



RESEARCH ARTICLE

Nef cones of fiber products and an application to the cone conjecture

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Dedicated to Professor Keiji Oguiso, on the occasion of his sixtieth birthday

Abstract

We prove a decomposition theorem for the nef cone of smooth fiber products over curves, subject to the necessary condition that their Néron–Severi space decomposes. We apply it to describe the nef cone of so-called Schoen varieties, which are the higher-dimensional analogues of the Calabi–Yau threefolds constructed by Schoen. Schoen varieties give rise to Calabi–Yau pairs, and in each dimension at least three, there exist Schoen varieties with nonpolyhedral nef cone. We prove the Kawamata–Morrison–Totaro cone conjecture for the nef cones of Schoen varieties, which generalizes the work by Grassi and Morrison.

1. Introduction

1.1. Cone conjecture

To understand the geometry of a smooth projective variety X , studying the Mori cone of curves $\overline{NE}(X)$ and its dual, the nef cone $\text{Nef}(X)$, is central, especially from the viewpoint of the minimal model program (MMP).

An important part of the relationship between the Mori cone and the MMP is captured by the cone theorem, and the contraction theorem. These theorems assert that the K_X -negative part of the Mori cone of a smooth projective variety X is rational polyhedral away from the K_X -trivial hyperplane, and the extremal rays of the K_X -negative part correspond to some morphisms from X involved in the MMP. In particular, when X is a Fano variety (namely, $-K_X$ is ample), the cone $\text{Nef}(X)$ is a rational polyhedral cone, and its extremal rays are generated by semiample classes. In general, however, it is difficult to describe the whole Mori cone, or dually the whole nef cone, even under the slightly weaker assumption that $-K_X$ is semiample. For instance, if X is the blowup of \mathbb{P}^2 at the base points of a general pencil of cubic curves in \mathbb{P}^2 , then $-K_X$ is semiample but $\text{Nef}(X)$ is not rational polyhedral.

When X is K -trivial, we expect nevertheless that some essential parts of the nef cone of X are rational polyhedral, up to the action of $\text{Aut}(X)$. A precise statement, known as the cone conjecture, was first

formulated by Morrison [35] and Kawamata [22]. It was later generalized by Totaro [48] to klt Calabi–Yau pairs (X, Δ) (see Section 2.2), thus including many more examples, already in dimension 2. When stated by these authors, the cone conjecture comprises predictions both on the nef cone and on the movable cone of varieties and has both an absolute and a relative version. In what follows, we will only consider the absolute cone conjecture for nef cones of certain Calabi–Yau pairs.

Let us recall the statement formulated by Totaro in [48, Conjecture 2.1], starting with some notations. For a pair (X, Δ) , we define

$$\text{Aut}(X, \Delta) := \{f \in \text{Aut}(X) \mid f(\text{supp}(\Delta)) = \text{supp}(\Delta)\}.$$

We also define the *nef effective cone* $\text{Nef}^e(X)$ as

$$\text{Nef}^e(X) := \text{Nef}(X) \cap \text{Eff}(X),$$

where $\text{Eff}(X)$ is the effective cone of X .

Conjecture 1.1 (Kawamata–Morrison–Totaro cone conjecture). *Let (X, Δ) be a klt Calabi–Yau pair. There exists a rational polyhedral cone Π in $\text{Nef}^e(X)$ which is a fundamental domain for the action of $\text{Aut}(X, \Delta)$ on $\text{Nef}^e(X)$, in the sense that*

$$\text{Nef}^e(X) = \bigcup_{g \in \text{Aut}(X, \Delta)} g^* \Pi,$$

and $\Pi^\circ \cap (g^* \Pi)^\circ = \emptyset$ unless $g^* = \text{id}$.

An important prediction of the cone conjecture for the MMP is that the number of $\text{Aut}(X, \Delta)$ -equivalence classes of faces of the nef effective cone $\text{Nef}^e(X)$ corresponding to birational contractions or fiber space structures is finite (see, e.g., [48, p.243]).

Note that it is standard to replace Conjecture 1.1 by the *a priori* stronger following conjecture. Let $\text{Nef}^+(X)$ denote the convex hull of

$$\text{Nef}(X) \cap N^1(X)_{\mathbb{Q}},$$

where $N^1(X)_{\mathbb{Q}}$ is the rational Néron–Severi space of X .

Conjecture 1.2. *Let (X, Δ) be a klt Calabi–Yau pair. Then the following statements hold.*

1. *There exists a rational polyhedral cone in $\text{Nef}^+(X)$ which is a fundamental domain for the action of $\text{Aut}(X, \Delta)$ on $\text{Nef}^+(X)$.*
2. *We have*

$$\text{Nef}^+(X) = \text{Nef}^e(X).$$

Thanks to the fundamental work of Looijenga [32], we prove that the two conjectures are equivalent (see Corollary 2.6).

1.2. Nef cones of fiber products

The starting point of this work is a decomposition theorem for the nef cone of a fiber product over a curve.

It begins with the following general question. Let W_1 and W_2 be projective varieties, and let $\phi_1 : W_1 \rightarrow B$ and $\phi_2 : W_2 \rightarrow B$ be surjective morphisms over a base B . Assume that the fiber product $W := W_1 \times_B W_2$ is irreducible.

Question 1.3. Denote by $p_i : W \rightarrow W_i$ the natural projections. When do we have

$$p_1^* \text{Nef}(W_1) + p_2^* \text{Nef}(W_2) = \text{Nef}(W)? \tag{1.1}$$

As the nef cone of a projective variety linearly spans the whole space of numerical classes of \mathbb{R} -divisors on the variety, the nef cone decomposition (1.1) exists only if

$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}. \tag{1.2}$$

We may then ask which fiber products satisfying the decomposition (1.2) also have the decomposition (1.1).

When B is a point, it is not hard to see that Equation (1.2) implies Equation (1.1). In this case, indeed, the decomposition (1.2) is a direct sum. Every divisor D decomposes uniquely as $p_1^*D_1 + p_2^*D_2$, where $D_1 = D|_{W_1 \times \{\text{pt}\}}$ and $D_2 = D|_{\{\text{pt}\} \times W_2}$. If D is nef, then so are its restrictions, and hence Equation (1.1) follows. When B is \mathbb{P}^1 and the varieties W_i are certain rational elliptic surfaces, the decomposition (1.1) was proven in [14, Proposition 3.1]. We show that the implication (1.2) \Rightarrow (1.1) continues to hold for an arbitrary irreducible fiber product over a curve.

Theorem 1.4. *For $i = 1, 2$, let $\phi_i : W_i \rightarrow B$ be a surjective morphism from a projective variety to a projective curve B . Assume that*

1. *The fiber product $W = W_1 \times_B W_2$ is irreducible.*
2. *We have*

$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}.$$

Then

$$p_1^*\text{Nef}(W_1) + p_2^*\text{Nef}(W_2) = \text{Nef}(W).$$

As a consequence, we also have $p_1^*\text{Amp}(W_1) + p_2^*\text{Amp}(W_2) = \text{Amp}(W)$.

In Examples 3.5, 3.6 and 3.7, we build examples of fiber products over bases of higher dimension, that fail the implication (1.2) \Rightarrow (1.1). In Remark 3.8, we recall a classical example emphasizing that a similar decomposition does not hold for the movable cone of divisors of a fiber product over a curve.

We establish the following corollary to this first theorem.

Corollary 1.5. *Keep the notations and assumptions of Theorem 1.4. Then the extremal rays of the convex cone $\text{Nef}(W)$ are exactly the pullbacks of the extremal rays of the two cones $\text{Nef}(W_1)$ and $\text{Nef}(W_2)$. In particular, the cone $\text{Nef}(W)$ is rational polyhedral if and only if the cones $\text{Nef}(W_1)$ and $\text{Nef}(W_2)$ are both rational polyhedral.*

This corollary can be seen as a means to construct fiber products over curves, whose nef cones are not rational polyhedral.

1.3. Cone conjecture for Schoen varieties

Among the strict Calabi–Yau manifolds (see Definition 2.2) whose nef cones are known to not be rational polyhedral, to our knowledge, the cone conjecture is only known so far in two special cases. One of them is the desingularized Horrocks–Mumford quintics, studied by Borcea in [4] (see also [12]); the other is the fiber product of two general rational elliptic surfaces with sections over \mathbb{P}^1 , constructed by Schoen in [44] and investigated by Namikawa and Grassi–Morrison [37, 14]. Both examples are of dimension three.

The main goal of this paper is to prove the cone conjecture for generalizations of Schoen’s Calabi–Yau threefolds, typically Calabi–Yau pairs, but also higher-dimensional strict Calabi–Yau varieties. In both cases, the underlying varieties, which we call *Schoen varieties*, are constructed as fiber products over \mathbb{P}^1 .

Let us first summarize our construction defining Schoen varieties; we refer to Subsections 4.1 and 4.2 for more details. We start with Fano manifolds Z_1 and Z_2 of dimension at least two, which respectively admit an ample and globally generated divisor D_i ($i = 1, 2$), such that $-(K_{Z_i} + D_i)$ is globally generated. We take $W_i \subset \mathbb{P}^1 \times Z_i$ to be a general member in the linear system $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_i}(D_i)|$. There is a fibration $\phi_i : W_i \rightarrow \mathbb{P}^1$. We put another mild condition on the fibrations ϕ_1 and ϕ_2 to be general with respect to one another (see the second paragraph of Subsection 4.2 for a precise statement). Consider

the following fiber product over \mathbb{P}^1

$$\phi : X := W_1 \times_{\mathbb{P}^1} W_2 \rightarrow \mathbb{P}^1.$$

Under our assumptions, the variety X is smooth and projective. All varieties obtained through this procedure are called *Schoen varieties*.

It follows from the construction that $-K_X$ is globally generated, so many effective \mathbb{Q} -divisors are \mathbb{Q} -linearly equivalent to $-K_X$. Any such \mathbb{Q} -divisor Δ yields a Calabi–Yau pair (X, Δ) , that we call a *Schoen pair*.

We prove the following result.

