3

Quasilinear Maps

In this chapter, we plunge into the non-linear aspects of the theory of twisted sums. Why should we do so? Because it turns out that exact sequences of quasi-Banach spaces correspond to certain non-linear maps, called quasilinear maps, that offer a convenient, useful and relatively simple tool to construct, describe and study such sequences. Of course a lazy reader (or author) may argue that a considerable stock of exact sequences has already been presented in Chapter 2. And that is indeed the case. However, a sober look at those constructions soon reveals that they all depend on the knowledge of one single operator: the embedding of the quotient map. And, more often than not, knowing an embedding entails no control over the quotient space, and knowing a quotient map does not provide much control on its kernel. Sometimes, but only sometimes, a stroke of luck makes the third space manageable, but this is not to be expected in general. We deplore to say that nothing in Chapter 2 is of great help for constructing a single non-trivial exact sequence $0 \longrightarrow \ell_2 \longrightarrow \cdot \longrightarrow \ell_2 \longrightarrow 0$ or proving that such sequence does not exist. One may also argue that, at the end of the day, all short exact sequences arise as, say, pushouts, and thus one only need consider a quotient map $Q\colon \ell_1 \longrightarrow \ell_2$ and study whether all operators $\ker Q \longrightarrow \ell_2$ extend to ℓ_1. That may be true but is, for various reasons, unfeasible: as a rule, kernels of projective presentations are a complete mystery, and the extension of operators is a problem of its own. Thus, one of our objectives in this chapter is to provide the reader with practical ways to construct non-trivial exact sequences $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ when only the spaces Y and X are known. The central idea here is that such exact sequences correspond to quasilinear maps $\Phi\colon X \longrightarrow Y$.

The chapter has been organised so that the reader can reach at an early stage a number of important applications. It begins with an informal discussion of quasilinear maps in order to immediately work through two classical examples:

- Ribe's solution to the 3-space problem for local convexity, namely a quasilinear functional $\varrho\colon \ell_1 \longrightarrow \mathbb{R}$ yielding a non-trivial exact sequence of the form $0 \longrightarrow \mathbb{R} \longrightarrow \cdot \longrightarrow \ell_1 \longrightarrow 0$. The middle space, obviously, cannot be locally convex.
- The Kalton–Peck spaces: a family of quasilinear maps $\Phi\colon X \longrightarrow X$ that generate non-trivial exact sequences $0 \longrightarrow X \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ for most quasi-Banach spaces X with unconditional basis, including the spaces ℓ_p for $0 < p < \infty$. The reader who welcomes a challenge is invited to reflect on the meaning of the cases $0 < p \leq 1$ and $p = 2$.

Other applications of quasilinear maps, many of them presented in this chapter, but not all, include finding pairs of quasi-Banach spaces X, Y such that all exact sequences $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ split; natural representations (natural equivalences is the right word) for the functor Ext; getting valuable insight into the structure of exact sequences and twisted sum spaces; simplifications for pullback, pushout and other homological constructions; a duality theory for exact sequences; uniform boundedness principles for exact sequences; a local theory for exact sequences … Enough talk. Let's dive in.

3.1 An Introduction to Quasilinear Maps

Let us try to explain why quasilinear maps are so useful for both the description of twisted sums and the construction of relevant examples. Suppose that for some reason we are given a short exact sequence of quasi-Banach spaces $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$. We ask the reader to imagine that X and Y are the data (that is, they are known) while the space Z is not: one only knows that it fits in the exact sequence above. As vector spaces, all exact sequences split; this quickly follows from the existence of a Hamel basis in X. This enables us to regard Z as the direct product $Y \times X$ equiped with some quasinorm transferred from Z. The embedding is $\imath(y) = (y, 0)$ and the quotient map is $\pi(y, x) = x$. Replacing the original quasinorms of X and Y by suitable equivalent ones, we may assume that

- $\|y\| = \|(y, 0)\|$ for all $y \in Y$.
- $\|x\| = \min_{y \in Y} \|(y, x)\|$ for all $x \in X$.

For each $x \in X$, let us choose $y = \Phi(x)$ such that $\|x\| = \|(\Phi(x), x)\|$. We may assume that the map $\Phi\colon X \longrightarrow Y$ is homogenous since $\|\lambda x\| = \|(\lambda\Phi(x), \lambda x)\|$ for any $\lambda \in \mathbb{K}$. Inspecting Φ more closely reveals that if one compares $\Phi(x + x')$ and $\Phi(x) + \Phi(x')$ for two different $x, x' \in X$, the choice of Φ yields

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$$\begin{aligned}\|\Phi(x+x')-\Phi(x)-\Phi(x')\| &= \|(\Phi(x+x')-\Phi(x)-\Phi(x'),0)\|\\ &= \|(\Phi(x+x'),x+x')-(\Phi(x),x)-(\Phi(x'),x')\|\\ &\le (\Delta_Y+\Delta_Y^2)(\|x+x'\|+\|x\|+\|x'\|)\\ &\le (\Delta_Y+\Delta_Y^2)(1+\Delta_X)(\|x\|+\|x'\|).\end{aligned}$$

Recapitulating; a certain map $\Phi\colon X \longrightarrow Y$ arises as soon as one considers an extension of X by Y. Most probably, this map is not linear, or bounded, but it is homogeneous and obeys the estimate

$$\|\Phi(x+x')-\Phi(x)-\Phi(x')\| \le M(\|x\|+\|x'\|). \tag{3.1}$$

Moreover, Φ can be used to describe an equivalent quasinorm on Z: indeed, each point of the product space can be decomposed as $(y,x) = (y-\Phi(x),0) + (\Phi(x),x)$ and thus $\|(y,x)\| \le \Delta(\|(y-\Phi(x),0)\| + \|(\Phi(x),x)\|) = \Delta(\|y-\Phi(x)\| + \|x\|)$. But since $\|(y,x)\| \ge \|x\|$ and $\|y-\Phi x\| = \|(y,x)-(\Phi x,x)\| \le \Delta(\|(y,x)\| + \|(\Phi x,x)\|) = \Delta(\|(y,x)\|+\|x\|)$, we get $\|y-\Phi(x)\|+\|x\| \le (1+\Delta)\|(y,x)\|$, and thus we have

$$\Delta^{-1}\|(y,x)\| \le \|y-\Phi(x)\|+\|x\| \le (1+\Delta)\|(y,x)\|.$$

Since any positively homogeneous functional that is equivalent to a quasinorm is itself a quasinorm, what has been shown is that the functional

$$\|y,x)\|_\Phi = \|y-\Phi(x)\|+\|x\| \tag{3.2}$$

gives an equivalent quasinorm on Z. Hence, for most practical purposes, the space Z is just the product space $Y\times X$ endowed with that quasinorm. If we agree to call a homogeneous map $\Phi\colon X\longrightarrow Y$ satisfying (3.1) *quasilinear*, we hit the nail on the head, since now we can follow the chain of inequalities backwards to get that if $\Phi\colon X\longrightarrow Y$ is a quasilinear map then $\|(y,x)\|_\Phi = \|y-\Phi(x)\|+\|x\|$ is a quasinorm on $Y\times X$: the homogeneity is trivial and

$$\begin{aligned}\|(y+y',x+x')\|_\Phi &= \|(y+y'-\Phi(x+x')\|+\|x+x'\|\\ &\le \|(y-\Phi(x)+y'-\Phi(x')-\Phi(x+x')+\Phi(x)+\Phi(x')\|+\|x+x'\|\\ &\le M(\|y-\Phi x\|+\|y'-\Phi x'\|+\|\Phi(x+x')-\Phi x-\Phi x'\|+\|x\|+\|x'\|)\\ &\le M(\|(y,x)\|_\Phi+\|(y',x')\|_\Phi),\end{aligned}$$

as required. Let us write $Y\oplus_\Phi X$ for the product space $Y\times X$ equipped with the quasinorm $\|(\cdot,\cdot)\|_\Phi$. We have the exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Y\oplus_\Phi X \xrightarrow{\ \pi\ } X \longrightarrow 0,$$

in which $\imath(y) = (y,0)$, and $\pi(y,x) = x$. It is obvious that the kernel of π and the image of $\imath$ agree, that $\imath$ is an isometric embeding and that π maps the unit ball of $Y \oplus_\Phi X$ onto that of X. The middle space is automatically complete, thus a quasi-Banach space, in view of Proposition 2.3.4.

3.2 Quasilinear Maps in Action

The moral of the preceding discussion is clear: if we want to construct an extension of X by Y, we have to define a quasilinear map $\Phi\colon X \longrightarrow Y$. We must warn the reader that this can be a nigh impossible task. we cannot 'explicitly' define a quasilinear map on the whole quasi-Banach space unless it is bounded (in which case the corresponding extension simply splits), for the same reason that we cannot explicitly define a linear map on a quasi-Banach space unless it is bounded: Banach proved in [30, Théorème 4, Chapitre 1] that Borel linear maps between F-spaces are continuous. Fortunately, this blow is not fatal since in practice it suffices to define quasilinear maps on some dense subspace. Let us make it official:

Definition 3.2.1 Let X and Y be quasinormed spaces. A map $\Phi\colon X \longrightarrow Y$ is quasilinear if it is homogeneous and there is a constant Q such that

$$\|\Phi(x+y) - \Phi(x) - \Phi(y)\| \leq Q\,(\|x\| + \|y\|)$$

for every $x, y \in X$. The quasilinearity constant of Φ, denoted by $Q(\Phi)$, is the infimum of the numbers Q above.

