BULL. AUSTRAL. MATH. SOC. 18A30, 18C10, 18D20
VOL. 18 (1978), 125-135.

On characterisation of finitary algebraic categories

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The aim of this article is to characterise categories which are V-algebraic (equals V-theoretical) over V where V is a symmetric monoidal closed category satisfying suitable limit-colimit commutativity conditions (basicly axiom π).

Introduction

In the theory of finitary V-algebraic categories over a category V satisfying axiom π (Borceux and Day [3]) there are two basic characterisation theorems. The first of these is discussed in Borceux and Day [4], Section 2.5, and is based on the concept of *rank* of a functor.

The aim of this paper is to describe the second characterisation theorem which is closer to the original characterisation theorem of Lawvere for $V = Ens$ (see Diers [7], Corollary 5.5.6). This second theorem is based on the notion of a suitable *strong projective generator* in the category; namely the free algebra on $I \in V$ when the category is known to be algebraic.

In Section 3 we develop the theory of near-cartesian closed categories. The principal example of such a category is the category of pointed k-spaces; the tensor product in this category is the "smash product" $X \# Y$ of pointed spaces X and Y and while this is not the cartesian product there are canonical diagonals $X \to X \# X \# X \# \ldots \# X$. This allows us to deduce, from the characterisation theorem, that *all* operadic categories on pointed k-spaces are algebraic (that is,

Received 11 November 1977.

https://doi.org/10.1017/S0004972700007875 Published online by Cambridge University Press

theoretical). The main interest here derives from the well-known fact that in the theory of infinite loop spaces it is possible to use theories (Boardman and Vogt [2] and Beck [1]) *or* operads (May [11] and Kelly [9]). We also point out other instances where this phenomenon occurs.

Throughout the article we assume that $V = (V, \otimes, I, [-,-], \ldots)$ is a complete and cocomplete symmetric monoidal closed category satisfying axiom π, and we assume that all categorical algebra is *relative* to V unless otherwise stated. We assume some familiarity with Borceux and Day [3] and [4]. The basic algebra appears in [6] and [7].

1. Preliminaries

We recall that a (finitary) V-theory is a finite-product-preserving functor $t : V_f^{\mathrm{op}} \to T$ which is one-one on objects, where V_f denotes the full subcategory of V comprising the finite copowers of $I \in V$. Each V-theory (T, t) generates a monad $\mathrm{T} = \mathrm{T}(T)$ on V which has the property

$$\int^{V_f} [n, X] \otimes [m, Tn] \cong [m, TX]$$

for all $m \in V_f$, and is thus the "restriction to V" of the monadic adjunction $t^* \dashv [t^{\mathrm{op}}, 1] : [T, V] \to \left[V_f^{\mathrm{op}}, V\right]$:

$$\begin{array}{ccc} T^b \simeq V^{\mathrm{T}} & \hookrightarrow & [T, V] \\ {\scriptstyle F_{\mathrm{T}}} \uparrow \downarrow {\scriptstyle U_{\mathrm{T}}} & & {\scriptstyle t^*} \uparrow \downarrow {\scriptstyle [t^{\mathrm{op}}, 1]} \\ V & \xrightarrow{\hat{J}} & \left[V_f^{\mathrm{op}}, V\right] \end{array} ,$$

where $J : V_f \to V$ is the canonical inclusion. We say that $\mathrm{T}(T)$ has algebraic rank J. By Day [5], Theorem 2.1, and the density of J, it follows that V^{T} is category equivalent to the full subcategory T^b comprising the finite-product-preserving functors from T to V.

PROPOSITION 1.1. *Let* $t : V_f \to T$ *be a* V*-theory and let* A *be a small category with finite products. Let* $G : A \to V$ *be a finite-product-*

preserving functor and let $H : A^{op} \to T^b$ *be any functor. Then the mean tensor product* $GA * HA$ *exists in* T^b *and is isomorphic to* $\int^A GA \otimes HA$ *in* $[T, V]$.