Theorem 1.6. *Let (X, Δ) be a Schoen pair. Then there exists a rational polyhedral fundamental domain for the action of $\text{Aut}(X, \Delta)$ on $\text{Nef}^e(X) = \text{Nef}^+(X) = \text{Nef}(X)$.*

Note that, by Corollary 1.5, the cone $\text{Nef}(X)$ is not rational polyhedral as soon as one of the cones $\text{Nef}(W_i)$ ($i = 1, 2$) is not, typically if one of the factors W_i is a rational elliptic surface with $Z_i \simeq \mathbb{P}^2$ and $D_i = \mathcal{O}_{\mathbb{P}^2}(3)$. Using this remark, we provide in Example 5.6 the first series of strict Calabi–Yau manifolds (and Calabi–Yau pairs) of arbitrary dimension for which the cone conjecture holds, with nef cones that are not rational polyhedral.

We finally mention two unsurprising consequences of Theorem 1.6 (see Corollary 5.7): The finite presentation of the group of components $\pi_0\text{Aut}(X)$ and the finiteness of real forms on X , up to isomorphism.

1.4. Relation to other work

1.4.1. Cone conjecture

We refer to [30] and the references therein for a survey of the cone conjecture for varieties (as opposed to pairs). As for the cone conjecture for Calabi–Yau pairs, its two-dimensional case was proven by Totaro [48]. Kopper [27] also proved the cone conjecture for Calabi–Yau pairs arising from Hilbert schemes of points on certain rational elliptic surfaces; the underlying varieties in his work may have non rational polyhedral nef cones, but they only appear in even dimensions. The references [11, 31] also contain some recent results.

1.4.2. Cone conjectures for varieties with rational polyhedral nef cones

One way of proving the cone conjecture for a smooth projective variety X is to show that $\text{Nef}(X)$ is a rational polyhedral cone and that $\text{Nef}(X) = \text{Nef}^e(X)$ (see, e.g., [29, Proposition 6.5]). This is the case whenever X is a smooth anticanonical hypersurface in a Fano manifold Y of dimension at least 4, by the following theorem, due to Kollár [3, Appendix].

Theorem 1.7. *Let D be a smooth anticanonical hypersurface in a smooth Fano variety Y of dimension at least 4. Then the natural restriction map $\text{Nef}(Y) \rightarrow \text{Nef}(D)$ is an isomorphism. In particular, $\text{Nef}(D)$ is a rational polyhedral cone, generated by classes of semiample divisors.*

Other Calabi–Yau pairs (X, Δ) for which $\text{Nef}(X) = \text{Nef}^e(X)$ is rational polyhedral are described in the work of Coskun and Prendergast-Smith [40, 8, 9].

1.4.3. Fiber product constructions

Constructing Calabi–Yau threefolds as fiber products of two general rational elliptic surfaces with sections over \mathbb{P}^1 was first considered and investigated by Schoen [44]. It recently came back to light as Suzuki considered a certain higher-dimensional generalization of Schoen’s construction and studied its arithmetic properties in [46]. Similar ideas are also involved in Sano’s constructions of non-Kähler Calabi–Yau manifolds with arbitrarily large second Betti number in [42].

1.4.4. Cone conjecture for movable cones

We have already mentioned that there is a part of the cone conjecture concerned with movable cones [48, Conjecture 2.1.(2)]. It predicts that a Calabi–Yau variety should have finitely many minimal models, up to isomorphism [7, Theorem 2.14]. See [36, 22, 48, 30] for related references. This part of the cone conjecture was verified for some cases. Notably, in [6], Cantat and Oguiso produced the first series of strict Calabi–Yau manifolds in arbitrary dimension whose movable cones are not rational polyhedral and for which the cone conjecture for movable cones holds. We refer to [17, 20, 28, 50] and references therein for more results.

In [37], Namikawa showed that a certain strict Calabi–Yau threefold, constructed as a Schoen variety, has finitely many minimal models, up to isomorphism. Nonetheless, the cone conjecture is still unknown for the movable cone of divisors of this Calabi–Yau threefold. The cone conjecture for the nef cones of each of these minimal models is not known either. Similar questions could be asked for the Schoen varieties of higher dimension constructed here.

1.5. Structure of the paper

Section 2 is devoted to some preliminaries and fundamental results. We prove Theorem 1.4 in Section 3. After constructing Schoen varieties and Schoen pairs in Section 4, we prove Theorem 1.6 in Section 5.

2. Preliminaries

We work over the field \mathbb{C} of complex numbers throughout this paper. For notions of birational geometry, we refer to [26].

2.1. Notations

Let X be a projective variety. We write $N^1(X)$ for the free abelian group generated by the classes of Cartier divisors modulo numerical equivalence.

Inside the vector space $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$, we denote by $\text{Nef}(X)$ the *nef cone*, that is, the closure of the ample cone $\text{Amp}(X)$, and by $\text{Eff}(X)$ the *effective cone*. The *nef effective cone* $\text{Nef}^e(X)$ is defined as

$$\text{Nef}^e(X) := \text{Nef}(X) \cap \text{Eff}(X).$$

Let $\text{Nef}^+(X)$ denote the convex hull of

$$\text{Nef}(X) \cap N^1(X)_{\mathbb{Q}},$$

where $N^1(X)_{\mathbb{Q}} := N^1(X) \otimes \mathbb{Q}$. We denote by $N_1(X)$ the group of 1-cycles modulo numerical equivalence. The intersection product defines a perfect pairing between the two vector spaces $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$. Under this pairing, the nef cone $\text{Nef}(X)$ is dual to the *Mori cone* $\overline{\text{NE}}(X)$, which is by definition the closure of the convex cone of effective 1-cycles in $N_1(X)_{\mathbb{R}}$.

The group of automorphisms of X is denoted by $\text{Aut}(X)$ and acts on $N^1(X)$ by pullback. This action

$$\rho : \text{Aut}(X) \rightarrow \text{GL}(N^1(X))$$

linearly extends to $N^1(X)_{\mathbb{R}}$, preserving the cones $\text{Nef}^e(X)$ and $\text{Nef}^+(X)$. The connected component of the identity in $\text{Aut}(X)$ is a normal subgroup $\text{Aut}^0(X)$, which acts trivially on $N^1(X)$ [5, Lemma 2.8].

2.2. Klt Calabi–Yau pairs

A *pair* is the data (X, Δ) of a normal projective variety X together with an effective \mathbb{R} -divisor Δ on X such that $K_X + \Delta$ is \mathbb{R} -Cartier.

Definition 2.1. Following [48], we say that a pair (X, Δ) is *Calabi–Yau* if X is \mathbb{Q} -factorial and $K_X + \Delta$ is numerically trivial.

Let us briefly recall the definition of a Kawamata log terminal (klt) pair. We start with a notation. For any pair (X, Δ) and any birational morphism $\mu : \tilde{X} \rightarrow X$, there exists a unique \mathbb{R} -divisor $\tilde{\Delta}$ on \tilde{X} such that

$$K_{\tilde{X}} + \tilde{\Delta} = \mu^*(K_X + \Delta) \text{ and } \mu_*\tilde{\Delta} = \Delta.$$

A pair (X, Δ) is called *klt* if, for any birational morphism $\mu : \tilde{X} \rightarrow X$, when defining the divisor $\tilde{\Delta}$ as above, each irreducible component of $\tilde{\Delta}$ has coefficient less than one.

Note that if we can find one resolution of singularities $\mu : \tilde{X} \rightarrow X$ whose corresponding divisor $\tilde{\Delta}$ has simple normal crossings, with irreducible components of coefficients less than one, then (X, Δ) is klt.

Definition 2.2. Let X be a smooth projective variety. We say that X is a *Calabi–Yau manifold* if its canonical line bundle K_X is trivial and $h^i(X, \mathcal{O}_X) = 0$ for any $0 < i < \dim X$. If in addition, X is simply connected, we call it a *strict Calabi–Yau manifold*.

2.3. Looijenga’s result

The following result is crucial in this paper.

Proposition 2.3. *Let X be a projective variety, and let $H \leq \text{Aut}(X)$ be a subgroup. Assume that there is a rational polyhedral cone $\Pi \subset \text{Nef}^+(X)$ such that $\text{Amp}(X) \subset H \cdot \Pi$. Then*

1. $H \cdot \Pi = \text{Nef}^+(X)$, and the H -action on $\text{Nef}^+(X)$ has a rational polyhedral fundamental domain.
2. The group $\rho(H)$ is finitely presented.

This result should be well known to experts, but we include a proof for the sake of completeness. It relies on the fundamental results due to Looijenga [32, Proposition 4.1, Application 4.14, and Corollary 4.15], which we extract and formulate here as Lemma 2.4. Recall that a convex cone $C \subset N_{\mathbb{R}}$ in a finite dimensional \mathbb{R} -vector space $N_{\mathbb{R}}$ is called *strict* if its closure $\overline{C} \subset N_{\mathbb{R}}$ contains no line.

Lemma 2.4. *Let N be a finitely generated free \mathbb{Z} -module, and let C be a strict convex open cone in the \mathbb{R} -vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Let C^+ be the convex hull of $\overline{C} \cap N_{\mathbb{Q}}$. Let $(C^{\vee})^{\circ} \subset N_{\mathbb{R}}^{\vee}$ be the interior of the dual cone of C . Let Γ be a subgroup of $\text{GL}(N)$ which preserves the cone C . Suppose that*

- there is a rational polyhedral cone $\Pi \subset C^+$ such that $C \subset \Gamma \cdot \Pi$;
- there exists an element $\xi \in (C^{\vee})^{\circ} \cap N_{\mathbb{Q}}^{\vee}$ whose stabilizer in Γ (with respect to the dual action $\Gamma \curvearrowright N_{\mathbb{Q}}^{\vee}$) is trivial.

Then $\Gamma \cdot \Pi = C^+$ and the Γ -action on C^+ has a rational polyhedral fundamental domain. Moreover, the group Γ is finitely presented.

To prove Proposition 2.3, it is key to connect abstract convex geometry as in Lemma 2.4 with the specifics of an automorphism group acting on an ample cone. That is the goal of the next lemma.