The functional $\|(y,x)\|_\Phi = \|y - \Phi(x)\| + \|x\|$ is a quasinorm on $Y \times X$, and if we denote by $Y \oplus_\Phi X$ the corresponding quasinormed space, then $0 \longrightarrow Y \xrightarrow{\imath} Y \oplus_\Phi X \xrightarrow{\pi} X \longrightarrow 0$ with $\imath(y) = (y, 0)$ and $\pi(y, x) = x$ is an isometrically exact sequence. Assume now that X and Y are quasi-Banach spaces, that X_0 is a dense subspace of X so that $X_0 \longrightarrow X$ is a completion of X_0 and that $\Phi\colon X_0 \longrightarrow Y$ is quasilinear. Let us form the exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus_\Phi X_0 \longrightarrow X_0 \longrightarrow 0$. Let $\kappa\colon Y \oplus_\Phi X_0 \longrightarrow Z(\Phi)$ denote *a* completion of $Y \oplus_\Phi X_0$ (please peek back at Note 1.8.1 if the word 'a' came as a surprise). We know from 2.3.5 that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Phi X_0 \arrow[r] \arrow[d, "\kappa"] & X_0 \arrow[r] \arrow[d, "\text{inclusion}"] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z(\Phi) \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

where the lower sequence is also exact. There are good reasons to say that this lower sequence is generated by Φ. To be honest, this exact sequence depends

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also on the completion of $Y \oplus_\Phi X_0$ we choose, but it is clear that all completions provide equivalent exact sequences. We shall see soon (Section 3.3) that the space $Z(\Phi)$ can be obtained as $Y \oplus_{\widetilde{\Phi}} X$, where $\widetilde{\Phi}\colon X \longrightarrow Y$ is a quasilinear map extending Φ. We now pause our study of quasilinear maps to follow Pełzyński's advice: examples first.

Ribe's Map

Ribe's map is a real-valued quasilinear map ϱ defined on the subspace ℓ_1^0 of finitely supported sequences of ℓ_1. It generates a non-trivial exact sequence $0 \longrightarrow \mathbb{R} \longrightarrow Z(\varrho) \longrightarrow \ell_1 \longrightarrow 0$. The construction depends on the properties of the function of a single variable $\omega(t) = t \log |t|$ (assuming that $\omega(0) = 0 \cdot \log 0 = 0$).

Lemma 3.2.2 *For all $s, t \in \mathbb{R}$, we have $|\omega(s+t)-\omega(s)-\omega(t)| \leq (\log 2)(|s|+|t|)$.*

Proof Let us consider first the case in which s and t have the same sign. Assuming for instance that $s, t \geq 0$, we have

$$\begin{aligned}
|\omega(s+t) - \omega(s) - \omega(t)| &= |(s+t)\log(s+t) - s\log s - t\log t| \\
&= |s\log(s+t) + t\log(s+t) - s\log s - t\log t| \\
&= \left| s\log\left(\frac{s}{s+t}\right) + t\log\left(\frac{t}{s+t}\right)\right| \\
&\leq (|s|+|t|) \underbrace{\left|\frac{s}{s+t}\log\left(\frac{s}{s+t}\right) + \frac{t}{s+t}\log\left(\frac{t}{s+t}\right)\right|}_{(\star)}.
\end{aligned}$$

To maximise $(\star)$, simply write it as

$$|s'\log s' + t'\log t'|, \quad \text{with} \quad s' = \frac{s}{s+t} \quad \text{and} \quad t' = \frac{t}{s+t},$$

and observe that, since $s' + t' = 1$, the maximum value is $\log 2$ (attained at $s' = t' = 1/2$). It follows that $|\omega(s+t) - \omega(s) - \omega(t)| \leq (\log 2)(|s|+|t|)$. Now, if s and t have distinct signs, we may assume that s is positive, t is negative and $s + t > 0$. Taking into account that ω is an odd map, the proof concludes with

$$\begin{aligned}
|\omega(s+t) - \omega(s) - \omega(t)| &= |\omega(s) - \omega(-t) - \omega(s+t)| \\
&\leq (\log 2)(|-t| + |s+t|) \leq (\log 2)(|s|+|t|). \qquad \square
\end{aligned}$$

Proposition 3.2.3 *The map $\varrho\colon \ell_1^0 \longrightarrow \mathbb{R}$ given by*

$$\varrho(x) = \sum_i x(i)\log|x(i)| - \left(\sum_i x(i)\right)\log\left|\sum_i x(i)\right|$$

is quasilinear with quasilinearity constant $2\log 2$ *and induces a nontrival sequence*

$$0 \longrightarrow \mathbb{R} \longrightarrow Z(\varrho) \longrightarrow \ell_1 \longrightarrow 0.$$

Proof The homogeneity of ϱ is obvious. Writing $S(x) = \sum_i x(i)$ for $x \in \ell_1$, we have $\varrho(x) = S(\omega \circ x) - \omega(S(x))$, so

$$\begin{aligned}
&|\varrho(x+y) - \varrho(x) - \varrho(y)| \\
&= |S(\omega \circ (x+y)) - \omega(S(x+y)) - S(\omega \circ x) + \omega(Sx) - S(\omega \circ y) + \omega(Sy)| \\
&\leq |\omega(Sx + Sy)) - \omega(Sx) - \omega(Sy))| + |S(\omega \circ (x+y)) - S(\omega \circ x) - S(\omega \circ y)| \\
&\leq (\log 2)\left(|S(x)| + |S(y)| + \sum_i (|x(i)| + |y(i)|)\right) \\
&\leq (2\log 2)(\|x\| + \|y\|).
\end{aligned}$$

To prove that the sequence does not split, observe that the estimate $\left\| \sum_i x_i \right\| \leq C\left(\sum_i \|x_i\|\right)$ in Lemma 1.1.2 is impossible in $\mathbb{R} \oplus_\varrho \ell_1^0$ because if (e_i) is the unit basis of ℓ_1 then $\|(0, e_i)\|_\varrho = 1$ for all $i \in \mathbb{N}$, which makes $\sum_{i=1}^n \|(0, e_i)\|_\varrho = n$, while

$$\left\| \left(0, \sum_{i=1}^n e_i\right) \right\|_\varrho = n \log n. \qquad \square$$

This construction from Ribe [401] shows that local convexity is not a 3-space property in the domain of quasi-Banach spaces. Other counterexamples were presented by Kalton [251] and Roberts [403], independently and more or less simultaneously; Smirnov and Sheikhman [437] gave one more, entirely alien, example. Roberts' example is obtained by a different technique not involving quasilinear maps. It is not coincidence that all four counterexamples are twisted sums of $\mathbb{R}$ and ℓ_1: one of the conclusions of Section 3.4 is that *any* counterexample for the 3-space problem for local convexity leads, by simple algebraic manipulations, to one in which the subspace is 1-dimensional and the quotient space is ℓ_1.

Kalton–Peck Maps

Kalton and Peck found a way to transform Ribe's scalar map into a vector-valued quasilinear map that can be defined on every quasi-Banach space with unconditional basis. The ground field can be either $\mathbb{R}$ or $\mathbb{C}$ from now on.

3.2.4 By a (quasinormed) sequence space, we understand a linear space X of functions $x\colon \mathbb{N} \longrightarrow \mathbb{K}$ equipped with a quasinorm $\|\cdot\|$ such that

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- if $|y| \leq |x|$ and $x \in X$, then $y \in Y$ and $\|y\| \leq \|x\|$,
- the unit vectors are normalised and the finitely supported sequences form a dense subspace of X.

Of course, all that means that the unit vectors form a 1-unconditional basis of X. If X is complete, we call it a quasi-Banach sequence space. We shall invariably denote by X^0 the dense subspace of finitely supported sequences in X. The simplest sequence spaces are ℓ_p for $0 < p < \infty$ and c_0. Every quasi-Banach space with a (normalised) 1-unconditional basis can also be seen as a sequence space in the obvious way. The space ℓ_∞ acts on every sequence space by pointwise multiplication in such a way that $\|ax\| \leq \|a\|\|x\|$ for every $a \in \ell_\infty$ and every $x \in X$ and such that $\|ax\| = \|x\|$ when a is unitary, that is, when $|a(k)| = 1$ for all $k \in \mathbb{N}$. Let $\mathrm{Lip}_0(\mathbb{R})$ be space of Lipschitz functions $\varphi\colon \mathbb{R} \longrightarrow \mathbb{K}$ vanishing at zero, and let $\mathrm{Lip}(\varphi)$ denote its Lipschitz constant. Similar conventions apply to $\mathrm{Lip}_0(\mathbb{R}^+)$. Lipschitz functions can be used to produce a variety of Ribe-like functions that share the quasiadditivity property appearing in Lemma 3.2.2:

Lemma 3.2.5 *Let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{K}$ be a Lipschitz function vanishing at zero, and let $\omega_\varphi\colon \mathbb{K} \longrightarrow \mathbb{K}$ be the map $\omega_\varphi(z) = z\,\varphi(-\log|z|)$. For all scalars z, z', one has*

$$|\omega_\varphi(z+z') - \omega_\varphi(z) - \omega_\varphi(z)| \leq 2\mathrm{Lip}(\varphi)e^{-1}(|z| + |z'|). \tag{3.4}$$