Proof. Iterated use of axiom π gives us

$$\left[m, \int^A GA \otimes HA(t1)\right] \cong \int^A GA \otimes HA(t1)^m$$
$$\cong \int^A GA \otimes HA(tm) ,$$

as required for $\int^A GA \otimes HA$ in $[T, V]$ to in fact be a T-algebra. //

We also recall from Borceux and Day [4] that if (T, t) is a *commutative* V-theory then T has a canonical symmetric monoidal structure $\otimes : T \otimes T \to T$ such that $t : V_f^{op} \to T$ preserves tensor products.

PROPOSITION 1.2. *If* (T, t) *is a commutative* V*-theory then* T^b *is a symmetric monoidal closed category enriched over* V .

Proof. Clearly T^b is closed under exponentiation in $[T, V]$, because the internal-hom is given by $[A, B] = \int_T [A(tn), B(tn \otimes -)]$ which preserves finite products whenever B is a T-algebra. The unit object is the free T-algebra on $I \in V$, namely $T(t1, -)$. The tensor product of two algebras A and B is given by

$$A \bar{\otimes} B = \int^{T \otimes T} A(tm) \otimes B(tn) \otimes T(tm \otimes tn, -)$$
$$\cong \int^T A(tm) \otimes \int^T B(tn) \otimes T(tm \otimes tn, -) .$$

But, for each fixed m , $\int^T B(tn) \otimes T(tm \otimes tn, -)$ is a T-algebra; so let it be $H(tm)$ in Proposition 1.1. This then shows that $A \bar{\otimes} B$ is again a T-algebra. Thus the convolution structure on $[T, V]$ *restricts* to T^b .//

This result was established in Borceux and Day [4] but is recalled here for convenience in Section 3.

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2. Structure-semantics and characterisation

We denote by $Adg = Adg(J)$ the category whose objects are functors $U : B \to V$ having a left J-adjoint and whose morphisms are functors $M : B \to B'$ such that $U'M = U$ (see Diers [6], Section 4). The functor $(J-)$ semantics

$$\text{Sem} : Th^{op} \to Adg$$

is given by $\text{Sem}(T) = \left\{V^{T}, U_{T}\right\}$.

THEOREM 2.1. *Semantics* Sem : $Th^{op} \to Adg$ *is fully faithful and has a left adjoint.*

Proof. This is just the V-analogue of Diers [6], Theorem 4.2. //

The left adjoint is the structure functor

$$\text{Str} : Adg \to Th^{op}$$

which maps $F \underset{J}{\dashv} U : B \to V$ to the obvious algebraic theory generated by $F \underset{J}{\dashv} U$. We have

$$\varepsilon : \text{Str Sem} \cong 1 : Th^{op} \to Th^{op} ,$$

$$\eta : 1 \Rightarrow \text{Sem Str} : Adg \to Adg .$$

THEOREM 2.2. *Given* $F \underset{J}{\dashv} U : B \to V$, *then* B *is algebraic with respect to* U *if* B *is cocomplete,* U *reflects isomorphisms, and* U *preserves* $GA * HA$ *whenever* A *is a small category with finite products,* $G : A \to V$ *is a finite-product-preserving functor, and* $H : A^{op} \to B$ *is a functor.*

Proof. Note first that, using the fact that Str (Sem Str) $\cong$ Str, we obtain a functor $H : T^{op} \to B$ such that

$$(*) \qquad \begin{array}{ccc} T^{op} & \xrightarrow{\;H\;} & B \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle U} \\ T^{b} & \xrightarrow[\;U_T\;]{} & V \end{array}$$

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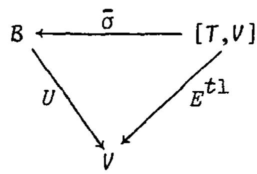
commutes, and such that $\mathrm{Lan}_Y H$ is left adjoint to $\eta_B : B \to T^b$. Thus, since B is cocomplete, η_B has a left adjoint σ which is the restriction to algebras of a left adjoint $\bar{\sigma}$ to $B \to [T, V]$; namely $\bar{\sigma}(G) = \int^T G(tn).H(tn)$. We require

(i) $\sigma\eta \cong 1 : B \to B$, and

(ii) $1 \cong \eta\sigma : T^b \to T^b$.