Lemma 2.5. *Let X be a projective variety. Then there exists an ample Cartier divisor on X , whose numerical class $\eta \in N^1(X)$ satisfies: For every $g \in \text{Aut}(X)$, if $g^*\eta = \eta$, then g^* is the identity on $N^1(X)$.*

Proof. Our proof is inspired by the argument of [29, Proposition 6.5].

Let $\Gamma := \rho(\text{Aut}(X)) \subset \text{GL}(N^1(X))$. For every $\theta \in N^1(X)_{\mathbb{Q}}$, let Γ_{θ} denote the subgroup of Γ stabilizing θ . We want to find an element $\eta \in \text{Amp}(X) \cap N^1(X)$ such that Γ_{η} is trivial. By linearity, it is sufficient to find such an element in $\text{Amp}(X) \cap N^1(X)_{\mathbb{Q}}$.

By Fujiki–Liebermann’s theorem [5, Theorem 2.10], for every element θ in $\text{Amp}(X) \cap N^1(X)_{\mathbb{Q}}$, the stabilizer Γ_{θ} is finite. Pick an element $\eta \in \text{Amp}(X) \cap N^1(X)_{\mathbb{Q}}$ such that Γ_{η} has the smallest possible order. Since the discrete set $N^1(X)$ is preserved by the action of Γ , there is an open neighborhood $U \subset \text{Amp}(X)$ of η such that, for every $\gamma \in \Gamma \setminus \Gamma_{\eta}$, the intersection $\gamma U \cap U$ is empty. In particular, for

every $\theta \in U \cap N^1(X)_{\mathbb{Q}}$, we have $\Gamma_{\theta} \subset \Gamma_{\eta}$, so $\Gamma_{\theta} = \Gamma_{\eta}$ by the minimality assumption on η . Hence, we have

$$\gamma|_{U \cap N^1(X)_{\mathbb{Q}}} = \text{id}_{U \cap N^1(X)_{\mathbb{Q}}},$$

which extends by linearity to $\gamma = \text{id}$. So the stabilizer Γ_{η} is trivial, which concludes the proof. □

We can now establish Proposition 2.3.

Proof of Proposition 2.3. Let us set $N = N^1(X)$, $C = \text{Amp}(X)$ and $\Gamma = \rho(H)$. To apply Lemma 2.4 in this setup, it suffices to construct an element $\xi_0 \in (C^{\vee})^{\circ} \cap N_{\mathbb{Q}}^{\vee}$ with trivial stabilizer with respect to the dually induced Γ -action. Start by picking any $\xi \in (C^{\vee})^{\circ} \cap N_{\mathbb{Q}}^{\vee}$.

The idea is to find a minimizer η for the linear functional ξ on the set

$$\Sigma := \{\eta \in C \cap N \mid \Gamma_{\eta} \text{ is trivial}\}$$

and to relate the stabilizer of η (which is then trivial by construction) to the stabilizer of ξ (which we want to be trivial).

Note that Σ is nonempty by Lemma 2.5 and discrete. By definition, the linear form ξ takes positive values on the whole convex set $\overline{C} \setminus \{0\}$. Picking a large enough positive integer r , the intersection

$$\Sigma \cap \{x \in \overline{C} \mid \xi(x) \leq r\}$$

is now nonempty and finite. Minimizing ξ on this finite set is equivalent to minimizing it on Σ , and thus ξ has finitely many minimizers in Σ .

Since $C \cap N$ is discrete, we can now perturb ξ into a new linear form $\xi_0 \in (C^{\vee})^{\circ} \cap N_{\mathbb{Q}}^{\vee}$, which has exactly one minimizer η on Σ . As the set Σ is Γ -invariant and as Γ_{η} is trivial, we have, for any nontrivial $\gamma \in \Gamma$, that $\gamma\eta \in \Sigma \setminus \{\eta\}$, and in particular

$$(\gamma\xi_0)(\eta) = \xi_0(\gamma\eta) > \xi_0(\eta).$$

So the stabilizer of ξ_0 in Γ is trivial. □

We prove a simple corollary of Proposition 2.3.

Corollary 2.6. *Conjecture 1.1 and Conjecture 1.2 are equivalent.*

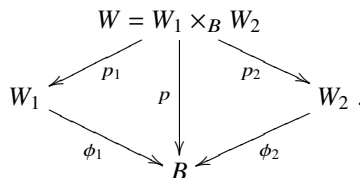
Proof. Clearly, Conjecture 1.2 implies Conjecture 1.1. Now, fix a pair (X, Δ) for which Conjecture 1.1 holds. Let $\Pi \subset \text{Nef}^e(X)$ be a rational polyhedral fundamental domain for the action of $\text{Aut}(X, \Delta)$ on $\text{Nef}^e(X)$. Then $\Pi \subset \text{Nef}^+(X)$ by definition of $\text{Nef}^+(X)$. By Proposition 2.3.(1),

$$\text{Nef}^e(X) = \text{Aut}(X, \Delta) \cdot \Pi = \text{Nef}^+(X).$$

So Conjecture 1.2 holds. □

3. The nef cone of a fiber product over a curve

In this section, we prove Theorem 1.4. Let us recall the notations. For $i = 1, 2$, the map $\phi_i : W_i \rightarrow B$ is a surjective morphism from a projective variety to a projective curve B . We consider the fiber product



and work under the following assumptions:

1. The fiber product $W = W_1 \times_B W_2$ is irreducible;
2. For every $D \in N^1(W)_{\mathbb{R}}$, there exist $D_1 \in N^1(W_1)_{\mathbb{R}}$ and $D_2 \in N^1(W_2)_{\mathbb{R}}$ such that

$$D = p_1^*D_1 + p_2^*D_2.$$

Proof of Theorem 1.4. Let us fix $D \in \text{Nef}(W)$ and consider a decomposition in real classes

$$D = p_1^*D_1 + p_2^*D_2 \in N^1(W)_{\mathbb{R}}$$

as in Assumption (2) right above. We prove three lemmas regarding the positivity of these two summands D_1 and D_2 .

Lemma 3.1. *Fix $i = 1, 2$. Let C_i be a curve contained in a fiber of $\phi_i : W_i \rightarrow B$. Then $D_i \cdot C_i \geq 0$.*

Proof. By symmetry, we can focus on $i = 1$. Fix any point $s \in \phi_2^{-1}(\phi_1(C_1))$ and consider the fiber product $\widetilde{C}_1 := C_1 \times_B \{s\}$, which can be seen as a curve in W_1 . We have

$$0 \leq D \cdot \widetilde{C}_1 = (p_1^*D_1 + p_2^*D_2) \cdot \widetilde{C}_1 = D_1 \cdot p_{1*}\widetilde{C}_1 + D_2 \cdot p_{2*}\widetilde{C}_1 = D_1 \cdot C_1.$$

This proves the lemma. □

Lemma 3.2. *Either D_1 or D_2 is nef.*

Proof. Assume by contradiction that both D_1 and D_2 are not nef. Then for each i , there exists a curve C_i in W_i such that $D_i \cdot C_i < 0$. Note that since the fiber product W is assumed to be irreducible, the base B is also irreducible. Hence, and by Lemma 3.1, we have $\phi_i(C_i) = B$. So the (possibly reducible) fiber product $C_1 \times_B C_2$ contains a curve \widetilde{C} dominating B . Let $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ be such that $p_{i*}\widetilde{C} = \beta_i C_i$. Then on one hand,

$$\beta_1 D_1 \cdot C_1 + \beta_2 D_2 \cdot C_2 < 0,$$

and on the other hand,

$$\beta_1 D_1 \cdot C_1 + \beta_2 D_2 \cdot C_2 = (p_1^*D_1 + p_2^*D_2) \cdot \widetilde{C} = D \cdot \widetilde{C} \geq 0.$$

This is a contradiction. □

For the third lemma, we fix a point $b \in B$.

Lemma 3.3. *Fix $i = 1, 2$. Then there exists $N_i \in \mathbb{R}$ such that for any real number $n \geq N_i$, the divisor $D_i + n\phi_i^*\mathcal{O}_B(b)$ is nef.*

Proof. By symmetry, we can focus on $i = 2$. Let C_1 be a curve in W_1 such that $\phi_1(C_1) = B$. Set

$$N_2 := \frac{D_1 \cdot C_1}{\deg(C_1 \xrightarrow{\phi_1} B)}$$

and consider the following classes

$$D'_1 := D_1 - N_2\phi_1^*\mathcal{O}_B(b) \quad \text{and} \quad D'_2 := D_2 + N_2\phi_2^*\mathcal{O}_B(b).$$

By construction, we have $D'_1 \cdot C_1 = 0$ and $D = p_1^*D'_1 + p_2^*D'_2$.

We want to show that D'_2 is nef. Let C_2 be a curve in W_2 . If it is contained in a fiber of ϕ_2 , then $D'_2 \cdot C_2 \geq 0$ by Lemma 3.1. Suppose now that $\phi_2(C_2) = B$, let \widetilde{C} be a curve in the fiber product $C_1 \times_B C_2$

dominating B , and define $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ such that $p_{i*}\tilde{C} = \beta_i C_i$. We have

$$\begin{aligned} \beta_2 D'_2 \cdot C_2 &= \beta_1 D'_1 \cdot C_1 + \beta_2 D'_2 \cdot C_2 \\ &= (p_1^* D'_1 + p_2^* D'_2) \cdot \tilde{C} \\ &= D \cdot \tilde{C} \geq 0. \end{aligned}$$

So D'_2 is nef. A fortiori, for $n \geq N_2$, the following class

$$D_2 + n\phi_2^* \mathcal{O}_B(b) = D'_2 + (n - N_2)\phi_2^* \mathcal{O}_B(b)$$

is also nef. □

Let us resume the proof of Theorem 1.4. For any $t \in \mathbb{R}$, let

$$D_1(t) := D_1 - t\phi_1^* \mathcal{O}_B(b) \quad \text{and} \quad D_2(t) := D_2 + t\phi_2^* \mathcal{O}_B(b).$$

By Lemma 3.3, we can define intervals

$$I_1 =] - \infty, -N_{1,\min}] \quad \text{and} \quad I_2 = [N_{2,\min}, +\infty[$$

such that $D_i(t)$ is nef if and only if $t \in I_i$. Since we have for all $t \in \mathbb{R}$,

$$D = p_1^* D_1(t) + p_2^* D_2(t),$$

Lemma 3.2 shows that either $D_1(t)$ or $D_2(t)$ is nef, that is, $I_1 \cup I_2 = \mathbb{R}$. Hence, $I_1 \cap I_2$ is nonempty and fixing an element t in this intersection, both $D_1(t)$ and $D_2(t)$ are now nef, giving a desired decomposition.