Proof The proof is based on the trivial fact that $|t\log t| \leq e^{-1}$ for $0 \leq t \leq 1$. We first assume that $|z+z'| \geq \max(|z|, |z'|)$. Then

$$\begin{aligned}
&\frac{|\omega_\varphi(z+z') - \omega_\varphi(z) - \omega_\varphi(z')|}{|z| + |z'|}\\
&\leq \frac{|z(\varphi(-\log|z+z'|) - \varphi(-\log|z|) - z'(\varphi(-\log|z+z'|) - \varphi(-\log|z'|)|}{|z+z'|}\\
&\leq \frac{|z|}{|z+z'|}\,\mathrm{Lip}(\varphi)\log\frac{|z+z'|}{|z|} + \frac{|z'|}{|z+z'|}\,\mathrm{Lip}(\varphi)\log\frac{|z+z'|}{|z'|}\\
&\leq 2\mathrm{Lip}(\varphi)e^{-1}.
\end{aligned}$$

If, on the contrary, $|z+z'| < \max(|z|, |z'|)$ then we may assume that $|z| \geq \max(|z+z'|, |z'|)$. Replacing z by $z+z'$ and z' by $-z'$ and taking into account that ω_φ is odd, we have

$$\begin{aligned}
|\omega_\varphi(z+z') - \omega_\varphi(z) - \omega_\varphi(z')| &= |\omega_\varphi(z) - \omega_\varphi(-z') - \omega_\varphi(z+z')|\\
&\leq 2\mathrm{Lip}(\varphi)e^{-1}(|z'| + |z+z'|)\\
&\leq 2\mathrm{Lip}(\varphi)e^{-1}(|z| + |z'|).
\end{aligned}$$ □

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Given a sequence space X and $\varphi \in \mathrm{Lip}_0(\mathbb{R}^+)$, we define the map $\mathsf{KP}_\varphi\colon X^0 \longrightarrow X$ by

$$\mathsf{KP}_\varphi(x) = x \cdot \varphi\left(\log \frac{\|x\|}{|x|}\right) \tag{3.5}$$

and call it the Kalton–Peck map induced by φ on X. For the avoidance of doubt, the definition means

$$\mathsf{KP}_\varphi(x)(k) = \begin{cases} x(k) \cdot \varphi\left(\log(\|x\|/|x(k)|)\right) & \text{if } x(k) \neq 0 \\ 0 & \text{if } x(k) = 0. \end{cases}$$

These maps commute with the action of the unitary group: if u is unitary, then $\mathsf{KP}_\varphi(ux) = u\mathsf{KP}_\varphi(x)$ for all $x \in X^0$ and thus $\|(uy, ux)\|_{\mathsf{KP}_\varphi} = \|(y, x)\|_{\mathsf{KP}_\varphi}$. This property will be used over and over.

Proposition 3.2.6 *The map $\mathsf{KP}_\varphi\colon X^0 \longrightarrow X$ is quasilinear and $Q(\mathsf{KP}_\varphi)$ depends only on the Lipschitz constant of φ and the modulus of concavity of X.*

Proof We can assume that φ is defined on the whole line by taking $\varphi(t) = 0$ for all $t < 0$; this extension has the same Lipschitz constant as the original φ. Consider the non-homogeneous map $\mathsf{kp}_\varphi\colon X^0 \longrightarrow X$ given by

$$\mathsf{kp}_\varphi(x) = x \cdot \varphi(-\log|x|),$$

with the same meaning as above. Since $\mathsf{kp}_\varphi(x)(k) = \omega_\varphi(x(k))$, the preceding Lemma provides the pointwise estimate

$$|\mathsf{kp}_\varphi(x+y)(k) - \mathsf{kp}_\varphi(x)(k) - \mathsf{kp}_\varphi(y)(k)| \leq 2\,\mathrm{Lip}(\varphi)e^{-1}(|x(k)| + |y(k)|); \tag{3.6}$$

hence,

$$\|\mathsf{kp}_\varphi(x+y) - \mathsf{kp}_\varphi(x) - \mathsf{kp}_\varphi(y)\| \leq \frac{2\,\mathrm{Lip}(\varphi)}{e}\big\|\,|x| + |y|\,\big\| \leq \frac{2\Delta\,\mathrm{Lip}(\varphi)}{e}(\|x\| + \|y\|),$$

where Δ is the modulus of concavity of X. To complete the proof, observe that

$$\begin{aligned} \|\mathsf{KP}_\varphi(x) - \mathsf{kp}_\varphi(x)\| &= \left\| x \cdot \left\{ \varphi\left(\log\frac{\|x\|}{|x|}\right) - \varphi\left(-\log|x|\right)\right\}\right\| \\ &\leq \big\| \mathrm{Lip}(\varphi) \cdot \log\|x\| \cdot x \big\|, \end{aligned} \tag{3.7}$$

since the term between braces is pointwise dominated by $\mathrm{Lip}(\varphi)\log\|x\|$. In particular, for $\|x\| \leq 1$, we have

$$\|\mathsf{KP}_\varphi(x) - \mathsf{kp}_\varphi(x)\| \leq e^{-1}\,\mathrm{Lip}(\varphi). \tag{3.8}$$

If $\|x\|, \|y\| \leq (2\Delta)^{-1}$ then $\|x+y\| \leq 1$, so adding and substracting $\mathsf{kp}_\varphi(x+y) - \mathsf{kp}_\varphi(x) - \mathsf{kp}_\varphi(y)$, applying the quasinorm inequality twice and applying (3.8) to the first three chunks, we get

$$
\begin{aligned}
&\|\mathsf{KP}_\varphi(x+y) - \mathsf{KP}_\varphi x - \mathsf{KP}_\varphi y\| \\
&\le \|(\mathsf{KP}_\varphi - \mathsf{kp}_\varphi)(x+y) - (\mathsf{KP}_\varphi - \mathsf{kp}_\varphi)x - (\mathsf{KP}_\varphi - \mathsf{kp}_\varphi)y + \mathsf{kp}_\varphi(x+y) - \mathsf{kp}_\varphi x - \mathsf{kp}_\varphi y\| \\
&\le \Delta^2\left(3\,\mathrm{Lip}(\varphi)/e + 2\,\mathrm{Lip}(\varphi)/e\right),
\end{aligned}
$$

and thus $\|\mathsf{KP}_\varphi(x+y) - \mathsf{KP}_\varphi(x) - \mathsf{KP}_\varphi(y)\| \le 5\Delta^2\,\mathrm{Lip}(\varphi)e^{-1}$ whenever $\|x\|, \|y\| \le (2\Delta)^{-1}$. Being KP_φ homogeneous, for any $x, y \in X^0$, we have

$$
\begin{aligned}
&\frac{\|\mathsf{KP}_\varphi(x+y) - \mathsf{KP}_\varphi(x) - \mathsf{KP}_\varphi(y)\|}{2\Delta(\|x\| + \|y\|)} \\
&= \left\|\mathsf{KP}_\varphi\left(\frac{x+y}{2\Delta(\|x\| + \|y\|)}\right) - \mathsf{KP}_\varphi\left(\frac{x}{2\Delta(\|x\| + \|y\|)}\right) - \mathsf{KP}_\varphi\left(\frac{y}{2\Delta(\|x\| + \|y\|)}\right)\right\| \\
&\le \frac{5\Delta^2\,\mathrm{Lip}(\varphi)}{e},
\end{aligned}
$$

hence KP_φ is quasilinear, with constant at most $10\Delta^3\,\mathrm{Lip}(\varphi)e^{-1}$. □

We now form the exact sequences and twisted sum spaces generated by the Kalton–Peck maps KP_φ. The twisted sum space $Z(\mathsf{KP}_\varphi)$ will be denoted by $X(\varphi)$ to emphasise that it depends on both the Lipschitz function φ and the space X where KP_φ acts. An especially interesting case is when $X = \ell_p$, in which we obtain the exact sequences

$$
0 \longrightarrow \ell_p \longrightarrow \ell_p(\varphi) \longrightarrow \ell_p \longrightarrow 0 \tag{3.9}
$$

The following result bluntly precludes the possibility of these sequences being trivial for unbounded φ.

Proposition 3.2.7 *If φ is unbounded on $\mathbb{R}^+$ then $\ell_p(\varphi)$ is not isomorphic to ℓ_p for any $p \in (0, \infty)$.*

Proof Set $s_n = \sum_{1\le i\le n} e_i$ such that $\mathsf{KP}_\varphi(s_n) = \varphi(\|s_n\|)s_n = \varphi(\log(n^{1/p})s_n$. The Lipschitz condition on φ implies that $\sup_n |\varphi(\log(n^{1/p})| = \infty$ since the points of the form $\log n^{1/p}$ form a $1/p$-net on $\mathbb{R}^+$. Now, the proof is different for different values of p: we show that $\ell_p(\varphi)$ is not (isomorphic to) a p-Banach space for $p \in (0, 1]$, that it does not have type p for $p \in (1, 2]$ and that it does not have cotype p for $p \in [2, \infty)$.

Case $p \in (0, 1]$. Since $\mathsf{KP}_\varphi(e_i) = 0$, $\|(0, e_i)\|_{\mathsf{KP}_\varphi} = 1$ for every $i \in \mathbb{N}$. However,

$$
\Big\|\sum_{i=1}^n (0, e_i)\Big\|_{\mathsf{KP}_\varphi} = \|(0, s_n)\|_{\mathsf{KP}_\varphi} = \left(\left|\varphi(\log(n^{1/p})\right| + 1\right)n^{1/p}.
$$

The estimate in Lemma 1.1.2 shows that $\ell_p(\varphi)$ cannot be isomorphic to a p-normed space.

