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Applicability of the Bulirsch-Stoer algorithm in the circular restricted three-body problem

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Abstract. The dynamics of massless planetesimals in the gravitational field of a star with a planet in a circular orbit is considered. The invariant of this problem is the Jacobi integral. Preserving the value of the Jacobi integral can be a test for numerical algorithms solving the equation of motion. The invariant changes for particles in the planetary chaotic zone due to numerical errors that occur during close encounters with the planet. The limiting distances from the planet, upon reaching which the value of the Jacobi integral changes, are determined for Bulirsch-Stoer algorithm. It is shown that violation of the Jacobi integral can be used to define the boundaries of the planetary chaotic zone.

Keywords. Bulirsch-Stoer algorithm, circular restricted three-body problem, simulation

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The size of the planetary chaotic zone depends on the mass of the planet (Wisdom (1980)). The asymmetry of the zone with respect to the planet's orbit was shown in Morrison & Malhotra (2015). The stepped form of the dependence was determined in Demidova & Shevchenko (2020). Over time, the zone is freed from the matter. Therefore, the dependence can be used to interpret observations of debris disks in the presence of a matter free gap and to determine the mass of an invisible planet. Thus, the study of the structure of the planetary chaotic zone remains an urgent task.

Since the debris disks are flat and low-mass structures, it is appropriate to carry out calculations in the approximation of the plane restricted three-body problem. Energy and angular momentum are not conserved in the problem (Murray & Dermott (1999)). But there is an invariant called the Jacobi integral and defined as follows:

$$C_J = 2\left(\frac{GM_*}{R} + \frac{m_p}{r}\right) + 2n(x\dot{y} - y\dot{x}) - \dot{x}^2 - \dot{y}^2, \tag{0.1}$$

where M_*, m_p are masses of a star and a planet, n is the mean motion of the planet, G is the gravitation constant, R, r are distances from a particle to the star and the planet, x, y and $\dot{x}, \dot{y}$ are coordinates and velocities of the particle. When one simulates the dynamics of a swarm of particles in the gravitational field of a single star with a planet, the value of the Jacobi integral must be constant throughout the calculations.

At the initial moment of time, the star and the planet are located on the same axis with the distance of a_p, and 10^4 particles are equally distributed along the radius within $[-4R_h, 4R_h]$, where $R_h = \left(\frac{m_p}{3M_*}\right)^{1/3} a_p$ is Hill radius, along one line inclined at an angle ϕ to the axis. The particle with coordinates $(x, y) = (r \cdot cos\phi, r \cdot sin\phi)$ has velocity components $(v_x, v_y) = (-[G(M+m)/r]^{1/2} sin\phi, [G(M+m)/r]^{1/2} cos\phi)$. The calculations were performed in a rectangular barocentric non-rotating coordinate system during 10^4 of

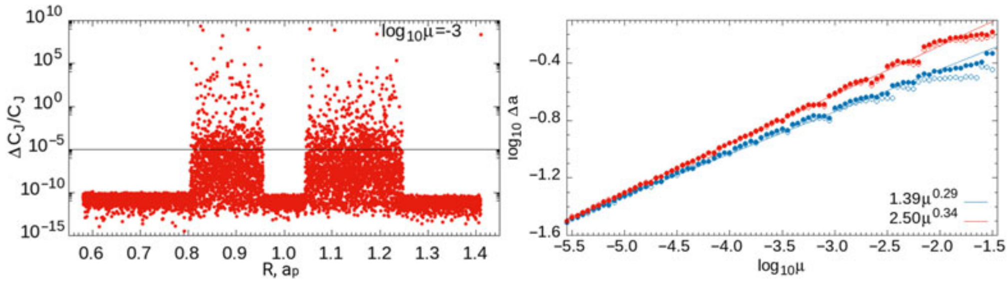

Figure 1. On the left is relative deviation of the value of the Jacobi integral from the initial value after 10^4 periods of the planet depending on the initial position of the particles for the case $\phi = 180°$. On the right the width of the inner (blue) and outer (red) chaotic zone for $\phi = 90°$ (filled circles) and $\phi = 180°$ (open circles), depending on the ratio of the masses of the planet and the star. The approximations of dependence for $\phi = 90°$ are in right bottom corner of the picture.

the planet convolutions. To solve the differential equations, we used Bulirsch-Stoer algorithm (Press *et al.* (1992)) with a fractional truncation error tolerance of $\epsilon = 10^{-14}$. During this time, the major semi-axis of the planet is preserved with an error of less than 10^{-10}.

Calculations have shown that with close approaches of a particle to a planet, the value of the Jacobi integral can change. It can be seen in Fig. 1 that the values of the Jacobi integral vary substantially in the vicinity of the planet's orbit, where, due to the overlap of the first-order mean motion resonances, there is a chaotic zone (Wisdom (1980)). The violations of the Jacobi integral for some particles were also identified in Morrison & Malhotra (2015).

It is interesting that the position of the boundaries of the chaotic zone can be traced from the violations of the Jacobi integral. The boundary positions of the particles were determined if the relative violation of the Jacobi integral was $> 10^{-5}$. The results show that the position of the boundaries of the chaotic zone is weak, but depends on ϕ, especially for the case of large planet masses. The averaged dependences of the size of the inner part of the chaotic zone on the ratio of the masses of the planet and the star (μ) is $\Delta a_{in} = 1.39\mu^{0.29} a_p$, and of the outer one is $\Delta a_{out} = 2.5\mu^{0.34} a_p$ (Fig. 1). They are close to those obtained in Morrison & Malhotra (2015); Demidova & Shevchenko (2020).

Thus, when solving the restricted three-body problem using the Bulirsch-Stoer method, it is necessary to set the accretion radius around the planet. Particles entering this radius should be considered to have left the system, because the Bulirsch-Stoer integrator failed in case of close encounters. The maximum deviation of the Jacobi integral from the initial value depends on the accretion radius. The calculations show if the accretion radius does not exceed $0.01R_h$ the maximum deviation of the integral is $\sim 10^{-10}$. It should be noted that maximum of the distribution of the physical radius of the planets is about $5 \cdot 10^{-2} R_h$ (Morrison & Malhotra (2015)). So, it is good enough for simulation if the accretion radius equals to the planet radius.

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