The decomposition of the ample cone of W finally follows from the decomposition of the nef cone by [41, Corollary 6.6.2]. □

Remark 3.4. In the setup of Theorem 1.4, we also have a decomposition of the relative nef cone

$$\text{Nef}(W/B) = p_1^* \text{Nef}(W_1/B) + p_2^* \text{Nef}(W_2/B)$$

by the projection formula – this is exactly Lemma 3.1.

As a consequence of Theorem 1.4, we prove Corollary 1.5.

Proof of Corollary 1.5. First, consider $E \in \text{Nef}(W)$ spanning an extremal ray of $\text{Nef}(W)$. Then by Theorem 1.4, there is a decomposition $E = p_1^* E_1 + p_2^* E_2$, with $E_i \in \text{Nef}(W_i)$, and either E_1 or E_2 is nonzero. By extremality, E is thus either in $p_1^* \text{Nef}(W_1)$, or in $p_2^* \text{Nef}(W_2)$. By symmetry, we can assume that $E = p_1^* D$, for some $D \in \text{Nef}(W_1)$. Let us show that D spans an extremal ray in $\text{Nef}(W_1)$. Let $D = F + F'$ be any decomposition with $F, F' \in \text{Nef}(W_1)$. Then $E = p_1^* D = p_1^* F + p_1^* F'$ with $p_1^* F, p_1^* F' \in \text{Nef}(W)$, and thus by extremality, $p_1^* F$ and $p_1^* F'$ are proportional. Since $p_1^* : N^1(W_1)_{\mathbb{R}} \rightarrow N^1(W)_{\mathbb{R}}$ is injective, F and F' are proportional as well. This shows that D spans an extremal ray.

We thus know that every extremal ray of $\text{Nef}(W)$ is obtained by pulling back an extremal ray of either $\text{Nef}(W_1)$ or $\text{Nef}(W_2)$.

Next, assume that $D \in \text{Nef}(W_1)$ is extremal, and let us prove that $p_1^* D$ is extremal in $\text{Nef}(W)$. Let $p_1^* D = E + E'$ be a decomposition with $E, E' \in \text{Nef}(W)$. Up to adding terms to E' , we can assume that E spans an extremal ray of $\text{Nef}(W)$. By Theorem 1.4, we can write

$$E = p_1^* E_1 + p_2^* E_2, \quad \text{and} \quad E' = p_1^* E'_1 + p_2^* E'_2$$

with $E_i, E'_i \in \text{Nef}(W_i)$. As E is extremal, the divisors $E, p_1^* E_1$ and $p_2^* E_2$ are proportional. Moreover $p_1^*(D - E_1 - E'_1) = p_2^*(E_2 + E'_2) \in \text{Nef}(W)$. Hence, by the projection formula, $D - E_1 - E'_1$ is nef. But D

is extremal in the cone $\text{Nef}(W_1)$, so D, E_1 , and E'_1 are proportional. In particular, $p_1^*D, p_1^*E_1, p_1^*E'_1$ and $p_2^*E_2$ are all proportional, which shows that E and E' are proportional and thus concludes the proof. \square

We now construct various fiber products showing that Theorem 1.4 fails in general over bases B of dimension at least 2. Two types of constructions are provided: In Example 3.5, the surjective maps ϕ_1 and ϕ_2 are birational morphisms; in Example 3.6, they are smooth fibrations. The first construction simply involves (-1) -curves on blow-ups of \mathbb{P}^2 ; the second construction uses Serre's construction of vector bundles of rank two.

Example 3.5. Take $S := \mathbb{P}^2$, and take four points P_1, P_2, P_3, P_4 in S so that no three of them lie on a line. Let ℓ_1 be the line through P_1, P_2 and let ℓ_2 be the line through P_3, P_4 . Take

$$W_1 := \text{Bl}_{P_1, P_2}(S) \quad \text{and} \quad W_2 := \text{Bl}_{P_3, P_4}(S).$$

We let

$$W := W_1 \times_S W_2.$$

As the blown-up points are distinct, W is isomorphic to $\text{Bl}_{P_1, P_2, P_3, P_4}(S)$, which is smooth. Moreover, the decomposition of the Picard group

$$\text{Pic}(W) = p_1^*\text{Pic}(W_1) + p_2^*\text{Pic}(W_2)$$

clearly holds.

Denote by ℓ'_1 and ℓ'_2 the strict transforms of ℓ_1 and ℓ_2 in W_1 and W_2 , respectively. Then ℓ'_i is an effective non-nef divisor on W_i as $(\ell'_i)^2 = -1$. Let

$$D := p_1^*\ell'_1 + p_2^*\ell'_2.$$

We show that D is nef; this also shows that Lemma 3.2 fails when $\dim B \geq 2$. As D is effective, it is enough to check that its intersections with its components are all nonnegative. By symmetry, it is enough to compute

$$D \cdot p_1^*\ell'_1 = (\ell'_1)^2 + \ell'_2 \cdot \phi_2^*\ell_1 = -1 + 1 = 0.$$

So D is nef and has vanishing intersection with the curves $p_1^*\ell'_1$ and $p_2^*\ell'_2$.

Now, assume by contradiction that D has another decomposition $D = p_1^*D_1 + p_2^*D_2$ with $D_i \in \text{Nef}(W_i)$. Then we have

$$p_1^*(\ell'_1 - D_1) = p_2^*(D_2 - \ell'_2).$$

As $p_1^*N^1(W_1)_{\mathbb{R}} \cap p_2^*N^1(W_2)_{\mathbb{R}}$ clearly has dimension one, it equals $\mathbb{R}[p^*\mathcal{O}_{\mathbb{P}^2}(1)]$, where p is the natural projection $W \rightarrow S$. Hence, for some $c \in \mathbb{R}$, we have

$$p_1^*(\ell'_1 - D_1) = p_2^*(D_2 - \ell'_2) = cp^*\mathcal{O}_{\mathbb{P}^2}(1).$$

Since

$$p_1^*D_1 \cdot p_i^*\ell'_i + p_2^*D_2 \cdot p_i^*\ell'_i = D \cdot p_i^*\ell'_i = 0,$$

and both $p_1^*D_1$ and $p_2^*D_2$ are nef, we have $p_i^*D_i \cdot p_i^*\ell'_i = 0$. Thus,

$$-1 = p_1^*\ell'_1 \cdot p_1^*(\ell'_1 - D_1) = cp_1^*\ell'_1 \cdot p^*\mathcal{O}_{\mathbb{P}^2}(1) = c$$

and similarly,

$$1 = p_2^*\ell'_2 \cdot p_2^*(D_2 - \ell'_2) = cp_2^*\ell'_2 \cdot p^*\mathcal{O}_{\mathbb{P}^2}(1) = c,$$

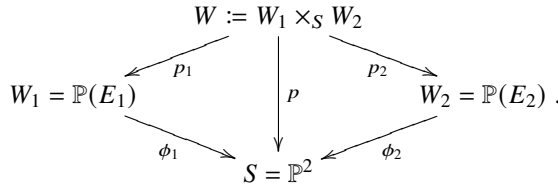
which is a contradiction.

Example 3.6. Take $S := \mathbb{P}^2$. Let us fix a closed subscheme Z_2 of \mathbb{P}^2 consisting of two distinct (reduced) points. We fix another closed subscheme Z_1 of \mathbb{P}^2 consisting of two distinct (reduced) points, chosen generally with respect to Z_2 .

For each $i = 1, 2$, Serre’s construction (see, e.g., [19, Theorem 5.1.1]) produces a locally free sheaf E_i of rank 2 on \mathbb{P}^2 , which fits into the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow E_i \rightarrow \mathcal{I}_{Z_i}(i) \rightarrow 0. \tag{3.1}$$

Set $W_i := \mathbb{P}(E_i)$; see [16, Definition in p.162]. Consider



As a projectivized vector bundle, each W_i is endowed with a tautological line bundle ζ_i satisfying $\phi_{i*}\zeta_i = E_i$. In particular, this line bundle has a distinguished section given by the inclusion morphism in Equation (3.1), whose zero locus we denote by S_i . We will describe the geometry of S_i later.

Note that the Néron–Severi space of W decomposes. Indeed, the smooth fibration $p_1 : W \rightarrow W_1$ identifies with the projectivization of the vector bundle $\phi_1^*E_2$ over W_1 , which has tautological line bundle $p_2^*\zeta_2$, so

$$\begin{aligned}
 N^1(W)_{\mathbb{R}} &= p_1^*N^1(W_1)_{\mathbb{R}} + \mathbb{R} \cdot p_2^*[\zeta_2] \\
 &= \mathbb{R} \cdot p_1^*[\zeta_1] + p^*N^1(S)_{\mathbb{R}} + \mathbb{R} \cdot p_2^*[\zeta_2] \\
 &= p_1^*N^1(W_1) + p_2^*N^1(W_2).
 \end{aligned}$$

Define the line bundle

$$D = p_1^*\zeta_1 + p_2^*\zeta_2$$

on W . It is effective, as the ζ_i both are. To prove that D is nef, let us describe the geometry of the zero loci S_i .

By [15, Proposition 3.6.2], and since the closed subschemes Z_i are locally complete intersections, each zero locus S_i is in fact a (reduced irreducible) surface, isomorphic to $\text{Bl}_{Z_i}\mathbb{P}^2 \simeq \mathbb{P}(\mathcal{I}_{Z_i}(i))$ naturally embedded in $\mathbb{P}(E_i)$ through the surjection in Equation (3.1). Through this identification, the restricted line bundle $\zeta_i|_{S_i}$ corresponds to the tautological line bundle of $\mathbb{P}(\mathcal{I}_{Z_i}(i))$, which in $\text{Bl}_{Z_i}\mathbb{P}^2$ corresponds to the dual of the exceptional line bundle twisted by $\phi_i^*\mathcal{O}_{\mathbb{P}^2}(i)$. For $i = 1$, this line bundle corresponds to the divisor obtained by strict transform of the line ℓ_1 passing through the two points of Z_1 with the following properties:

- It is effective and has a unique section which is irreducible;
- It has negative square.