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Case $p \in (1,2]$. We use a randomised version of the preceding argument. For $\varepsilon_i = \pm 1$, one has $\mathsf{KP}_\varphi(\sum_{1\le i\le n} \varepsilon_i e_i) = |\varphi(\log(n^{1/p})| \sum_{1\le i\le n} \varepsilon_i e_i$. Hence, if r_i are the Rademacher functions, then

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)(0, e_i) \right\|_{\mathsf{KP}_\varphi}^p dt = \int_0^1 \left(\left\| \mathsf{KP}_\varphi \left(\sum_{i=1}^n r_i(t) e_i \right) \right\|_p + \left\| \sum_{i=1}^n r_i(t) e_i \right\|_p \right)^p dt$$
$$= \left(\left| \varphi(\log(n^{1/p}) \right| + 1 \right)^p n, \tag{3.10}$$

and thus $\ell_p(\varphi)$ does not have type p and cannot be isomorphic to ℓ_p.
Case $p \in [2,\infty)$. We show that $\ell_p(\varphi)$ does not have cotype p by exploiting the symmetries of KP_φ. We have

$$\mathsf{KP}_\varphi(s_n) = \varphi(\log n^{1/p}) s_n \qquad \Longrightarrow \qquad \mathsf{KP}_\varphi \left(\frac{s_n}{\varphi(\log n^{1/p})} \right) = s_n.$$

For $i \in \mathbb{N}$, set $z_i = (e_i, \varphi(\log n^{1/p})^{-1} e_i)$ so that $\|z_i\|_{\mathsf{KP}_\varphi} = 1 + \varphi(\log n^{1/p})^{-1} \ge 1$ and $\left(\sum_{1\le i\le n} \|z_i\|^p \right)^{1/p} \ge n^{1/p}$, while

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) z_i \right\|_{\mathsf{KP}_\varphi}^p dt \right)^{1/p} = \left\| \left(s_n, \frac{s_n}{\varphi(\log n^{1/p})} \right) \right\| = \frac{n^{1/p}}{\varphi(\log n^{1/p})}. \qquad \square$$

Observe that by taking $p = 1$, we obtain new counterexamples to the 3-space problem for local convexity, namely $\ell_1(\varphi)$ is not a Banach space. But, wait! Are $\ell_p(\varphi)$ Banach spaces when $p > 1$? The topic is studied in Section 3.4, and an affirmative and non-trivial answer is given in Proposition 3.4.5: any twisted sum of ℓ_p can be renormed to be a Banach space when $1 < p < \infty$. Thus, in the end, the following notion refers only to Banach spaces:

Definition 3.2.8 A twisted Hilbert space is a twisted sum of two Hilbert spaces.

The choice $p = 2$ in Proposition 3.2.7 produces our first non-trivial (i.e. not isomorphic to a Hilbert space) twisted Hilbert spaces. The spaces $\ell_p(\varphi)$ obtained from the simplest unbounded Lipschitz function $\varphi(t) = t$ play a very special role in the theory and in this book. We have reserved for them their original name: Z_p. The twisted Hilbert Z_2 space stands apart: it is so special that it will close this book by explaining its many hidden talents. Summing up, none of the sequences (3.9) for $0 < p < \infty$ is trivial when φ is unbounded. In general, it is hard to find isomorphic invariants capable of distinguishing a particular twisted sum from the corresponding direct sum. Actually, it may be an impossible task since there are non-trivial exact sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in which $B \simeq A \times C$ (see the comments after Definition 2.1.4), the Foiaş–Singer sequence (2.5) being perhaps the most natural example. In other cases, however, an isomorphism $B \simeq A \times C$ immediately forces the splitting of the sequence, as is the case for sequences $0 \longrightarrow \ell_p \longrightarrow \cdot \longrightarrow \ell_p \longrightarrow 0$ with $p = 2$ or $0 < p \leq 1$, due to the peculiarities of those spaces; see also Proposition 7.5.2 for another result along these lines. It is an open problem to determine whether non-trivial sequences $0 \longrightarrow \ell_p \longrightarrow \ell_p \longrightarrow \ell_p \longrightarrow 0$ exist for $p \in (1, \infty)$ different from 2.

3.3 Quasilinear Maps versus Exact Sequences

We need to develop an operational theory of quasilinear maps. As the reader may expect, the first order of business is to find criteria that detect when a quasilinear map induces a trivial extension and when two quasilinear maps induce the same extension. The following delightful result yields both criteria showing that two quasilinear maps generate equivalent extensions if and only if the extension generated by their difference splits. Moreover, it hints towards an underlying vector space structure.

Equivalence and Triviality

Let X, Y be quasi-Banach spaces and let X_0 be a dense subspace of X. The following assertion is somehow implicit in 2.3.5:

Lemma 3.3.1 *If $\Phi\colon X \longrightarrow Y$ is quasilinear map and X_0 is a dense subspace of X_0 then $Y \oplus_\Phi X_0$ is a dense subspace of $Y \oplus_\Phi X$.*

Proof Fix $(y, x) \in Y \oplus_\Phi X$ and $\varepsilon > 0$. Take $x' \in X_0$ such that $\|x - x'\| < \varepsilon$ and then set $y' = y - \Phi(x - x')$. Then, even if y' might be far from y in Y, we have

$$\|(y, x) - (y', x')\|_\Phi = \|y - y' - \Phi(x - x')\| + \|x - x'\| = \|x - x'\| < \varepsilon. \qquad \square$$

In other words, the canonical inclusion $Y \oplus_\Phi X_0 \longrightarrow Y \oplus_\Phi X$ is a completion of $Y \oplus_\Phi X_0$. Now assume Φ and Ψ are quasilinear maps $X_0 \longrightarrow Y$ (defined only on the dense subspace X_0).

Lemma 3.3.2 *There is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z(\Phi) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z(\Psi) \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \tag{3.11}$$

if and only if $\Phi - \Psi = B + L$, *where* $B\colon X_0 \longrightarrow Y$ *is (homogeneous) bounded and* $L\colon X_0 \longrightarrow Y$ *is linear. Consequently, the extension induced by* Φ *is trivial if and only if* $\Phi = B + L$, *where* B *is homogenous bounded and* L *is linear.*

Proof The proof is a simple recipe: one grain of the universal property of the completion (Diagram (1.4)) and three drops of the behaviour of its associated exact sequence 2.3.5, plus the fact that $Y \oplus_\Phi X_0 \longrightarrow Z(\Phi)$ and $Y \oplus_\Psi X_0 \longrightarrow Z(\Psi)$ are completions, show that if the commutative diagram (3.11) exists then there is also a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Phi X_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z(\Phi) \arrow[r] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z(\Psi) \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] & Y \oplus_\Psi X_0 \arrow[r] \arrow[u] & X_0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which all vertical unnamed arrows are inclusions. The operator u necessarily maps $Y \oplus_\Phi X_0$ onto $Y \oplus_\Psi X_0$ and makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Phi X_0 \arrow[r] \arrow[d, "u"] & X_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Y \oplus_\Psi X_0 \arrow[r] & X_0 \arrow[r] & 0
\end{tikzcd}$$

On the other hand, a linear map u makes the preceding diagram commute if and only if it has the form $u(y, x) = (y - L(x), x)$ for some linear $L\colon X_0 \longrightarrow Y$, in which case u is invertible, with inverse $u^{-1}(y, x) = (y + L(x), x)$. We then just need to show that such a u is continuous if and only if $\|\Phi - \Psi - L\| < \infty$. The two implications are easy. If u is continuous, then $\|u(y, x)\|_\Psi \leq \|u\|\,\|(y, x)\|_\Phi$, that is, $\|y - L(x) - \Psi(x)\| + \|x\| \leq \|u\|\big(\|y - \Phi(x)\| + \|x\|\big)$. Taking $y = \Phi(x)$, we get $\|\Phi(x) - L(x) - \Psi(x)\| + \|x\| \leq \|u\|\,\|x\|$, and thus $\|\Phi - \Psi - L\| \leq \|u\| - 1$. To get the converse,

$$\begin{aligned}
\|u(y, x)\|_\Psi &= \|(y - L(x), x)\|_\Psi \\
&= \|y - L(x) - \Psi(x)\| + \|x\| \\
&= \|y - \Phi(x) + \Phi(x) - L(x) - \Psi(x)\| + \|x\| \\
&\leq \Delta_Y\big(\|y - \Phi(x)\| + \|\Phi(x) - L(x) - \Psi(x)\|\big) + \|x\|,
\end{aligned}$$

hence $\|u\| \leq 1 + \Delta_Y \|\Phi - \Psi - L\|$. Reversing the roles of Φ and Ψ and using $u^{-1}(y, x) = (y + L(x), x)$, we obtain the bound $\|u^{-1}\| \leq 1 + \Delta_Y \|\Phi - \Psi - L\|$. □

Definition 3.3.3 A quasilinear map $\Phi\colon X \longrightarrow Y$ is said to be trivial if it is the sum of a bounded homogeneous map $B\colon X \longrightarrow Y$ and a linear map $L\colon X \longrightarrow Y$. Two quasilinear maps Φ, Ψ are said to be equivalent, written $\Phi \sim \Psi$, if $\Phi - \Psi$ is trivial.

