ of endpoint a containing $\eta(z)$. It follows that

$$\tau(C \times \{0\}) = \cup_{x \epsilon \alpha}\, ax = a\alpha.$$

Since every line intersects X at no more than two points, τ maps $C \times \{0\}$ homeomorphically onto $a\alpha$.

Let β be an oriented simple closed curve in $X - A$ such that the local orientation of X at a point x of β is reversed when x moves along β once. Let $R^4 \cup \{\infty\}$ and $a\alpha \cup \{\infty\}$ be oriented. By Lemma 2, the linking number of the integral fundamental cycle b on β and the integral fundamental cycle c on $a\alpha \cup \{\infty\}$ does not vanish. Making use of the map τ constructed above, we can have a singular cycle c' in $\tau(C \times \{1\}) \cup \{\infty\}$ which is homologous to c in $\tau(C \times [0, 1]) \cup \{\infty\} \subset R^4 - \beta$. The linking number of b and c' is equal to that of b and c so that it does not vanish.

Let U be the component of $W - L$ containing β. Then b, as a singular cycle in U, is bounding. For a fixed $z \in \alpha'$, τ maps the half line $\{(rz, 1)|r \geqq 0\}$ into the union of the arc $\{\eta r \eta^{-1} qh(\eta(z), 1)|0 \leqq r \leqq 1\}$ in E and the half line of endpoint $qh(\eta(z), 1)$ containing $ph(\eta(z), 1)$. Since

$$(ph(\eta(z), 1), qh(\eta(z), 1)) = h(\eta(z), 1) \in I,$$

there is some $(x, y) \in J$ such that $|\xi^{-1}ph(\eta(z), 1) - \xi^{-1}(x)| < \epsilon'$ and $|\xi^{-1}qh(\eta(z), 1) - \xi^{-1}(y)| < \epsilon'$. By our choice of ϵ', (x, y) belongs to J. It follows that the line joining $ph(\eta(z), 1)$ and $qh(\eta(z), 1)$ does not intersect $L \cap W$ and then does not intersect U either. From this result, we infer that $\tau(C \times \{1\})$ does not intersect U. Hence b is bounding in $R^4 \cup \{\infty\} - \tau(C \times \{1\}) \cup \{\infty\}$, contrary to the fact that b and c' are linking. This completes the proof of our theorem.

Remark. For the case that $n = 4$ and X is *not* the projective plane, a simpler proof of our theorem may be given as follows. Since the first mod 2 Betti number of X is >1, there are two disjoint simple closed curves β and γ in X, such as β_1 and β_2' in the proof of Lemma 1, such that the local orientation of X at a point x of each of the curves is reversed if x moves along the curve once. Let D be the unit circular disk $|z| \leqq 1$ in the complex plane. It is easy to construct a map

$$f: D \times [0, 1] \to X$$

such that

(i) $f(0, 0) = f(0, 1)$ and $f(0, t)$ moves along γ once as t varies from 0 to 1;

(ii) for every $t \in [0, 1]$, f maps $D \times \{t\}$ homeomorphically onto a closed 2-cell E_t in X;

(iii) $f(D \times [0, 1]) \subset X - \beta$;

(iv) for all $z \in D, f(z, 0) = f(\bar{z}, 1)$.

Let $a_t = f(0, t)$ and let α_t be the boundary of E_t. Let $R^4 \cup \{\infty\}$ and β be oriented and let $a_t\alpha_t \cup \{\infty\}$ be oriented such that the orientation is continuous in t, $0 \leqq t \leqq 1$. By Lemma 2, the linking number l_t of the integral fundamental cycle on $a_t\alpha_t \cup \{\infty\}$ and that on β is an odd number. Since l_t is continuous in t, it is independent of t so that $l_0 = l_1$. On the other hand, $a_0\alpha_0 \cup \{\infty\}$ and $a_1\alpha_1 \cup \{\infty\}$ are identical but have opposite orientations. It follows that $l_0 = -l_1 = -l_0$. Hence $l_0 = 0$, contrary to the fact that it is odd.

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