For $i = 2$, it is the strict transform of any conic through the two points of Z_2 with the following properties:

- It is effective and admits an irreducible section;
- It has positive square.

Let us summarize: On one hand, ζ_1 has exactly one negative curve ℓ'_1 on W_1 , which is contained in S_1 and has negative square there. On the other hand, ζ_2 is nef on W_2 .

We now prove that D is nef, arguing by contradiction: Assume that there is a curve C in W such that $D \cdot C < 0$. We just proved that $p_2^*\zeta_2$ is nef, so $\zeta_1 \cdot p_{1*}C < 0$, and so there is a positive integer m such that $p_{1*}C = m\ell'_1$; moreover, C must lie in $p_1^{-1}(\ell'_1)$. The restricted map $\phi_1 : \ell'_1 \rightarrow \ell_1$ is an isomorphism,

so its base change $p_2 : p_1^{-1}(\ell'_1) \rightarrow \phi_2^{-1}(\ell_1) \simeq \mathbb{P}(E_{2|\ell_1})$ is an isomorphism too. Thus, $p_{2*}C$ is a reduced curve C_2 in W_2 , and $\phi_{2*}C_2 = m\ell_1$. By the projection formula,

$$D \cdot C = m\zeta_1 \cdot \ell'_1 + \zeta_2 \cdot C_2.$$

We have $\zeta_1|_{S_1} = \ell'_1$, so $\zeta_1 \cdot \ell'_1 = -1$. Moreover, by [39, Example 1 in §5.2, Chapter 1], and since we chose Z_1 generally with respect to Z_2 , we have

$$E_{2|\ell_1} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1), \tag{3.2}$$

so $E_{2|\ell_1} \otimes \mathcal{O}_{\ell_1}(-1)$ is nef, in particular $\zeta_2 \cdot C_2 \geq \phi_2^* \mathcal{O}_{\ell_1}(1) \cdot C_2 = m$. Hence, we finally have $0 > D \cdot C \geq -m + m = 0$, contradiction. So D is nef.

We conclude this example by picking a decomposition of D as $p_1^*D_1 + p_2^*D_2$ with $D_i \in N^1(W_i)_{\mathbb{R}}$, and proving that at least one of the D_i is not nef. Since the intersection of $p_1^*N^1(W_1)_{\mathbb{R}}$ with $p_2^*N^1(W_2)_{\mathbb{R}}$ is the subspace $\mathbb{R} \cdot p^*[\mathcal{O}_{\mathbb{P}^2}(1)]$, and since we already have $D = p_1^*\zeta_1 + p_2^*\zeta_2$, there exists $a \in \mathbb{R}$ such that

$$D_1 = \zeta_1 + a\phi_1^* \mathcal{O}_{\mathbb{P}^2}(1), \quad D_2 = \zeta_2 - a\phi_2^* \mathcal{O}_{\mathbb{P}^2}(1).$$

In particular,

$$D_1 \cdot \ell'_1 = \zeta_1 \cdot \ell'_1 + a\mathcal{O}_{\mathbb{P}^2}(1) \cdot \phi_{1*}\ell'_1 = -1 + a.$$

Moreover, by [39, Example 1 in §5.2, Chapter 1] again, there exists a line ℓ_2 in \mathbb{P}^2 such that

$$E_{2|\ell_2} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \tag{3.3}$$

and so there is a section ℓ'_2 of the fibration $\phi_2 : \mathbb{P}(E_{2|\ell_2}) \rightarrow \ell_2$ such that $\zeta_2 \cdot \ell'_2 = 0$. In particular,

$$D_2 \cdot \ell'_2 = \zeta_2 \cdot \ell'_2 - a\ell_2 \cdot \phi_{2*}\ell'_2 = -a.$$

Since at least one of the two numbers $a - 1$ and $-a$ is negative, D_1 and D_2 cannot both be nef.

We now use Examples 3.5 and 3.6 to build similar counterexamples over bases of higher dimension.

Example 3.7. Take W, W_1, W_2 and S as in Example 3.5 or Example 3.6. Note that they all are rationally connected: It is clear in Example 3.5 and follows from [13, Corollary 1.3] in Example 3.6. Introduce

$$W \times T = (W_1 \times T) \times_{(S \times T)} (W_2 \times T),$$

where T is an arbitrary smooth projective variety. Since W, W_1 and W_2 are rationally connected and smooth, they have trivial irregularity so that

$$N^1(Z \times T)_{\mathbb{R}} = p_Z^* N^1(Z)_{\mathbb{R}} \oplus p_T^* N^1(T)_{\mathbb{R}},$$

for $Z = W, W_1$ or W_2 . This implies that

$$N^1(W \times T)_{\mathbb{R}} = (p_1 \times \text{id}_T)^* N^1(W_1 \times T)_{\mathbb{R}} + (p_2 \times \text{id}_T)^* N^1(W_2 \times T)_{\mathbb{R}}.$$

Note that by the projection formula,

$$\text{Nef}(Z \times T) = p_Z^* \text{Nef}(Z) \oplus p_T^* \text{Nef}(T),$$

for $Z = W, W_1$ or W_2 . So, if we assume by contradiction that

$$\text{Nef}(W \times T) = (p_1 \times \text{id}_T)^* \text{Nef}(W_1 \times T) + (p_2 \times \text{id}_T)^* \text{Nef}(W_2 \times T),$$

we get $\text{Nef}(W) = p_1^* \text{Nef}(W_1) + p_2^* \text{Nef}(W_2)$, which contradicts Example 3.5 or Example 3.6.

Remark 3.8. We note that Theorem 1.4 also fails if the nef cones are replaced by the movable cones. In general, let X be a smooth projective variety and recall that a divisor D on X is called *movable*, if there is a positive integer m such that mD is effective and the base locus of the linear system $|\mathcal{O}_X(mD)|$ has no component of codimension 1. The closed movable cone $\overline{\text{Mov}}(X)$ is then defined as the closure of the convex cone in $N^1(X)_{\mathbb{R}}$ generated by the classes of movable divisors. It always holds $\text{Nef}(X) \subset \overline{\text{Mov}}(X)$, and, if moreover X is a surface, then $\text{Nef}(X) = \overline{\text{Mov}}(X)$.

Take a general fiber product $W = W_1 \times_{\mathbb{P}^1} W_2$ of two very general rational elliptic surfaces $W_1 \rightarrow \mathbb{P}^1$ and $W_2 \rightarrow \mathbb{P}^1$ with sections. Then W is a strict Calabi–Yau threefold and has nontrivial algebraic flops (see [37]). Thus, $\text{Nef}(W) \subsetneq \overline{\text{Mov}}(W)$. But since the W_i are surfaces, we have

$$p_1^* \overline{\text{Mov}}(W_1) + p_2^* \overline{\text{Mov}}(W_2) = p_1^* \text{Nef}(W_1) + p_2^* \text{Nef}(W_2) = \text{Nef}(W) \subsetneq \overline{\text{Mov}}(W),$$

where the second equality follows from Theorem 1.4.

Even in this particular case, the version of the cone conjecture stated in [48, Conjecture 2.1.(2)] is not known.

We conclude this section with a corollary of Theorem 1.4 that will be key in the proof of Theorem 1.6. For a morphism $\pi : X \rightarrow Y$, we define

$$\text{Aut}(X/Y) = \{g \in \text{Aut}(X) \mid \pi \circ g = \pi\}.$$

Corollary 3.9. For $i = 1, 2$, let $\phi_i : W_i \rightarrow B$ be a surjective morphism from a projective variety to a projective curve B ; let H_i be a subgroup of $\text{Aut}(W_i/B)$. Assume that

1. The fiber product $W = W_1 \times_B W_2$ is irreducible;
2. It holds

$$p_1^* N^1(W_1)_{\mathbb{R}} + p_2^* N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}},$$

where p_i denotes the projection from W onto W_i ;

3. For each $i = 1, 2$, there exists a rational polyhedral cone Π_i in $\text{Nef}^+(W_i)$ such that $\text{Amp}(W_i) \subset H_i \cdot \Pi_i$.

Then, for any subgroup H of $\text{Aut}(W)$ containing $H_1 \times H_2$, there is a rational polyhedral fundamental domain for the H -action on $\text{Nef}^+(W)$.

Proof. Let Π be the convex hull of $p_1^* \Pi_1 + p_2^* \Pi_2$. Then Π is a rational polyhedral cone contained in $\text{Nef}^+(W)$. Moreover,

$$\text{Amp}(W) \subset (H_1 \times H_2) \cdot \Pi \subset H \cdot \Pi$$

as $p_1^* \text{Amp}(W_1) + p_2^* \text{Amp}(W_2) = \text{Amp}(W)$ by Theorem 1.4. The existence of a rational polyhedral fundamental domain then follows from Proposition 2.3.(1). □

4. Construction of Schoen varieties

Schoen varieties are constructed as fiber products of two fibrations over \mathbb{P}^1 . Let us first construct these fibrations.

4.1. The factor W with a fibration over \mathbb{P}^1

This construction relies on a pencil of ample hypersurfaces in a Fano manifold.

Let Z be a Fano manifold of dimension at least 2, and let D be an ample divisor in Z such that both $\mathcal{O}_Z(D)$ and $\mathcal{O}_Z(-K_Z - D)$ are globally generated. Note that $\mathcal{O}_Z(-K_Z)$ is then globally generated as well.

Example 4.1. Take any toric Fano manifold Z of dimension at least 2. Since nef line bundles on a projective toric manifold are globally generated, any decomposition $-K_Z = D + D'$ as the sum of an ample divisor D and a nef divisor D' yields a pair (Z, D) satisfying the above condition.