Because U reflects isomorphisms, we require for (i) that $U\sigma\eta \cong U$. But $U_T\eta \cong U$, so we need $U\sigma \cong U_T : T^b \to V$; this also guarantees (ii). Finally, to establish the result, consider



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Then $\bar{\sigma}(G) = \int^T G(tn).H(tn)$ for all $G \in [T, V]$. If $G \in T^b$, then

$$U\left(\int^T G(tn).H(tn)\right) \cong B\left(F1, \int^T G(tn).H(tn)\right) \quad \text{since } F \dashv_J U ,$$

$$\cong \int^T G(tn) \otimes B(F1, H(tn)) \quad \text{by hypothesis,}$$

$$\cong \int^T G(tn) \otimes UH(tn)$$

$$\cong \int^T G(tn) \otimes T(Tn, T1) \quad \text{by } (*),$$

$$\cong G(t1) \quad \text{by the representation theorem,}$$

$$= U_T(G) \quad \text{as required.} \qquad //$$

An object $P \in B$ is called an *abstractly finite projective generator of* B if $B(P, -) : B \to V$ reflects isomorphisms and preserves $GA * HA$ whenever A is a small category with finite products, $G : A \to V$ is a finite-product-preserving functor, and $H : A^{op} \to B$ is any functor.

COROLLARY 2.3. *Let* B *be cocomplete with an abstractly finite*

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projective generator P . *Then* $U = \mathcal{B}(P, -) : \mathcal{B} \to V$ *is algebraic.*

Proof. The adjoint $F \underset{J}{\dashv} U$ is given by $F(n) = {}^{n}P$, so the result follows from the theorem.

COROLLARY 2.4. *Let* V *be a* π*-category* (see Borceux and Day [4], Definition 2.1.1). *Then* $\mathcal{B}$ *is algebraic over* V *if and only if* $\mathcal{B}$ *is cocomplete and has an abstractly finite projective generator.*

Proof. Over a π-category any algebraic category is cocomplete, since it has coequalisers of reflective pairs. Moreover, FI is an abstractly finite projective generator of T^b by Proposition 1.1. //

In conclusion we note that if $F \underset{J}{\dashv} U : \mathcal{B} \to V$ and $UF : V_f \to V$ has the structure of a monoidal functor then the theory of the structure of U is *commutative*.

THEOREM 2.5. *If* V *is a* π*-category, then* $\mathcal{B}$ *is commutatively* V*-algebraic over* V *if and only if* $\mathcal{B}$ *is cocomplete and has a symmetric monoidal closed structure* $(\mathcal{B}, I, \otimes, [-, -], \ldots)$ *whose identity object* I *is an abstractly finite projective generator of* $\mathcal{B}$. //

3. Example: near-cartesian closed categories

The category of pointed compactly generated spaces (k-spaces) is more than just algebraic over compactly generated spaces. It is equipped with a canonical identification map $A \times B \to A \otimes B$ and this permits us to consider *diagonals* $A \to A \otimes \ldots \otimes A$. The key theoretical observation at this point is that if T is a commutative V-theory over a closed category V which satisfies axiom π then, in the presence of a suitable diagonal functor $T \to T \otimes T$, the functor $\int^{T} A(tn) \otimes T(tn \otimes \ldots \otimes tn, -)$ is again a T-algebra and is, in fact, the mth tensor power of A .

In order to formalise what we have in mind here, we introduce the following definition.

