Factorial Moments and Frequencies of Charlier's Type B.

By J. T. Campbell, Edinburgh University.

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§ 1. *Introductory.*

When a given frequency distribution is to be graduated, it is customary to express the constants of the fitted curve in terms of the moments of the frequency distribution. The r^{th} moment of the distribution, in which the relative frequency of a measure x is $\phi(x)$, or, in the case of a continuous variable, the differential of frequency is $\phi(x)dx$, is defined in the respective cases by

$$\Sigma x^r \phi(x), \quad \text{or by} \quad \int x^r \phi(x) dx,$$

the summation or integration being taken over the whole range of possible values of x. In the present paper we make use of another kind of moment, the factorial moment, which has already been considered by several writers,[1] and which is specially suited to the case when the frequencies of the distribution are given for discrete, equidistant values of the variable. The $(r+1)^{\text{th}}$ factorial moment, for the case where x, measured from some arbitrary origin, can increase by increments h, $2h$, $3h$, ..., will be defined to be

$$\Sigma \phi(x)\, x(x-h)(x-2h)\ldots.(x-rh),$$

where the summation extends over all possible values of x; it will be denoted by $m_{(r+1)}$. By a suitable choice of scale the increments of x may be taken as equal to unity in any given case.

I propose to deduce the Charlier series of Type B by means of a factorial moment generating function, to find the relations connecting factorial moments and the constants in the Type B series, and to use factorial moments in a numerical example.

If $F(t)$ is the frequency generating function of a distribution, that is, if we have respectively

$$F(t) = \Sigma \phi(x)\, t^x, \quad F(t) = \int \phi(x)\, t^x\, dx,$$

[1] W. Palin Elderton, "*Frequency Curves and Correlation*" (2nd Ed., 1927), 20. W. F. Sheppard, *Proc. London Math. Soc.* (2), 13 (1913), 81. J. F. Steffensen, "*Interpolation*" (Baltimore, 1927), § 6.

then the generating function of ordinary moments is obtained by the substitution $t = e^{a}$. The generating function of factorial moments (referred to afterwards as "factorial m.g.f.") is obtained by the substitution $t = 1 + a$. Thus, from the first form of $F(t)$ given above, we derive

$$\begin{aligned} F(1+a) &= \Sigma\phi(x)(1+a)^{x} \\ &= \Sigma\phi(x) + \Sigma x\phi(x) \,.\, a + \Sigma x(x-1)\phi(x)\,.\,a^2/2! + \ldots \end{aligned}$$

where the coefficient of $a^r/r!$ is $m_{(r)}$; that is, we have the factorial m.g.f. for $\phi(x)$.

Evidently the factorial m. g. f. of the sum, or of a linear combination, of two or more frequency functions is the same sum or linear combination of the factorial m.g.f.'s of the several frequency functions. Another result that will be required is this: if $F(t)$ is the factorial m.g.f. of $\phi(x)$, then $(t-1)^r F(t)$ is the factorial m.g.f. of $(-)^r \nabla^r \phi(x)$, where $\nabla f(x) = f(x) - f(x-1)$, a receding finite difference. This follows at once from the fact that, since $\phi(x)$ is the coefficient of t^x in $F(t)$, the coefficient of t^x in $(t-1)^r F(t)$ is

$$(-)^r \left\{ \phi(x) - r\phi(x-1) + \frac{r(r-1)}{2!}\phi(x-2) - \ldots + (-)^r \phi(x-r) \right\}.$$

The expression in brackets is recognised from the calculus of finite differences as the expansion of $\nabla^r \phi(x)$.

Supposing then that we have obtained the factorial m.g.f. of some frequency distribution as a sum of terms such as $a^r\psi(a)$, and that we know $\psi(a)$ to be the factorial m.g.f. of a certain frequency function $\phi(x)$, we may infer from the above that the frequency function in question is a sum of terms like $(-)^r \nabla^r \phi(x)$.

In a later section it will be shewn that the factorial moments of a given distribution can be obtained by a simple method of successive summation.

§ 2. *Derivation of the Charlier Series of Type B.*

The dominant part of the Charlier Series of Type B is the well known Poisson limit to the binomial distribution for probabilities of small order; but both the Poisson function and the Charlier series arise under more general conditions.