Let $W \subset \mathbb{P}^1 \times Z$ be a general member of the ample and basepoint-free linear system $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Z(D)|$. We have a fibration $\phi : W \rightarrow \mathbb{P}^1$ via the first projection, and the second projection $\varepsilon : W \rightarrow Z$ is the blow-up of Z along the smooth subvariety Y of codimension two cut out by the members of the pencil in $|D|$ defined by W . Since Z is Fano, W is rationally connected. By construction, any point $y \in Y$ defines a rational curve $\varepsilon^{-1}(y)$ which is a section of $\phi : W \rightarrow \mathbb{P}^1$.

By the adjunction formula,

$$\mathcal{O}_W(-K_W) = (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Z(-K_Z - D))|_W, \tag{4.1}$$

so $\mathcal{O}_W(-K_W)$ is globally generated, *a fortiori* nef and effective.

If Z is chosen to be a del Pezzo surface, then the surface W is described by the following lemma. Recall that a smooth projective surface S is called *weak del Pezzo* if its anticanonical divisor $-K_S$ is nef and big.

Lemma 4.2. *If Z has dimension 2, then either $D \in |-K_Z|$ and $W \xrightarrow{\phi} \mathbb{P}^1$ is a rational elliptic surface with globally generated anticanonical line bundle, or W is a weak del Pezzo surface.*

Proof. Since W is rationally connected and $\dim W = 2$, we know that W is rational. If $D \in |-K_Z|$, then $\mathcal{O}_W(-K_W) = \phi^* \mathcal{O}_{\mathbb{P}^1}(1)$, which is globally generated, and which makes W into a rational elliptic surface.

Suppose now that $D \notin |-K_Z|$. As $-K_Z - D$ is effective and nontrivial, and as $-K_Z$ and D are ample, we have $-K_Z(-K_Z - D) > 0$ and $D(-K_Z - D) > 0$, and thus,

$$K_Z^2 > -K_Z \cdot D > D^2.$$

As W is the blowup of Z at (D^2) points, we have $K_W^2 = K_Z^2 - D^2 > 0$. Since $-K_W$ is nef, W is a weak del Pezzo surface. □

Remark 4.3. Note that, in the case where W is a rational elliptic surface, the fact that it has a section and that it is chosen general in its pencil on $\mathbb{P}^1 \times Z$ implies that it is isomorphic to \mathbb{P}^2 blown-up in the base locus of a general pencil of cubics. In particular, W has topological Euler characteristics 12, the canonical fibration $W \rightarrow \mathbb{P}^1$ has some singular fibers but no multiple fibers. The fact that the rational elliptic surface W is general implies that the singular fibers of $W \rightarrow \mathbb{P}^1$ are exactly 12 nodal rational curves ([34, p.8]).

Considering the j -invariant in family for the fibration $W \rightarrow \mathbb{P}^1$, we obtain a proper surjective map $j : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which is finite of degree 12 and has 12 simple poles which occur at the 12 image points of the 12 singular fibers ([34, Lemma (IV.4.1), Corollary (IV.4.2)]).

In general, the construction of W described above ensures the following properties.

Proposition 4.4. *We have*

$$\text{Nef}^e(W) = \text{Nef}^+(W) = \text{Nef}(W).$$

Moreover, if $\dim W \geq 3$ or if W is a weak del Pezzo surface, then the cone $\text{Nef}(W)$ is rational polyhedral, spanned by classes of semiample divisors.

Proof. We start with the ‘moreover’ part. It is a corollary of some known results. If W is a weak del Pezzo surface, then W is log Fano (see, e.g., [33, Proposition 2.6]). Hence, by the cone theorem [26, Theorem 3.7], its nef cone is a rational polyhedral cone spanned by classes of semiample divisors.

Assume that $\dim W \geq 3$. Since $\mathbb{P}^1 \times Z$ is a smooth Fano variety of dimension at least four, and since $W \subset \mathbb{P}^1 \times Z$ is a smooth ample divisor such that

$$\mathcal{O}_{\mathbb{P}^1 \times Z}(-K_{\mathbb{P}^1 \times Z} - W) = \mathcal{O}_Z(-K_Z - D) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$$

is nef, we can apply [1, Proposition 3.5] (which generalizes [3, Appendix]). It yields an isomorphism

$$j_* : \overline{\text{NE}}(W) \xrightarrow{\sim} \overline{\text{NE}}(\mathbb{P}^1 \times Z)$$

induced by the inclusion $j : W \hookrightarrow \mathbb{P}^1 \times Z$. Dually, we obtain an isomorphism

$$j^* : \text{Nef}(\mathbb{P}^1 \times Z) \xrightarrow{\sim} \text{Nef}(W).$$

As $\text{Nef}(\mathbb{P}^1 \times Z)$ is rational polyhedral and spanned by classes of semiample divisors, so is $\text{Nef}(W)$.

We now prove the equality of the three cones $\text{Nef}^e(W)$, $\text{Nef}^+(W)$, and $\text{Nef}(W)$. If $\dim W \geq 3$ or if W is a weak del Pezzo surface, the equality clearly follows from the fact that $\text{Nef}(W)$ is rational polyhedral, spanned by classes of semiample divisors. So by Lemma 4.2, we can focus on the case where W is a rational elliptic surface.

Clearly, $\text{Nef}^e(W)$ and $\text{Nef}^+(W)$ are subcones of $\text{Nef}(W)$. Moreover, $\text{Nef}^+(W) \subset \text{Nef}^e(W)$ by [48, Lemma 4.2]. We only need to show that $\text{Nef}(W) = \text{Nef}^+(W)$. By [38, Corollary 3.3. (c)], the cone $\overline{\text{NE}}(W)$ is generated by curve classes, so dually, $\text{Nef}(W)$ is spanned by Cartier divisors. So $\text{Nef}(W) = \text{Nef}^+(W)$ indeed. \square

Let us conclude the description of W by describing the general fiber of $\phi : W \rightarrow \mathbb{P}^1$, under the assumption that $D \in |-K_Z|$.

Lemma 4.5. *Suppose that $D \in |-K_Z|$. Then the general fiber F of $\phi : W \rightarrow \mathbb{P}^1$ is a Calabi–Yau manifold (as in Definition 2.2).*

Proof. Since $D \in |-K_Z|$, the general fiber F is linearly equivalent to the anticanonical divisor $-K_W$ by Equation (4.1). By adjunction, F has trivial canonical bundle. We also have an exact sequence

$$0 \rightarrow \mathcal{O}_W(-K_W) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_F \rightarrow 0.$$

Since W is rationally connected, we have

$$h^{\dim W - i}(W, -K_W) = h^i(W, \mathcal{O}_W) = 0$$

for $i \geq 1$. Hence, $h^i(F, \mathcal{O}_F) = 0$ whenever $1 \leq i \leq \dim W - 2 = \dim F - 1$. \square

4.2. The fiber product $X = W_1 \times_{\mathbb{P}^1} W_2$

We are ready to generalize Schoen’s construction and obtain Calabi–Yau pairs in arbitrary dimension. For $i = 1, 2$, let Z_i, D_i, W_i be as in §4.1. We denote by $\phi_i : W_i \rightarrow \mathbb{P}^1$ the associated fibration and recall that it has a section.

We add one assumption, which is automatically satisfied by taking the fibrations ϕ_i for $i = 1, 2$ to be general with respect to one another:

For every $t \in \mathbb{P}^1$, the fiber of at least one of the ϕ_i above t is smooth.

In the case where both W_i are rational elliptic surfaces, this assumption has an important consequence.

Lemma 4.6. *Let W_1 and W_2 be general rational elliptic surfaces, with their canonical fibrations $\phi_i : W_i \rightarrow \mathbb{P}^1$, each admitting a section. Assume that for every $t \in \mathbb{P}^1$, there is i such that the fiber $\phi_i^{-1}(t)$*

is smooth. Then, for a very general point $t \in \mathbb{P}^1$, the fibers $\phi_1^{-1}(t)$ and $\phi_2^{-1}(t)$ are smooth, nonisogenous elliptic curves.

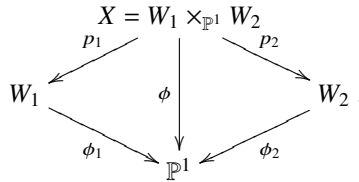
Proof. For $i = 1, 2$, consider the finite morphism $j_i : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ induced by the j -invariant of the elliptic fibration ϕ_i (see Remark 4.3). Define the morphism $J := (j_1, j_2) : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Its image is an irreducible curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

For each positive integer n , let $F_n(x, y) \in \mathbb{Z}[x, y]$ be the polynomial as in [45, Theorem 6.3 in p.146]. Then by [45, Exercise 2.19. (a) in p.182], we have $F_n(j(\phi_1^{-1}(t)), j(\phi_2^{-1}(t))) = 0$ if and only if there is an isogeny $\phi_1^{-1}(t) \rightarrow \phi_2^{-1}(t)$ of degree n . By [45, Exercise 2.18. (e) in p.181], each $F_n(x, y)$ is a product of some polynomials $\Phi_m(x, y)$ indexed by positive integers. By the expression in [45, Exercise 2.18 in p.181], each $\Phi_m(x, y)$ viewed as a polynomial in the single variable x has leading coefficient 1. Together with [45, Exercise 2.18.(b) in p.181], we obtain the irreducibility of $\Phi_m(x, y)$ in $\mathbb{C}[x, y]$.

Let $\Sigma_m \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the irreducible curve defined by the homogenization of $\Phi_m(x, y)$ using $x = s/t$ and $y = u/v$. We claim that $J(\mathbb{P}^1)$ intersects with each Σ_m at finitely many points. Indeed, by our assumption, we can take $t \in \mathbb{P}^1$ such that the fiber $\phi_1^{-1}(t)$ is singular, while the fiber $\phi_2^{-1}(t)$ is an elliptic curve, so $J(t) = ([1 : 0], [\alpha : 1])$ for some $\alpha \in \mathbb{C}$. As we mentioned before, each $\Phi_m(x, y)$ viewed as a polynomial in the single variable x has leading coefficient 1, so $([1 : 0], [\alpha : 1]) \notin \Sigma_m$. This implies $J(\mathbb{P}^1) \not\subset \Sigma_m$, and the claim holds because both Σ_m and $J(\mathbb{P}^1)$ are irreducible.