Thus, a quasilinear map is trivial if and only if it is equivalent to the zero map. Lemma 3.3.2 can be rephrased as saying that a quasilinear map is trivial if and only if the induced sequence is trivial, while two quasilinear maps defined between the same quasinormed spaces are equivalent if and only if they induce equivalent exact sequences. When necessary, we will say that $\Phi\colon X \longrightarrow Y$ is μ-trivial to mean that $\|\Phi - L\| \leq \mu$ for some linear map $L\colon X \longrightarrow Y$.

A timely intermission about twisted sums generated by bounded or linear maps: The second part of Lemma 3.3.2 provides a remarkably simple criterion for the splitting of the sequence induced by a quasilinear map; namely that the quasilinear map be the sum of a bounded and a linear map. It is thus obvious that bounded and linear maps are the simplest quasilinear maps there are. Let us see what occurs when Φ is taken to be one of them.

- If L is linear then $Y \oplus_L X$ is isometric to the direct sum $Y \oplus_1 X$ under the isometry $(y, x) \mapsto (y - L(x), x)$. The induced extension splits, but beware: the projection $Y \oplus_L X \longrightarrow Y$ is not the obvious one $(y, x) \longmapsto y$, which turns out to be discontinuous unless L is bounded, in which case $\|(\cdot, \cdot)\|_L$ is even equivalent to the sum quasinorm. No, the correct projection is $P(y, x) = y - L(x)$, which is even contractive.
- If B is bounded then $\|(\cdot, \cdot)\|_B$ is merely equivalent to $\|(\cdot, \cdot)\|_1$. The sequence splits, and $(y, x) \longmapsto y$ is a bounded projection, although its norm depends on $\|B\|$.

If $\Phi = B + L$ is the sum of both then it takes a few seconds of pondering to realise that the induced exact sequence also splits through the retraction $(y, x) \longrightarrow y - L(x)$ or the section $x \longmapsto (L(x), x)$. More precisely:

Lemma 3.3.4 *Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map and let $L\colon X \longrightarrow Y$ be a linear map. Then $\Phi - L$ is bounded $\iff$ $P(y, x) = y - L(x)$ is bounded from $Y \oplus_\Phi X$ to Y $\iff$ $S(x) = (L(x), x)$ is bounded from X to $Y \oplus_\Phi X$. Moreover, $\|\Phi - L\| \leq \|P\| \leq \max(\Delta_Y, \|\Phi - L\|)$ and $\|S\| = 1 + \|\Phi - L\|$.*

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Proof It clearly suffices to check the part involving the bounds. As $\Phi(x) - L(x) = P(\Phi(x), x)$ and $\|(\Phi(x), x)\|_\Phi = \|x\|$, we have $\|\Phi - L\| \le \|P\|$. The other inequality follows from

$$\|y - L(x)\| \le \Delta_Y(\|y - \Phi(x)\| + \|\Phi(x) - L(x)\|) \le \Delta_Y(\|y - \Phi(x)\| + \|\Phi - L\|\,\|x\|).$$

The equality is just that $\|S(x)\| = \|(L(x), x)\|_\Phi = \|L(x) - \Phi(x)\| + \|x\|$. □

Non-triviality of the Kalton–Peck Maps

Here is a serious application of the splitting criterion:

Proposition 3.3.5 *Let X be a sequence space that is either*

(a) *a Banach space different from c_0, or*
(b) *a quasi-Banach space such that no subsequence of (e_n) is equivalent to the unit basis of c_0.*

Then $\mathsf{KP}_\varphi\colon X^0 \longrightarrow X$ is trivial if and only if φ is bounded on $\mathbb{R}^+$.

Proof The proof is different depending on whether one assumes (a) or (b). We begin with (a). Assume there is a linear map $\ell\colon X^0 \longrightarrow X$ for which $\mathsf{KP}_\varphi - \ell$ is bounded. The idea is to use the symmetries of X and KP_φ to replace ℓ by a bounded operator L with the same symmetries in order to conclude that KP_φ is bounded, which makes φ bounded. Consider the real unitary group $U = \{\pm 1\}^{\mathbb{N}}$ with the product topology and let m denote the Haar measure on U, which is actually a probability. Since X is a Banach space, every continuous function $f\colon U \longrightarrow X$ can be averaged using the Bochner integral $\int_U f(u)dm(u)$. The convexity of the norm yields the bound

$$\left\|\int_U f(u)dm(u)\right\| \le \int_U \|f(u)\|dm(u) \le \max_{u\in U} \|f(u)\|,$$

while the invariance of the Haar measure yields

$$\int_U f(u)dm(u) = \int_U f(uv)dm(u).$$

We will go ahead and use the hypothesised ℓ to define a new map $L\colon X^0 \longrightarrow X$,

$$L(x) = \int_U u^{-1}\ell(ux)dm(u).$$

The map L is clearly linear and correctly defined since for each $x \in X^0$ the orbit $\{ux\colon u \in U\}$ is finite, thus it spans a finite-dimensional space, on which the restriction of any linear map, as well as the map $u \in U \longmapsto u^{-1}\ell(ux) \in X$, must be continuous. Actually, if $x(k) = 0$ for $k > n$ then

$$L(x) = \int_U u^{-1}\ell(ux)dm(u) = \frac{1}{2^n}\sum\nolimits_{u\in U_n} u\ell(ux),$$

where $U_n = \{u\colon u(k) = 1 \text{ for all } k > n\}$. To estimate $\|\mathsf{KP}_\varphi - L\|$, pick $x \in X^0$ and observe that $\mathsf{KP}_\varphi(ux) = u\mathsf{KP}_\varphi(x)$ for every $u \in U$. Hence

$$\begin{aligned}
\|\mathsf{KP}_\varphi(x) - L(x)\| &= \left\|\int_U u^{-1}\mathsf{KP}_\varphi(ux)dm(u) - \int_U u^{-1}\ell(ux)dm(u)\right\| \\
&\le \int_U \|u^{-1}\mathsf{KP}_\varphi(ux) - u^{-1}\ell(ux)\|dm(u) \\
&\le \int_U \|\mathsf{KP}_\varphi(ux) - \ell(ux)\|dm(u) \\
&\le \int_U \|\mathsf{KP}_\varphi - \ell\|\|ux\|dm(u) \\
&= \|\mathsf{KP}_\varphi - \ell\|\|x\|,
\end{aligned}$$

which yields $\|\mathsf{KP}_\varphi - L\| \le \|\mathsf{KP}_\varphi - \ell\|$. Moreover, by the invariance of m, for each $v \in U$ we have

$$L(vx) = \int_U u\ell(uvx)dm(u) = v\int_U vu\ell(uvx)dm(u) = vL(x).$$

Writing $e_n = \frac{1}{2}(u + v)$ with $u, v \in U$ (say, $u = 1$ and $v = -1 + 2e_n$), we get

$$L(e_n) = L\left(\frac{u+v}{2}e_n\right) = \frac{u+v}{2}L(e_n) = e_nL(e_n).$$

Hence $L(e_n) = a_ne_n$ for some scalar a_n. Since $\mathsf{KP}_\varphi(e_n) = 0$ for all n, we have $|a_n| \le \|\mathsf{KP}_\varphi - \ell\|$, which means that the sequence $(a_n)_{n\ge1}$ is bounded. The map L is therefore bounded and thus KP_φ must also be bounded. Set $s_n = \sum_{1\le i\le n} e_i$. Since $\mathsf{KP}_\varphi(s_n) = \varphi(\log\|s_n\|)s_n$, we get that the sequence $\varphi(\log\|s_n\|)$ is bounded. But if $X \ne c_0$ then $\|s_n\| \to \infty$, and since $\|s_n\| \le \|s_{n+1}\| \le \|s_n\| + 1$, the Lipschitz condition implies that φ is bounded on $\mathbb{R}^+$, which proves the result under the assumption (a).

We now work under assumption (b). If X fails to be locally convex then we cannot reduce the complexity of the approximating linear map so easily, and it is necessary to fight for longer to arrive at the same point as before. So, before entering into the details, let us indicate how the hypotheses will be used to get the result:

(1) At a certain stage of the proof we will have to consider a subsequence $(e_{n(k)})$ of the basis, the differences $e_{n(1)} - e_{n(2)}, e_{n(3)} - e_{n(4)}, \dots$ and the quasinorms

$$\|e_{n(1)} - e_{n(2)} + \cdots + e_{n(2k-1)} - e_{n(2k)}\| = \left\|\sum_{i=1}^{2k}(-1)^{i+1}e_{n(i)}\right\| = \left\|\sum_{i=1}^{2k}e_{n(i)}\right\|.$$

The hypothesis (b) guarantees that these go to infinity as k increases (otherwise $(e_{n(k)})$ is equivalent to the unit basis of c_0).
(2) It is plain that any linear map $X^0 \longrightarrow Y$ is entirely defined by the sequence $(y_n) = (Le_n)$. If, moreover, Y is a quasi-Banach space and $\|y_n\| \leq 2^{-n}$ for all n, then L is bounded. Indeed, by the Aoki–Rolewicz theorem, we may assume that Y carries a p-norm and check that L actually extends to a bounded operator from ℓ_∞ to Y, since for $f \in \ell_\infty$ we have

$$\left\|\sum_n f(n)Le_n\right\| \leq \left(\sum_n |f(n)|^p \|y_n\|^p\right)^{1/p} \leq \|f\|_\infty \left(\sum_{n\geq 1} 2^{-pn}\right)^{1/p} = \left(\frac{2^{-p}}{1-2^{-p}}\right)^{1/p} \|f\|.$$

(3) The maps KP_φ do not increase supports: $\operatorname{supp} \mathsf{KP}_\varphi(x) \subset \operatorname{supp} x$; consequently, if $\mathsf{KP}_\varphi(x)$ is close to y then it is closer to $y1_{\operatorname{supp} x}$.