Let the variable x be the resultant of N fortuitous increments δx, where δx_j, the j^{th} of these increments, may take any one of k values, each with a certain probability. The assumptions are that the elementary increments δx_j may take the values 0, $1/N$, $2/N$, and

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so on; also that the probability of non-zero increments is of so small an order in N that, if $\delta m_{(j:r)}$ is the factorial moment of order r of the elementary distribution, then

$$\delta m_{(j:1)} = \Sigma \delta x_j p_j \text{ is of order } 1/N,$$
$$\delta m_{(j:2)} = \Sigma \delta x_j (\delta x_j - 1/N) p_j \text{ is of order } 1/N^2, \text{ etc.}$$

For reasons of convenience that will appear later we write the elementary factorial m. g. f., namely

$$F_j(\alpha) = 1 + \delta m_{(j:1)} \alpha + \delta m_{(j:2)} \alpha^2/2! + \ldots .$$

as

$$F_j(\alpha) = \exp[\alpha\, \delta m_{(j:1)}] (1 + b_2\alpha^2 + b_3\alpha^3 + \ldots .),$$

where b_2 is of order $1/N^2$, b_3 is of order $1/N^3$, and so on.

By compound probability the factorial m. g. f. of a variable which is a sum of independent variables is the product of the factorial m. g. f.'s of the several variables. Hence the factorial m. g. f. $F(\alpha)$ of x in the present case is

$$F(\alpha) = \exp[\alpha\, \Sigma\, \delta m_{(j:1)}]\, \alpha\, \Pi\, (1 + b_2\, \alpha^2 + b_3\, \alpha^3 + \ldots .)$$
$$= e^{a\alpha} (1 + B_2\, \alpha^2 + B_3\, \alpha^3 + \ldots .),$$

where a is the mean value of x, given by $a = \Sigma \delta m_{(j:1)}$.

Reserving consideration of the question of the order in N of $B_2, B_3, \ldots .$, we shall find what frequency function corresponds to the factorial m. g. f. $F(\alpha)$, as given above. When N is large the dominant term, it will appear, is the first term, $e^{a\alpha}$. If $\psi(x)$ is the frequency function corresponding to this, we shall have

$$e^{a\alpha} = \sum_{x=0}^{\infty} \psi(x) \,.\, (1+\alpha)^x,$$

that is

$$e^{-a} e^{a(1+\alpha)} = \sum_{x=0}^{\infty} \psi(x)\, (1+\alpha)^x,$$

whence we have

$$\psi(x) = e^{-a} a^x / x!,$$

which is Poisson's frequency function referred to above. This is the usual form in which it is given. We ought, however, to take account here of the fact that x, being compounded of elements which can assume the values $1/N$, $2/N, \ldots .$, can also assume fractional values, and so we adopt the more general form

$$\psi(x) = e^{-a}\, a^x / \Gamma(x+1), \qquad (x > 0).$$

In order to obtain the more general frequency function which results from taking all the terms of $F(\alpha)$, use is made of the result given at the end of § 1. This yields the more general frequency function,

$$\phi(x) = \psi(x) + B_2 \nabla^2 \psi(x) + B_3 \nabla^3 \psi(x) + \ldots ,$$

where $\psi(x)$ is the Poisson function found above.

It remains to investigate the order in N of the coefficients B_r, where

$$1 + B_2 a^2 + B_3 a^3 + \ldots = (1 + b_2 a^2 + b_3 a^3 + \ldots)^N.$$

Clearly B_r will be composed of a sum of terms like $b_h b_k \ldots b_q$, where $h, k, \ldots, q$ may take the integer values 2, 3, 4,...., and where the sum of these indices $h, k, \ldots$ is r. The order of each such term is evidently $1/N^r$. Also, if the number of parts in such a partition of r is m, the number of m-part terms is ${}^N C_m$, which is of order N^m. This order is greatest when m is greatest; in other words, the order of B_r is determined by the partition of the integer r *which has most parts.* If r is even, equal to $2s$, such a partition is made up of s 2's; if r is odd, of the form $2s - 1$, the partition is of $(s - 2)$ 2's and a single 3, that is, has $s - 1$ parts. It follows at once that the order both of B_{2s-1} and of B_{2s} is $1/N^s$, so that the B's descend in order regularly in pairs,[1] B_2 being of order $1/N$, B_3 and B_4 of order $1/N^2$, and so on. This is why in fitting a curve of Type B it is advisable to stop after an even coefficient.