DEFINITION 3.1. The closed category V is called *near-cartesian* if there exists an ordinary natural transformation $e_{AB} : A \times B \to A \otimes B$ such that the following diagrams commute:

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(1)
$$\begin{array}{ccc} I & \xrightarrow{\delta} & I \times I \\ & \searrow^{\cong} & \downarrow e \\ & & I \otimes I \end{array} ,$$

(2)
$$\begin{array}{ccc} (A \times B) \otimes (C \times D) & \xrightarrow{\binom{p \otimes p}{p \otimes p}} & (A \otimes C) \times (B \otimes D) \\ {\scriptstyle e \otimes e} \downarrow & & \downarrow {\scriptstyle e} \\ (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\cong} & (A \otimes C) \otimes (B \otimes D) \end{array} ,$$

and

(3) $$\int^{n} [n, A] \otimes \left(\otimes^{m} n\right) \cong \otimes^{m} A \quad \text{for all } m > 0 \text{ and } A \in V .$$

Note that it is possible to write (3) because (1) and (2) imply the existence of a canonical functor $K = K(e) : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ for any V-categories $\mathcal{A}$ and $\mathcal{B}$. The following consequence is easily established.

PROPOSITION 3.2. *Let V be near-cartesian and let (T, t) be a commutative V-theory. Then the m-fold tensor power $(m > 0)$ of a T-algebra A is given by the formula*

$$A \otimes \ldots \otimes A = \int^{T} A(tn) \otimes T(tn \otimes \ldots \otimes tn, -) . \qquad //$$

THEOREM 3.3. *Let (T, t) be a commutative theory over the near-cartesian closed category V and suppose V is a π-category. Let R be a monad on T^{b} generated by an operad on T^{b}. Then $\left(T^{b}\right)^{\mathrm{R}}$ is algebraic over V.*

Proof. For the concept of an operad we refer to May [*11*]. The important aspect here is that the endofunctor R is given by an expression of the form $RA = \int^{n} Sn \mathbin{\bar{\otimes}} \left(\otimes^{n} A\right)$ where n runs over *either* the free V-category on the integers *or* the free V-category on the permutation category (the integers are greater than or equal to 0, with no morphisms $n \to m$ if $n \neq m$, and the morphisms $n \to n$ being the permutations on n). Let us denote the V-adjunctions involved by

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$$V \underset{U}{\overset{F}{\rightleftarrows}} T^b \underset{U'}{\overset{F'}{\rightleftarrows}} (T^b)^{\mathrm{R}} .$$

Both adjunctions $F \dashv U$ and $F' \dashv U'$ are monadic and both U and U' create coequalisers of reflective pairs; hence UU' reflects isomorphisms and $(T^b)^{\mathrm{R}}$ is cocomplete. Thus, by Theorem 2.2, it remains to check that UU' preserves $GA * HA$ whenever A is a small category with finite products, $G : \mathrm{A} \to V$ preserves finite products, and $H : \mathrm{A}^{\mathrm{op}} \to (T^b)^{\mathrm{R}}$ is any functor. But already U preserves $GA * U'HA$ so it remains to check that R on T^b preserves $GA * U'HA$. For any $H' : \mathrm{A}^{\mathrm{op}} \to T^b$ we have

$$R(GA * H'A) = \int^n Sn \mathbin{\bar{\otimes}} \left(\vec{\otimes}^n (GA * H'A)\right) ,$$

where

$$\vec{\otimes}^n B = \int^T B(tm) \otimes \mathrm{T}(tm \otimes \ldots \otimes tm, -) : \mathrm{T} \to V .$$

Thus

$$\begin{aligned} \vec{\otimes}^n (GA * H'A) &\cong \int^T \left\{\int^{\mathrm{A}} GA . H'A\right\}(tm) \otimes \mathrm{T}(tm \otimes \ldots \otimes tm, -) \\ &\cong \int^T \left\{\int^{\mathrm{A}} GA . H'A(tm)\right\} \otimes \mathrm{T}(tm \otimes \ldots \otimes tm, -) ; \end{aligned}$$

so

$$\begin{aligned} R(GA * H'A) &\cong \int^n Sn \mathbin{\bar{\otimes}} \left[\int^{\mathrm{A}} GA . \left\{\int^T H'A(tm) \otimes \mathrm{T}(tm \otimes \ldots \otimes tm, -)\right\}\right] \\ &\cong \int^{\mathrm{A}} GA . \left\{\int^n Sn \mathbin{\bar{\otimes}} \int^T H'A(tm) \otimes \mathrm{T}(tm \otimes \ldots \otimes tm, -)\right\} \\ &\cong GA * RH'A . \end{aligned}$$