(4) If X is a sequence space, $A \subset \mathbb{N}$, and Y denotes the subspace of those sequences of X with support contained in A, then we can regard Y as a sequence space (which satisfies (b) if X does) and KP_φ maps Y^0 to Y. Moreover, if KP_φ is trivial (as a quasilinear map from X^0 to X) then so is the restriction $\mathsf{KP}_\varphi \colon Y^0 \longrightarrow Y$: indeed, if $\ell \colon X^0 \longrightarrow X$ is a linear map at finite distance from KP_φ, then since the projection $P \colon X \longrightarrow Y$ given by $P(x) = 1_A x$ is contractive, we have

$$\|(\mathsf{KP}_\varphi - P\,\ell) \colon Y^0 \to Y\| \leq \|(\mathsf{KP}_\varphi - \ell) \colon X^0 \to X\|.$$

Ok, a little less conversation and a little more action, please. Assume there is a linear map $\ell \colon X^0 \longrightarrow X$ such that $\mathsf{KP}_\varphi - \ell$ is bounded. The difficulty to overcome is that ℓ need not preserve disjointness, and thus we need to pass to a certain subsequence of the basis where ℓ behaves better. First of all, observe that $\mathsf{KP}_\varphi(e_n) = 0$ for all n, so

$$\|\ell(e_n)\|_\infty \leq \|\ell(e_n)\| = \|\mathsf{KP}_\varphi(e_n) - \ell(e_n)\| \leq \|\mathsf{KP}_\varphi - \ell\|.$$

Hence $(\ell(e_n))$ has a subsequence that converges coordinatewise to some element of ℓ_∞. By our last remark, we can replace X by the subspace spanned by that subsequence and assume that $\ell(e_n)$ is pointwise convergent to a bounded sequence, which implies that $(\ell(e_{2n-1} - e_{2n}))_{n\geq 1}$ is bounded in X and converges to zero in every coordinate. Next we ‘disjointify’ the values of ℓ on a subsequence (f_k) of $(e_{2n-1} - e_{2n})$ using a gliding hump argument as follows. We start with $f_1 = e_1 - e_2$ and look at $\ell(f_1)$ to choose $p(1) > 2$ such that

$$\|\ell(f_1) - 1_{[1,p(1))}\ell(f_1)\| \leq 1/2.$$

Since $\ell(e_{2n-1} - e_{2n})$ converges to zero in every coordinate, we can select $2k-1 \geq p(1)$ and $p > 2k$ such that

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$$\|\ell(e_{2k-1} - e_{2k}) - 1_{[p(1),p(2))}\ell(f_{k(2)})\| \le 2^{-2}.$$

We then set $f_2 = (e_{2k-1} - e_{2k})$ and continue this way, yielding two sequences of integers $(k(n))_{n\ge1}$ and $(p(n))_{n\ge0}$ such that

- $k(1) = p(0) = 1$ and $\operatorname{supp} f_n \subset [p(n-1), p(n))$ for $n \ge 1$,
- $\|\ell(f_n) - 1_{[p(n-1),p(n))}\ell(f_n)\| \le 2^{-n}$ for $n \ge 1$,

where $f_n = e_{2k(n)-1} - e_{2k(n)}$. Let F denote the linear subspace spanned by $(f_n)_{n\ge1}$ in X^0 and let $L\colon F \longrightarrow X$ be the linear map given by $L(f_n) = 1_{[p(n-1),p(n))}\ell(f_n)$. By the remark made in (2), we know that the difference $\ell - L$ is bounded (from F to X), and so is $\mathsf{KP}_\varphi - L$. Since the elements of the sequence (Lf_n) have mutually disjoint supports, the map $\tilde{\ell}\colon F \longrightarrow X$ given by $\tilde{\ell}(f) = 1_{\operatorname{supp} f}L(f)$ is linear. Note that the action of $\tilde{\ell}$ on the elements of the basis of F is

$$\tilde{\ell}(f_n) = |f_n|\ell(f_n) = \ell(f_n)(2k(n)-1)e_{2k(n)-1} + \ell(f_{k(n)})(2k(n))e_{2k(n)}.$$

By (3), we get that $\|\mathsf{KP}_\varphi - \tilde{\ell}\| \le \|\mathsf{KP}_\varphi - L\|$, so $(\tilde{\ell}(f_n))_{n\ge1}$ is bounded in X, and thus $\tilde{\ell}(f) = af$ for some $a \in \ell_\infty$ and every $f \in F$. Now set $\tilde{s}_n = \sum_{k\le n} f_k$. Then since $\mathsf{KP}_\varphi(\tilde{s}_n) = \varphi(\log\|\tilde{s}_n\|)\tilde{s}_n$ and

$$\|\mathsf{KP}_\varphi(\tilde{s}_n) - \tilde{\ell}(\tilde{s}_n)\| = \|\varphi(\log\|\tilde{s}_n\|)\tilde{s}_n - a\tilde{s}_n\| \le M\|\tilde{s}_n\|,$$

we see that the numerical sequence $\varphi(\log\|\tilde{s}_n\|)$ is bounded and that φ is bounded on $\mathbb{R}^+$, by (1). □

From the observation $\mathsf{KP}_\varphi - \mathsf{KP}_\gamma = \mathsf{KP}_{\varphi-\gamma}$ we immediately have:

Corollary 3.3.6 *Under the same hypotheses on X as Proposition* 3.3.5*, let $\varphi, \gamma \in \mathrm{Lip}_0(\mathbb{R}^+)$. Then KP_φ and KP_γ are equivalent on X if and only if* $\sup_{t>0}|\varphi(t) - \gamma(t)| < \infty$.

While Proposition 3.3.5 shows that the Kalton–Peck maps produce non-trivial sequences $0 \longrightarrow X \longrightarrow X(\varphi) \longrightarrow X \longrightarrow 0$ for most quasi-Banach sequence spaces X and unbounded φ, the conclusion fails when $X = c_0$ for the trivial reason that all Kalton–Peck maps are bounded on c_0, which follows easily from the fact that $|t\log t|$ is bounded for $0 \le t \le 1$. Sobczyk's theorem gives a deeper explanation of why one cannot expect to see a non-trivial self-extension of c_0. However, the question remains whether there exist exact sequences $0 \longrightarrow c_0 \longrightarrow \cdot \longrightarrow c_0 \longrightarrow 0$ in which the middle space is not locally convex. There do not, but this is a *really* deep result of Kalton and Roberts stated later in 3.4.6. It has a surprisingly large number of connections, ramifications and applications through the Maharam problem on exhaustive submeasures, which go far beyond the scope of this volume. The hungry reader can find nutritious information in [257] and [181].

Existence of Quasilinear Maps

It is implicit in the discussion in Section 3.1 that every short exact sequence of quasi-Banach spaces arises from and gives rise to a quasilinear map. Let us make this statement precise:

Proposition 3.3.7 *Every exact sequence of quasi-Banach spaces is generated, up to equivalence, by a quasilinear map. More precisely, for every exact sequence* $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$*, there is a quasilinear map* $\Phi\colon X \longrightarrow Y$ *and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, equal] & Y \oplus_\Phi X \arrow[r, "\pi"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\jmath"] & Z \arrow[r, "\rho"] & X \arrow[r] & 0 \end{tikzcd} \tag{3.12}$$

Proof Let $B\colon X \longrightarrow Z$ be a homogeneous bounded section of ρ, and let $L\colon X \longrightarrow Z$ be a linear (possibly discontinuous) section. The difference $B - L$ takes values in $\ker\rho = \jmath[Y]$ since $\rho(B(x) - L(x)) = x - x = 0$, and composing with the inverse of $\jmath$, we get the map $\Phi = \jmath^{-1} \circ (B - L)$ from X to Y. The quasilinear character of Φ is obvious: $B - L$ has the same Cauchy differences as B and $\jmath^{-1}$ is bounded. Anyway, let us estimate $Q(\Phi)$. Pick $x, y \in X$. If $\Delta_X, \Delta_Y, \Delta_Z$ denote the respective concavity constants then

$$\begin{aligned} \|\Phi(x+y) - \Phi(x) - \Phi(y)\| &\le \|\jmath^{-1}\|\, \|(B-L)(x+y) - (B-L)(x) - (B-L)(y)\| \\ &= \|\jmath^{-1}\|\, \|B(x+y) - B(x) - B(y)\| \\ &\le \|\jmath^{-1}\|(\Delta_Z\|B(x+y)\| + \Delta_Z^2(\|B(x)\| + \|B(y)\|)) \\ &\le \|\jmath^{-1}\|\, \|B\|\, (\Delta_Z\Delta_X + \Delta_Z^2)\, (\|x\| + \|y\|). \end{aligned}$$