§ 3 *Relations between Coefficients and Factorial Moments in Type B.*

For the purpose of fitting a series of Type B to a given frequency distribution it is necessary to know the relations between the coefficients B_r and the factorial moments as computed from the distribution.

These are found by considering the factorial moment generating function found above. If in this we expand the exponential, remembering that $m_{(r)}$ is the coefficient of $a^r/r!$, we derive the relations

$$m_{(1)} = a,$$

$$m_{(2)} = a^2 + 2!\,B_2,$$

$$m_{(3)} = a^3 + \frac{3!}{1!} a B_2 + 3!\,B_3,$$

$$m_{(4)} = a^4 + \frac{4!}{2!} a^2 B_2 + \frac{4!}{1!} a B_3 + 4!\,B_4,$$

[1] *Cf.* C. V. L. Charlier, *Medd. f Lunds Astron. Observ.* (2), **51**, 3.

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and generally

$$m_{(r)} = a^r + \frac{r!}{(r-2)!}a^{r-2}B_2 + \frac{r!}{(r-3)!}a^{r-3}B_3 + \ldots + \frac{r!}{1!}aB_{r-1} + r!\,B_r.$$

These expressions may be made formally complete by inserting after the first term in each case $B_1, \frac{2!}{1!}aB_1, \frac{3!}{2!}a^2B_1, \ldots, \frac{r!}{(r-1)!}a^{r-1}B_1$ respectively, where B_1 is zero. The last of the relations may then be written

$$m_{(r)}/a^r = 1 + \binom{r}{1}\frac{1!\,B_1}{a} + \binom{r}{2}\frac{2!\,B_2}{a^2} + \ldots + \binom{r}{1}\frac{(r-1)!\,B_{r-1}}{a^{r-1}} + \frac{r!\,B_r}{a^r}.$$

If in this we take $m_{(s)}/a^s$ and $s!\,B_s/a^s$ as the variables, then by the application of a reciprocal result in matrices having binomial coefficients for elements we may invert the relations, and so obtain

$$r!\,B_r = m_{(r)} - \binom{r}{1} m_{(r-1)}a + \ldots + (-)^{r-1}\binom{r}{1} m_{(1)}a^{r-1} + (-)^r a^r.$$

Since $m_{(1)} = a$, the last two terms may be combined into $(-)^{r-1}(r-1)a^r$. The expression for B_r in terms of the moments can be represented symbolically by

$$B_r = \frac{([m]-a)^r}{r!},$$

where $[m]^s$ is to be understood as $m_{(s)}$.

In the special case of the series of Type B represented by the Poisson exponential, the factorial moment generating function is simply $e^{a\alpha}$, so that the successive factorial moments form a geometrical progression in a, the mean of the distribution. The fact that the second factorial moment is a^2 provides a criterion of the suitability of representing the distribution by the Poisson function.

§ 4. *The Numerical Process of Fitting a Series of Type B.*

Factorial moments can be calculated with rapidity and ease by a method of continued summation, as may be seen in the following example, in which a curve of Type B is fitted to data given in W. Palin Elderton's *Frequency Curves and Correlation*, 2nd Edition, page 131. We shall take only one correction term, that involving B_2.

The scheme for the calculation of the factorial moments can be arranged as follows:

x	(1) $f_{(x)}$	(2)	(3)	(4)
0	133	222		
1	55	89	140	
2	23	34	51	76
3	7	11	17	25
4	2	4	6	8
5	2	2	2	2

Column (2) is obtained from column (1) by continued summation, beginning from the bottom. The final total in (2) is evidently $\Sigma f(x)$. Column (3) is obtained from (2) in the same way, but the process is stopped one line below the final total in the previous column. This sum is $\Sigma\, xf(x)$, that is $Nm_{(1)}$. Column (4) is obtained similarly from (3), the process being stopped one line below the final total in (3). The last total in (4) is $\Sigma x(x-1)f(x)/2!$, that is $Nm_{(2)}/2!$. The procedure is general, and so the factorial moments of higher orders could be obtained in the form $Nm_{(r)}/r!$. We have then, in the present example,

$$m_{(1)} = 140/222 \qquad = 0{\cdot}631.$$
$$m_{(2)} = 76/222 \times 2! \qquad = 0{\cdot}685.$$
$$B_2 = \tfrac{1}{2}({\cdot}685 - ({\cdot}631)^2) = 0{\cdot}143.$$