Thus, by induction, we have $R^p(GA * H'A) \cong GA * R^p H'A$ for $p \geq 0$. Thus U' creates $GA * HA$, as required. //

In order to generate examples of near-cartesian closed categories we consider the following

DEFINITION 3.4. A symmetric monoidal monad $\mathrm{T} = (T, \mu, \eta)$ on a cartesian closed category is called *near-cartesian* if the transformation

$\tilde{T} : TX \times TY \to T(X \times Y)$ is left inverse to the canonical transformation $\kappa : T(X \times Y) \to TX \times TY$.

LEMMA 3.5. *Let V be cartesian closed and let $\mathsf{T} = (T, \mu, \eta)$ be a near-cartesian monad on V. Suppose T preserves coequalisers of reflective pairs and let $F \dashv U$ denote the associated monoidal adjunction over V. Then $\tilde{U}_{AB} : UA \times UB \to U(A \otimes B)$ is a (regular epimorphic) natural transformation in V.*

We leave the proof to the reader as an exercise.

THEOREM 3.6. *Let V be cartesian closed and let $\mathsf{T} = (T, \mu, \eta)$ be a finitary near-cartesian monad on V. Then V^{T} is a near-cartesian closed category.*

Proof. V^{T} satisfies axiom π by Borceux and Day [3]. To satisfy Definition 3.1 we choose $e_{AB} = \tilde{U}_{AB}$, using Lemma 3.5. Then, by Definition 3.1, (1) and (2) are simple consequences of applying U and using the naturality of $\tilde{U}$. It remains to prove that

$$\int^{Fn} [Fn, A] \otimes (\otimes^m Fn) \cong \otimes^m A$$

for all $m > 0$ and $A \in V^{\mathsf{T}}$. By virtue of the diagram

$$\begin{array}{l} F^2A \times F^2A \xrightarrow{e} F^2A \times F^2A \rightrightarrows FA\otimes FA \cong F(A\times A) \xrightarrow{\zeta} A\times A \\ \qquad\qquad\qquad\qquad\qquad FA\otimes FA \xrightarrow{\zeta\otimes\zeta} A\otimes A, \quad A\times A \xrightarrow{e} A\otimes A \end{array}$$

we have that e is the coequaliser in V^{T} of a pair of morphisms $F^2A \times F^2A \to A \times A$. We then have

$$\begin{array}{ccc} \int^{Fn} [Fn,A]\otimes(F^2Fn\times F^2Fn) & \longrightarrow & F^2A\times F^2A \\ \downarrow & & \downarrow\downarrow \\ \int^{Fn} [Fn,A]\otimes(Fn\times Fn) & \xrightarrow{\cong} & A\times A \\ \downarrow & & \downarrow \\ \int^{Fn} [Fn,A]\otimes(Fn\otimes Fn) & \longrightarrow & A\otimes A \end{array},$$

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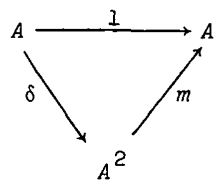
where the isomorphism follows from axiom π on V^T. For similar reasons the top morphism is an epimorphism, so $\int^{Fn} [Fn, A] \otimes (Fn \otimes Fn) \cong A \otimes A$. The proof is analogous for $m > 2$. //

EXAMPLE 3.7 (V cartesian closed). Let A be a commutative semigroup in V such that



commutes (sometimes such an object is called a *semilattice* (without a unit)). Then $TX = X + A$ is a near-cartesian unary monad on V. Thus the category A/V is near-cartesian closed.

EXAMPLE 3.8 (V cartesian closed). Let $G : V \to V$ be a symmetric monoidal finitary near-cartesian endofunctor on V and let $\varepsilon : G \Rightarrow 1$ be a monoidal natural transformation. Then $TX = X + GX$ is near-cartesian and finitary. Thus the category "G/V" is near-cartesian closed.

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