The linear map $u(y, x) = \jmath(y) + L(x)$ makes Diagram (3.12) commutative, as can be easily seen. We conclude the proof by showing that it is bounded:

$$\begin{aligned} \|u(y,x)\| &= \|\jmath(y) - \jmath(\Phi(x)) + \jmath(\Phi(x)) + L(x)\| \\ &\le \Delta_Z(\|\jmath(y) - \jmath(\Phi(x))\| + \|\jmath\Phi(x) + L(x)\|) \\ &= \Delta_Z(\|\jmath(y - \Phi(x))\| + \|B(x)\|) \\ &\le \Delta_Z(\|\jmath\|\, \|y - \Phi(x)\| + \|B\|\, \|x\|) \\ &\le \Delta_Z \max(\|\jmath\|, \|B\|) \cdot \|(y,x)\|_\Phi. \end{aligned}$$ □

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We know from Roelcke's lemma that the map u is an isomorphism. A direct computation shows that its inverse is given by $u^{-1}(z) = (\jmath^{-1}(z - L(\rho(z)), \rho(z))$ and satisfies

$$\begin{aligned}\|u^{-1}(z)\|_\Phi &= \|\jmath^{-1}(z - L(\rho(z)) - \Phi(\rho(z))\| + \|\rho z\| \\ &= \|\jmath^{-1}(z - L(\rho(z)) - \jmath^{-1}(B - L)(\rho(z))\| + \|\rho z\| \\ &= \|\jmath^{-1}(z - B(\rho(z))\| + \|\rho z\| \\ &\leq \left(\|\jmath^{-1}\|\Delta_Z(1 + \|B\|\,\|\rho\|) + \|\rho\|\right)\|z\|.\end{aligned}$$

Admittedly, the preceding computations are a bit nitpicky. In their defense, they provide very thin bounds for the Banach–Mazur distance between the space Z and the twisted sum $Y \oplus_\Phi X$:

Corollary 3.3.8 *If $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is an isometrically exact sequence of p-Banach spaces, then for every $\varepsilon > 0$, there is a quasilinear map $\Phi\colon X \longrightarrow Y$ with $Q(\Phi) < 2^{2/p-1} + \varepsilon$ and a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus_\Phi X \arrow[r] \arrow[d, "u"'] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \tag{3.13}$$

where u is an isomorphism such that $\|u\| < 2^{1/p-1} + \varepsilon$ and $\|u^{-1}\| < 2^{1/p} + \varepsilon$. In particular, the Banach–Mazur distance between Z and the twisted sum space $Y \oplus_\Phi Z$ is at most $2^{2/p-1}$.

Proof Just follow the proof of Proposition 3.3.7, taking into account that in the isometric setting we have $\|\jmath\| = \|\jmath^{-1}\| = \|\rho\| = 1$, that for every $\varepsilon > 0$ the bounded selection $B\colon X \longrightarrow Z$ can be chosen such that $\|B\| < 1 + \varepsilon$, and that the concavity of any p-normed space is at most $2^{1/p-1}$. □

In general, almost everything involving quasinormed spaces depends on the modulus of concavity of the quasinorm. The quasilinearity constant is not an exception.

Lemma 3.3.9 *Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map acting between quasinormed spaces. Then $Q(\Phi) - 1 \leq \Delta_{Y\oplus_\Phi X} \leq \max\left(\Delta_Y^2, Q(\Phi)\Delta_Y + \Delta_X\right)$.*

Proof Indeed, if $\|(y, x) + (y', x')\|_\Phi \leq \Delta\,(\|(y, x)\|_\Phi + \|(y', x')\|_\Phi)$ for some Δ then, setting $y = \Phi(x), y' = \Phi(x')$, we obtain

$$\|\Phi(x + x') - \Phi(x) - \Phi(x')\| + \|x + x'\| \leq (\Delta - 1)(\|x\| + \|x'\|).$$

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In particular, $Q(\Phi) \leq \Delta - 1$. Conversely,

$$\begin{aligned}
&\|(y,x) + (y',x')\|_\Phi \\
&= \|y + y' - \Phi(x+x')\| + \|x + x'\| \\
&\leq \Delta_Y \left(\|y + y' - \Phi x - \Phi x'\| + \|\Phi x + \Phi x' - \Phi(x+x')\|\right) + \Delta_X \left(\|x\| + \|x'\|\right) \\
&\leq \Delta_Y^2 \left(\|y_1 - \Phi(x)\| + \|y_2 - \Phi(x')\|\right) + (Q(\Phi)\Delta_Y + \Delta_X)(\|x\| + \|x'\|) \\
&\leq \max\left(\Delta_Y^2, Q(\Phi)\Delta_Y + \Delta_X\right) \cdot \left(\|(y,x)\|_\Phi + \|(y',x')\|_\Phi\right). \qquad \square
\end{aligned}$$

Extension of Quasilinear Maps from Dense Subspaces

And so we arrive to the core of the non-linear approach. While the equivalence between exact sequences and quasilinear maps explained in Proposition 3.3.7 is in a sense the final word concerning the theory of quasilinear maps, in practice there is still a loose end to be tied: namely, the question of how to obtain such quasilinear maps, since, as we remarked in Section 3.2, quasilinear maps $X \longrightarrow Y$ usually only come defined on a dense subspace of X. This is why Lemma 3.3.2 took the form it did. The introduction to Section 3.2 also claimed that 'fortunately, this blow is not fatal,' and concluded with an obscure reference to a certain 'quasilinear map $\widetilde{\Phi}$ extending Φ'. Time has come to put all this in order and show exactly how things must be done. Let us assume that X and Y are quasi-Banach spaces and that we are given a quasilinear map $\Phi\colon X_0 \longrightarrow Y$, where X_0 is a dense subspace of X. Recall Diagram (3.3),

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, equal] & Y \oplus_\Phi X_0 \arrow[r, "\pi"] \arrow[d, "\kappa"] & X_0 \arrow[r] \arrow[d, "\text{inclusion}"] & 0 \\
0 \arrow[r] & Y \arrow[r, "\imath"] & Z(\Phi) \arrow[r, "\widehat{\pi}"] & X \arrow[r] & 0
\end{tikzcd}$$

and call Proposition 3.3.7 onstage so that the lower sequence can be represented by a quasilinear map $\widetilde{\Phi}\colon X \longrightarrow Y$. It is already clear that the restriction of $\widetilde{\Phi}$ to X_0 is equivalent to Φ. The point, however, is that $\widetilde{\Phi}$ can be chosen to be a true extension of Φ, and this information is contained in the proof of Proposition 3.3.7: all maps that are equivalent to $\widetilde{\Phi}$ are obtained as the difference of a bounded homogeneous section $B\colon X \longrightarrow Z(\Phi)$ and a linear section $L\colon X \longrightarrow Z(\Phi)$ for $\widehat{\pi}$. So pick B as a homogeneous bounded true extension of $x_0 \longmapsto \kappa(\Phi(x_0), x_0)$ and take for L a true linear extension of $x_0 \longmapsto \kappa(0, x_0)$. Now, yes, $\imath^{-1} \circ (B - L)$ is our true quasilinear extension $\widetilde{\Phi}$ of Φ. Die-hard sceptics can have great fun checking the commutativity of the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & Y \arrow[r, "\iota"] \arrow[dl, equal] \arrow[dd, equal] & & Y \oplus_\Phi X_0 \arrow[rr, "\pi"] \arrow[dl] \arrow[dd] & & X_0 \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \\
0 \arrow[r] & Y \arrow[rr] \arrow[dr, equal] & & Y \oplus_{\widetilde{\Phi}} X \arrow[rr] & & X \arrow[rr] \arrow[dr, equal] & & 0 & \\
 & 0 \arrow[r] & Y \arrow[rr] & & Z(\Phi) \arrow[rr] \arrow[ul] & & X \arrow[rr] & & 0
\end{tikzcd}$$

A direct consequence is:

Corollary 3.3.10 *A quasilinear map is trivial if and only if its restriction to some (any) dense subspace is trivial.*

Proof It is clear that the restriction of a trivial quasilinear map to any subspace, dense or not, is again trivial. To prove the converse, let $\Phi\colon X \longrightarrow Y$ be quasilinear and let X_0 be a dense subspace of X. If $\Phi|_{X_0}$ is trivial then the lower quotient map in the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Y \oplus_\Phi X \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Y \oplus_\Phi X_0 \arrow[r] \arrow[u, "\kappa"] & X_0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

has a linear continuous section $s\colon X_0 \longrightarrow Y \oplus_\Phi X_0$. Any linear continuous extension of κs to X shows that the upper sequence splits. $\square$

A quantisation is always welcome:

Proposition 3.3.11 *For every $\Delta \geq 1$, there is a constant C such that if X and Y are quasi-Banach spaces with moduli of concavity Δ or less and X_0 is a dense subspace of X then every quasilinear map $\Phi\colon X_0 \longrightarrow Y$ has an extension $\widetilde{\Phi}\colon X \longrightarrow Y$ with $Q(\widetilde{\Phi}) \leq CQ(\Phi)$.*