The remaining arithmetical work in fitting the curve is given in the following table:

x	(1) $\psi(x)$	(2) $N\psi(x)$	(3) $N\Delta\psi(x)$	(4) $N\Delta^2\psi(x)$	(5) $B_2 \times$ (4)	(6) (2)+(5)	(7) Data
0	0·5320	118·1	118·1	118·1	16·9	135·0	133
1	·3357	74·5	− 43·6	− 161·7	− 23·1	51·4	55
2	·1059	23·5	− 51·0	− 7·4	− 1·1	22·4	23
3	·0223	4·9	− 18·6	32·4	4·6	9·5	7
4	·0035	0·8	− 4·1	14·5	2·1	2·9	2
5	·0004	0·1	− 0·7	3·4	0·5	0·6	2

The values of $\psi(x) = e^{-a}a^x/x!$, where $a = m_{(1)}$, can be obtained from tables. Columns (3) and (4) are obtained by considering the elements in column (2) corresponding to negative integer values of x to be zero. The Poisson function is indeed zero for such values.

Column (6) gives the graduated figures as obtained by using one correction term. A comparison with the original data, given in column (7), shows good agreement between the theoretical and the observed.

§ 5. *The Distribution and Factorial Moments of* $m_{(1)}$ *in Samples.*

In this section we shall investigate the distribution and the factorial moment generating function of the first factorial moment $m_{(1)}$, as computed from samples of N drawn from an infinite Poisson universe.

The universe or population being typified by a frequency generating function $e^{a(t-1)}$, the frequency generating function of samples of N is, by compound probability,

$$e^{Na(t-1)}. \qquad (A)$$

This is the distribution of $(x_1 + x_2 + \ldots + x_N)$, or $Nm_{(1)}$. Hence the distribution of $m_{(1)}$ is obtained by altering the scale of measurement in (A) in the ratio $N:1$, that is, we must replace t by $t^{1/N}$. Hence the distribution of $m_{(1)}$ is given by

$$e^{Na(t^{1/N}-1)}. \qquad (B)$$

From this we learn at once that the relative frequency, or probability, of a value x for $m_{(1)}$ is given by

$$e^{-Na}(Na)^{Nx}/(Nx)!.$$

Further, by substituting $1+\beta$ for t in (B) we derive the factorial moment generating function of $m_{(1)}$ in samples of N as

$$e^{-Na}\, e^{Na(1+\beta)^{1/N}}.$$

The first factorial moment of this sample distribution, which we may denote $\overline{m}_{(1)}$, is the coefficient of β in the expansion of this, and is seen to be equal to a. Thus the expected or mean value of the mean in samples of N is no different from the mean of the infinite Poisson universe itself.

The second factorial moment of the sample distribution differs from that of the universe; for the coefficient of $\beta^2/2!$ in the generating function is

$$e^{-Na}\left\{Na\,\frac{1}{N}\left(\frac{1}{N}-1\right)+\frac{N^2a^2}{2!}\,\frac{2}{N}\left(\frac{2}{N}-1\right)+\ldots\right\}$$

$$=ae^{-Na}\left\{\frac{1}{N}-1+\frac{Na}{1!}\left(\frac{2}{N}-1\right)+\frac{N^2a^2}{2!}\left(\frac{3}{N}-1\right)+\ldots\right\}$$

$$=ae^{-Na}\left\{\frac{1}{N}e^{Na}(Na+1)-e^{Na}\right\}$$

$$=a\,(a-1)+a/N.$$

For samples of 1, that is, single drawings, this gives the familiar result $m_{(2)}=a^2$.

§ 6. Since I established the relations of § 3 between factorial moments and the coefficients of a Type B series, my attention has been drawn to a recent paper[1] in which the same results are obtained. Since the method of derivation is different, it has been thought worth while to retain the section in question.

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[1] Hilda Pollaczek-Geiringer, *ZS. f. Math. u. Mech.*, **8** (1928), 292-309, § 4.

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