Let $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the union of the countably many curves Σ_m . Then the set $\mathbb{P}^1 \setminus (Z_1 \cup Z_2 \cup J^{-1}(J(\mathbb{P}^1) \cap \Sigma))$ is nonempty with the property that each of its elements is a very general point, say t , satisfying that the fibers $\phi_1^{-1}(t)$ and $\phi_2^{-1}(t)$ are smooth, nonisogenous elliptic curves. \square

Now that we better understand the fibrations ϕ_i relatively to one another, we can consider the fiber product over \mathbb{P}^1



As for every $t \in \mathbb{P}^1$, the fiber of at least one of the ϕ_i above t is smooth, the variety X is smooth too. We can also view X as a complete intersection of two hypersurfaces in $\mathbb{P}^1 \times Z_1 \times Z_2$, given by general members in the linear systems

$$|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_1}(D_1) \boxtimes \mathcal{O}_{Z_2}| \text{ and } |\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_1} \boxtimes \mathcal{O}_{Z_2}(D_2)|.$$

By adjunction, we obtain that

$$\mathcal{O}_X(-K_X) = (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{Z_1}(-K_{Z_1} - D_1) \boxtimes \mathcal{O}_{Z_2}(-K_{Z_2} - D_2))|_X, \tag{4.2}$$

which is globally generated, hence nef and effective.

Definition 4.7. A smooth projective variety X constructed as above is called a *Schoen variety*. A pair (X, Δ) is called a *Schoen pair* if X is a Schoen variety, and Δ is an effective \mathbb{Q} -divisor such that $K_X + \Delta \sim_{\mathbb{Q}} 0$.

Any Schoen variety X can be associated many Schoen pairs (X, Δ) as long as $-K_X$ is nontrivial. Every Schoen pair is by definition a Calabi–Yau pair (as in Definition 2.1). Moreover, if (X, Δ) is a Schoen pair, then there exists a positive integer m such that

$$\Delta = \frac{1}{m} \Delta_{m,X}, \text{ with } \Delta_{m,X} \in |-mK_X|. \tag{4.3}$$

If $m \geq 2$ and $\Delta_{m,X} \in |-mK_X|$ is general, the Calabi–Yau pair (X, Δ) is klt.

To conclude this section, we prove that, if for both $i = 1, 2$, the divisor D_i chosen when constructing W_i is in the linear system $| -K_{Z_i} |$, then the Schoen variety X is a strict Calabi–Yau manifold.

Lemma 4.8. *Any Schoen variety X is simply connected.*

Proof. The proof is similar to [43, Lemma 1] and [46, Lemma 2.1].

Let $U \subset \mathbb{P}^1$ be the open subset over which the morphism $\phi : X \rightarrow \mathbb{P}^1$ is smooth, and set $V := \phi^{-1}(U)$. Let $i : V \hookrightarrow X$ and $j : U \hookrightarrow \mathbb{P}^1$ be the natural inclusions. The restriction $\phi' := \phi|_V : V \rightarrow U$ is topologically locally trivial with a fiber, say F . Since both ϕ_1 and ϕ_2 have sections, $\phi : X \rightarrow \mathbb{P}^1$ also admits a section $\sigma : \mathbb{P}^1 \rightarrow X$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(V) & \begin{array}{c} \xleftarrow{\sigma_{U*}} \\ \xrightarrow{\phi'_*} \end{array} & \pi_1(U) & \longrightarrow & 1 \\
 & & & & i_* \downarrow & & j_* \downarrow & & \\
 & & & & \pi_1(X) & \begin{array}{c} \xleftarrow{\sigma_*} \\ \xrightarrow{\phi_*} \end{array} & \pi_1(\mathbb{P}^1) & &
 \end{array}$$

Here, the first row is exact by the homotopy long exact sequence, and $i_* : \pi_1(V) \rightarrow \pi_1(X)$ is surjective by [25, Proposition 2.10.1].

We claim that the image of $\pi_1(F)$ in $\pi_1(X)$ equals $\pi_1(X)$. Indeed, since $\pi_1(\mathbb{P}^1)$ is trivial, the composition $i_* \circ \sigma_{U*} = \sigma_* \circ j_*$ is trivial. Using that i_* is surjective, that this composition is trivial and that $\pi_1(V)$ is generated by the union of its subgroups $\pi_1(F)$ and $\sigma_{U*}\pi_1(U)$, we obtain

$$\pi_1(X) = i_*\pi_1(V) = i_*\pi_1(F).$$

We are now left to show that the image of $\pi_1(F)$ in $\pi_1(X)$ is trivial. Write $F = F_1 \times F_2$, where F_i is a general fiber of $\phi_i : W_i \rightarrow \mathbb{P}^1$ for $i = 1, 2$. Since $\pi_1(F) = \pi_1(F_1) \times \pi_1(F_2)$, it is enough to show that the image of $\pi_1(F_i)$ in $\pi_1(X)$ is trivial, which we prove for $i = 1$.

A section of $\phi_2 : W_2 \rightarrow \mathbb{P}^1$ gives rise to a section s of $p_1 : X \rightarrow W_1$. By construction, the homomorphism $\pi_1(F_1) \rightarrow \pi_1(X)$ is induced by $F_1 \hookrightarrow W_1 \xrightarrow{s} X$, thus factors through $\pi_1(W_1)$. Since it is rationally connected, W_1 is simply connected and hence the image of $\pi_1(F_1)$ in $\pi_1(X)$ is trivial. \square

Proposition 4.9. *Suppose that $D_i \in | -K_{Z_i} |$ for both $i = 1, 2$. Then the Schoen variety X is a strict Calabi–Yau manifold (see Definition 2.2).*

Proof. By Equation (4.2) and Lemma 4.8, X has trivial canonical bundle and trivial fundamental group. We are left showing that $h^p(X, \mathcal{O}_X) = 0$ for every $0 < p < \dim X$.

Lemma 4.10. *Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a surjective morphism between smooth projective varieties. Assume that a general fiber F of g is a Calabi–Yau manifold and that the canonical line bundle $\omega_{\mathcal{X}}$ is trivial. Then, for every positive integer q , we have*

$$R^q g_* \mathcal{O}_{\mathcal{X}} = \begin{cases} \omega_{\mathcal{Y}}, & \text{if } q = \dim \mathcal{X} - \dim \mathcal{Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Set $r := \dim \mathcal{X} - \dim \mathcal{Y}$. By [23, Theorem 2.1. (i)] and [24, Corollary 3.9], the sheaf $R^q g_* \omega_{\mathcal{X}} = R^q g_* \mathcal{O}_{\mathcal{X}}$ is reflexive. Since \mathcal{Y} is smooth, the invertibility of $R^q g_* \mathcal{O}_{\mathcal{X}}$ follows provided it has rank one. Its rank is explicitly given by the dimension of $H^q(F, \mathcal{O}_F)$, which is one if $q = 0$ or r , and zero otherwise. Hence, we have

$$R^q g_* \mathcal{O}_{\mathcal{X}} = \begin{cases} \text{a line bundle,} & \text{if } q = 0 \text{ or } r, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

By Grothendieck–Verdier duality [18, Theorem 3.34], we have

$$Rg_*\mathcal{O}_X \simeq Rg_*\omega_X \simeq R\mathcal{H}om(Rg_*\mathcal{O}_X, \omega_X[-r]).$$

The Grothendieck spectral sequence gives

$$E_2^{p,-q} := \mathcal{E}xt^p(R^q g_*\mathcal{O}_X, \omega_X) \Rightarrow R^{p-q+r} g_*\mathcal{O}_X$$

(see, e.g., [18, Example 2.70.ii]). But by Equation (4.4), the page E_2 has exactly two nonzero entries, namely $E_2^{0,0} = \omega_X$, and $E_2^{0,-r}$. So Lemma 4.10 follows. \square

We return to our Schoen variety X . For $i = 1, 2$, we let $w_i := \dim W_i$. By Lemma 4.5, and as $p_2 : X \rightarrow W_2$ is a base change of $\phi_1 : W_1 \rightarrow \mathbb{P}^1$, the general fiber of p_2 is a Calabi–Yau manifold. We can thus apply Lemma 4.10 to p_2 , and obtain that

$$R^q p_{2*}\omega_X = R^q p_{2*}\mathcal{O}_X = \begin{cases} \mathcal{O}_{W_2}, & \text{if } q = 0, \\ \omega_{W_2}, & \text{if } q = \dim w_1 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

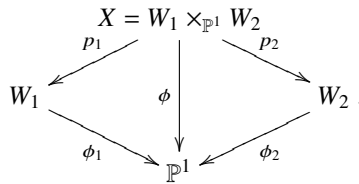
Together with [24, Corollary 3.2], this yields

$$h^p(X, \mathcal{O}_X) = h^p(W_2, \mathcal{O}_{W_2}) + h^{p-w_1+1}(W_2, \omega_{W_2})$$

for all $0 \leq p \leq \dim X$. Since W_2 is rationally connected, this is zero as soon as $p \neq 0$ and $p < w_1 + w_2 - 1 = \dim X$. \square

5. Application to the cone conjecture

In this section, we prove Theorem 1.6. The setup and the notations were defined in Section 4: We consider a Schoen variety X , fitting in a Schoen pair (X, Δ) . Let us recall the Cartesian diagram defining X :



Lemma 5.1. *We have*

$$p_1^* N^1(W_1)_{\mathbb{R}} + p_2^* N^1(W_2)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}.$$

Proof. Let $p \in \mathbb{P}^1$ be a very general point, and let $F_i := \phi_i^{-1}(p) \subset W_i$.

Claim 5.2. The map

$$\Psi : \text{Pic}(F_1) \times \text{Pic}(F_2) \rightarrow \text{Pic}(F_1 \times F_2)$$

defined by $\Psi(L, M) = L \boxtimes M$ is an isomorphism.

Proof. First, suppose that W_1 and W_2 are not both rational elliptic surfaces. If there is i such that Z_i has dimension at least 3, then F_i is a smooth ample hypersurface in Z_i , and so by Lefschetz hyperplane theorem, F_i has trivial irregularity. If there is i such that Z_i is a surface and $D_i \notin |-K_{Z_i}|$, then F_i is a smooth curve in Z_i , and by adjunction, it is in fact a rational curve, which again has trivial irregularity. In any case, Claim 5.2 follows from [16, Exercise III.12.6].

Assume now that both W_1 and W_2 are rational elliptic surfaces. Then, by Lemma 4.6, the fibers F_1 and F_2 are smooth, nonisogenous elliptic curves. We have a short exact sequence of abelian groups [2, Theorem 11.5.1]

$$0 \rightarrow \text{Pic}(F_1) \times \text{Pic}(F_2) \xrightarrow{\Psi} \text{Pic}(F_1 \times F_2) \rightarrow \text{Hom}(F_1, F_2) \rightarrow 0,$$

where $\text{Hom}(F_1, F_2)$ denotes the group of homomorphisms from F_1 to F_2 preserving both the variety and the group structure. Since F_1 and F_2 are nonisogenous, $\text{Hom}(F_1, F_2) = 0$, which proves Claim 5.2. \square

Let L be a line bundle on X . Claim 5.2 implies that

$$L|_{\phi^{-1}(p)} \simeq L|_{F_1 \times \{u\}} \boxtimes L|_{\{v\} \times F_2},$$

for any points $u \in F_2$ and $v \in F_1$.

For each $i = 1, 2$, we choose a section $s_i : \mathbb{P}^1 \rightarrow W_i$ and let $\sigma_i : W_i \rightarrow X$ be the induced section:

$$\sigma_1(w_1) := (w_1, s_2(\phi_1(w_1))) \in W_1 \times_{\mathbb{P}^1} W_2,$$

and similarly for σ_2 . We have

$$\begin{aligned} L|_{\phi^{-1}(p)} &\simeq L|_{F_1 \times \{s_1(p)\}} \boxtimes L|_{\{s_2(p)\} \times F_2} \\ &\simeq (\sigma_1^* L)|_{F_1} \boxtimes (\sigma_2^* L)|_{F_2} \\ &\simeq (p_1^* \sigma_1^* L \otimes p_2^* \sigma_2^* L)|_{\phi^{-1}(p)}. \end{aligned}$$

Since $p \in \mathbb{P}^1$ is very general, by applying [49, Theorem 3.1 and Remark 3.3] to the smooth part of the fibration $\phi : X \rightarrow \mathbb{P}^1$, we obtain

$$L \sim_{\mathbb{Q}} p_1^* \sigma_1^* L \otimes p_2^* \sigma_2^* L \otimes \mathcal{O}_X(D),$$

for a divisor D whose support is contained in a finite union of fibers of $\phi : X \rightarrow \mathbb{P}^1$. Note that an irreducible component R of a fiber of ϕ embeds in the product $\phi_1^{-1}(\phi(R)) \times \phi_2^{-1}(\phi(R))$, of which at least one factor $\phi_i^{-1}(\phi(R))$ is smooth, hence irreducible. It follows that there is an irreducible component R' of $\phi_j^{-1}(\phi(R))$ with $j = \{1, 2\} \setminus \{i\}$ such that $R = p_j^* R'$. Applying this to the irreducible components of D , we obtain that

$$N^1(W_1)_{\mathbb{R}} \times N^1(W_2)_{\mathbb{R}} \xrightarrow{p_1^* + p_2^*} N^1(X)_{\mathbb{R}}$$

is surjective. \square

Lemma 5.3. *For every $D \in \text{Nef}(X)$, one can write $D = p_1^* D_1 + p_2^* D_2$, where $D_i \in \text{Nef}(W_i)$.*

Proof. Lemma 5.3 follows from Lemma 5.1 and Theorem 1.4. \square

Theorem 5.4 (= Theorem 1.6). *Let (X, Δ) be a Schoen pair. Then*

$$\text{Nef}(X) = \text{Nef}^+(X) = \text{Nef}^e(X),$$

and moreover, there exists a rational polyhedral fundamental domain for the action of $\text{Aut}(X, \Delta)$ on $\text{Nef}^e(X)$.

Proof. Since $\text{Nef}(W_i) = \text{Nef}^+(W_i) = \text{Nef}^e(W_i)$ by Proposition 4.4, we have, by Lemma 5.3, $\text{Nef}(X) = p_1^* \text{Nef}^+(W_1) + p_2^* \text{Nef}^+(W_2) \subset \text{Nef}^+(X)$, so $\text{Nef}(X) = \text{Nef}^+(X)$. Similarly, we have $\text{Nef}(X) = \text{Nef}^e(X)$. This proves the first assertion.

Define the subgroups $H_i \leq \text{Aut}(W_i)$ by

$$H_i = \begin{cases} \text{Aut}(W_i/\mathbb{P}^1), & \text{if } W_i \text{ is a rational elliptic surface,} \\ \{\text{id}_{W_i}\}, & \text{otherwise.} \end{cases}$$

Then there exists a rational polyhedral cone $\Pi_i \subset \text{Nef}^+(W_i)$ such that $H_i \cdot \Pi_i$ contains $\text{Amp}(W_i)$. Indeed, the case where W_i is a rational elliptic surface with $-K_{W_i}$ semiample follows from [47, Theorem 8.2], and the other cases follow from Proposition 4.4.

We claim that $H_1 \times H_2 \leq \text{Aut}(X, \Delta)$. Note that there exists a positive integer m such that

$$\Delta = \frac{1}{m} \Delta_{m,X}$$

for some $\Delta_{m,X} \in |-mK_X|$. If neither W_1 nor W_2 is a rational elliptic surface, then $H_1 \times H_2$ is trivial by definition. If both W_1 and W_2 are rational elliptic surfaces, then $\Delta_{m,X} = 0$ and clearly, $H_1 \times H_2 \leq \text{Aut}(X)$. Finally, if one of the W_i , say W_1 , is a rational elliptic surface, and the other, say W_2 , is not, then $\mathcal{O}_X(-K_X) \simeq p_2^* \mathcal{O}_{W_2}(-K_{Z_2} - D_2)$. Since p_2 is proper surjective with connected fibers, the pullback p_2^* induces an isomorphism

$$H^0(X, p_2^* \mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))) \simeq H^0(W_2, \mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))).$$

So $\Delta_{m,X} = p_2^* \Delta_{m,W_2}$, for some divisor $\Delta_{m,W_2} \in |-m(K_{Z_2} + D_2)|$. Since $H_2 = \{\text{id}_{W_2}\}$ in this case, it follows that $\Delta_{m,X}$ is invariant under $H_1 \times H_2$. This proves the claim.

It then follows from Corollary 3.9 that $\text{Nef}^e(X) = \text{Nef}^+(X)$ has a rational polyhedral fundamental domain Π for the $\text{Aut}(X, \Delta)$ -action. □

Remark 5.5. In [14], the authors verified the cone conjecture for a strict Calabi–Yau threefold $X = W_1 \times_{\mathbb{P}^1} W_2$, where both W_i are general rational elliptic surfaces with sections. They use the following identification shown by Namikawa [37, Proposition 2.2 and Corollary 2.3]

$$\text{Aut}(X) \cong \text{Aut}(W_1) \times \text{Aut}(W_2),$$

which our proof bypasses, using Looijenga’s result (Lemma 2.4) instead.

Example 5.6. Fix an integer $n \geq 3$. Let us explain how to choose Z_1, Z_2, D_1 , and D_2 so that our construction produces a strict Calabi–Yau manifold X of dimension n such that $\text{Nef}(X)$ admits infinitely many extremal rays and X satisfies the cone conjecture. We take $Z_1 = \mathbb{P}^2$ and $D_1 = \mathcal{O}_{\mathbb{P}^2}(3)$ so that W_1 is a general rational elliptic surface. We take Z_2 to be a Fano variety of dimension $n - 1$ with $-K_{Z_2}$ globally generated (for example, $Z_2 = \mathbb{P}^{n-1}$), and we take $D_2 = -K_{Z_2}$.

The Schoen variety X obtained from these choices is a strict Calabi–Yau manifold by Proposition 4.9, and $\text{Nef}(X)$ admits infinitely many extremal rays by Lemma 5.1, by the fact that $\text{Nef}(W_1)$ admits infinitely extremal rays already, and by Corollary 1.5.

We conclude with an unsurprising corollary of the fact that Schoen varieties satisfy the cone conjecture.

Corollary 5.7. *Let X be a Schoen variety. Then the group $\pi_0 \text{Aut}(X)$ is finitely presented, and there are at most finitely many real forms for X , up to isomorphism.*

Proof. The linear action $\rho : \text{Aut}(X) \rightarrow \text{GL}(N^1(X))$ induces and factorizes through an action

$$\bar{\rho} : \pi_0 \text{Aut}(X) \rightarrow \text{GL}(N^1(X)).$$

We let $\text{Aut}^*(X) = \rho(\text{Aut}(X)) = \bar{\rho}(\pi_0 \text{Aut}(X))$.

Choose an effective \mathbb{Q} -divisor Δ on X such that (X, Δ) is a Schoen pair. By Theorem 1.6, there exists a rational polyhedral cone $\Pi \subset \text{Nef}^+(X)$ such that

$$\text{Amp}(X) \subset \text{Aut}(X, \Delta) \cdot \Pi \subset \text{Aut}^*(X) \cdot \Pi.$$

It follows from Proposition 2.3 that there is a rational polyhedral fundamental domain for the $\text{Aut}^*(X)$ -action on $\text{Nef}^+(X)$, and that the group $\text{Aut}^*(X)$ is finitely presented. By Fujiki–Liebermann’s theorem [5, Corollary 2.11], the kernel $\text{Ker}(\bar{\rho})$ is finite, and so the first claim follows from [21, Corollary 10.2].

The second claim follows from Theorem 5.8 below. \square

Theorem 5.8 [10, Theorem 1.6]. *Let V be a smooth complex projective variety. Assume that $\text{Nef}^+(V)$ contains a rational polyhedral cone Π such that*

$$\text{Amp}(V) \subset \text{Aut}(V) \cdot \Pi.$$

Then V has at most finitely many mutually nonisomorphic real forms.

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