Proof We need a simple amalgamation argument, once we know that individual maps extend from dense subspaces. Let $(X^i)_{i\in I}$ and $(Y^i)_{i\in I}$ be families of quasi-Banach spaces whose moduli of concavity are at most Δ. For each $i \in I$, let X_0^i be a dense subspace of X^i and let $\Phi_i\colon X_0^i \longrightarrow Y^i$ be a quasilinear map with $Q(\Phi_i) \leq 1$. Their c_0-sums $c_0(I, X^i)$ and $c_0(I, Y^i)$ are quasi-Banach spaces with moduli Δ. Let $c_0^0(I, X_0^i)$ be the dense subspace of finitely supported sequences of $c_0(I, X^i)$. The map $\Phi\colon c_0^0(I, X_0^i) \longrightarrow c_0(I, Y^i)$ defined by $\Phi((x_i)_{i\in I}) = (\Phi(x_i))_{i\in I}$ is quasilinear with $Q(\Phi) \leq 1$ and can be extended to a quasilinear map $\widetilde{\Phi}\colon c_0(I, X^i) \longrightarrow c_0(I, Y^i)$. If $\jmath_i\colon X_i \longrightarrow c_0(I, X^i)$ and $\pi_i\colon c_0(I, Y^i) \longrightarrow Y_i$ denote the obvious embedding and projection then $\pi_i \circ \widetilde{\Phi} \circ \jmath_i$ is an extension of Φ_i to X^i with quasilinearity constant at most $Q(\widetilde{\Phi})$. $\square$

https://doi.org/10.1017/9781108778312.005 Published online by Cambridge University Press

3.4 Local Convexity of Twisted Sums and $\mathscr{K}$-Spaces

We can no longer avoid the hard truth that no result proved so far guarantees that any of the twisted sums constructed in this chapter are Banach spaces. Quite the contrary, both the Ribe space and the Kalton–Peck spaces $\ell_1(\varphi)$ clearly show that twisted sums of Banach spaces can be non-locally convex. Thus, the topic that must be considered is the 3-space problem for local convexity; when we undertake this study, we encounter the following notions at the center of it all:

Definition 3.4.1 A minimal extension of X is a short exact sequence of the form $0 \longrightarrow \mathbb{K} \longrightarrow Z \longrightarrow X \longrightarrow 0$. A quasi-Banach space X is said to be a $\mathscr{K}$-space if every minimal extension is trivial, i.e. $\operatorname{Ext}(X, \mathbb{K}) = 0$.

Non-trivial minimal extensions appear whenever the Hahn–Banach theorem fails: if Z has trivial dual then every non-zero point $z \in Z$ gives rise to a minimal extension of $Z/[z]$. A minimal extension $0 \longrightarrow \mathbb{K} \longrightarrow Z \longrightarrow X \longrightarrow 0$ of a Banach space X splits if and only if the twisted sum space Z is locally convex; that is, isomorphic to a Banach space. Indeed, if Z is a Banach space, the Hahn–Banach theorem applied to $\mathbf{1}_{\mathbb{K}}$ yields a projection of Z onto $\mathbb{K}$. And, conversely, if the sequence splits then Z is isomorphic to $\mathbb{K} \times X$, which is locally convex.

Definition 3.4.2 A subspace Y of a quasi-Banach space X is said to have the Hahn–Banach extension property (HBEP) if each linear continuous functional on Y can be extended to a linear continuous functional on X.

Kalton proved in [246] that a quasi-Banach space all of whose subspaces have the HBEP must be locally convex. More modestly one has:

Lemma 3.4.3 *Let* $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ *be a short exact sequence in which X and Y are Banach spaces. Then Z is a Banach space if and only if Y has the HBEP in Z.*

Proof The 'only if' part is straightforward from the Hahn–Banach extension theorem. For the converse, observe that a quasi-Banach space A is (isomorphic to) a Banach space if and only if the natural evaluation map $\delta_A : A \longrightarrow A^{**}$ is an isomorphic embedding. Now, if Y has the HBEP in Z then the sequence $0 \longrightarrow X^* \longrightarrow Z^* \longrightarrow Y^* \longrightarrow 0$ is exact, and so is the bidual sequence $0 \longrightarrow Y^{**} \longrightarrow Z^{**} \longrightarrow X^{**} \longrightarrow 0$. Thus, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\delta_Y"] & Z \arrow[r] \arrow[d, "\delta_Z"] & X \arrow[r] \arrow[d, "\delta_X"] & 0 \\
0 \arrow[r] & Y^{**} \arrow[r] & Z^{**} \arrow[r] & X^{**} \arrow[r] & 0
\end{tikzcd}$$

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in which the vertical arrows are the natural evaluation maps. Since δ_Y and δ_X are isomorphic embeddings, so is δ_Z by Roelcke's lemma 2.1.8. $\square$

Thus, whenever one has an exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ in which both Y and X are Banach spaces and Z is not, there exists $y^* \in Y^*$ that does not extend to a bounded linear functional on Z; this means that the lower row in the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "y^*"'] & Z \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[r] & \mathrm{PO} \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

does not split. Hence, any counterexample for the 3-space problem for local convexity leads to a non-trivial minimal extension of the quotient space. Such is the context of the following classical result of Dierolf [149]:

3.4.4 Dierolf's Theorem *A Banach space X is a $\mathscr{K}$-space if and only if every extension of X by a Banach space is locally convex.*

Is there some obvious example of $\mathscr{K}$-space in sight? No, because proving that a given Banach or quasi-Banach space is a $\mathscr{K}$-space requires some work ... or, rather, a lot of work. Actually, the only large class of Banach $\mathscr{K}$-spaces we can isolate at this moment is:

Proposition 3.4.5 *Superreflexive Banach spaces are $\mathscr{K}$-spaces.*

Proof Let $\phi\colon X \longrightarrow \mathbb{K}$ be a quasilinear funcional on a Banach space X. For each finite-dimensional subspace $E \in \mathscr{F}(X)$, set $d_E = \operatorname{dist}(\phi|_E, E^*)$. Since these infima are attained, one can pick $L_E \in E^*$ such that $\|\phi|_E - L_E\| = d_E$. Now, if $\sup_E d_E < \infty$, pick $\mathfrak{U}$ an ultrafilter on $\mathscr{F}(X)$ refining the order filter and define $L\colon X \longrightarrow \mathbb{K}$ by $L(x) = \lim_{\mathfrak{U}(E)} L_E(x)$, which makes sense, since for each $x \in X$, the family $(L_E(x))_E$ is bounded because $|L_E(x)| \leq |\phi|_E(x)| + d_E$. The map L is clearly linear and $\|\Phi - L\| \leq \sup_E d_E$. If, on the contrary, $\sup_E d_E = \infty$, consider the function $f_E\colon E \longrightarrow \mathbb{K}$ given by

$$f_E(x) = \frac{\phi(x) - L_E(x)}{d_E}$$

(set $f_E = 0$ in the innocuous case $d_E = 0$). These maps are all bounded, with $\|f_E\| \leq 1$. If $\mathfrak{U}$ is once again an ultrafilter refining the order filter on $\mathscr{F}(X)$ then $\sup_E d_E = \infty$ necessarily implies $\lim_{\mathfrak{U}(E)} Q(f_E) = \lim_{\mathfrak{U}(E)} d_E^{-1} Q(\phi) = 0$. Form the ultraproduct $\mathscr{F}(X)_{\mathfrak{U}}$ and tentatively define a mapping $f\colon \mathscr{F}(X)_{\mathfrak{U}} \longrightarrow \mathbb{K}$ by the formula $f[(x_E)] = \lim_{\mathfrak{U}(E)} f_E(x_E)$.

Claim f is a bounded linear functional on $\mathscr{F}(X)_{\mathfrak{U}}$.

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Proof of the claim We must check that f is correctly defined. First assume $[(x_E)] = 0$, that is, $\|x_E\| \longrightarrow 0$ along $\mathfrak{U}$. Then,

$$\Big|\lim_{\mathfrak{U}(E)} f_E(x_E)\Big| = \lim_{\mathfrak{U}(E)} |f_E(x_E)| \le \lim_{\mathfrak{U}(E)} \|f_E\|\,\|x_E\| = 0.$$

Now, if $[(x_E)] = [(y_E)]$ then

$$\begin{aligned}\lim_{\mathfrak{U}(E)} \big|f_E(x_E) - f_E(y_E)\big| &= \lim_{\mathfrak{U}(E)} \big|f_E(x_E) - f_E(y_E) - f_E(x_E - y_E)\big| \\ &\le \lim_{\mathfrak{U}(E)} Q(f_E)(\|y_E\| + \|x_E - y_E\|) = 0.\end{aligned}$$

The map f is obviously bounded, and it is linear since it is homogeneous, and given bounded families (x_E) and (y_E), we have

$$\begin{aligned}\big|f([(x_E + y_E)]) - f[(x_E)] - f[(y_E)]\big| &= \lim_{\mathfrak{U}(E)} |f_E(x_E + y_E) - f_E(x_E) - f_E(y_E)| \\ &\le \lim_{\mathfrak{U}(E)} Q(f_E)(\|x_E\| + \|y_E\|) = 0.\end{aligned}$$ □

The hypothesis on X enters now: if X is superreflexive, the ultraproduct $\mathscr{F}(X)_{\mathfrak{U}}$ is reflexive and its dual agrees with the ultraproduct of the dual family $(E^*)_{E\in\mathscr{F}(X)}$ with respect to $\mathfrak{U}$. Thus, $f \in (\mathscr{F}(X)_{\mathfrak{U}})^*$ is represented by a bounded family of functionals $g_E \in E^*$ in the form

$$f[(x_E)] = \lim_{\mathfrak{U}(E)} f_E(x_E) = \lim_{\mathfrak{U}(E)} \langle g_E, x_E\rangle$$

for every bounded family (x_E). This clearly implies $\lim_{\mathfrak{U}(E)} \|f_E - g_E\| = 0$ and thus the set $\{E \in \mathscr{F}(X) \colon \|f_E - g_E\| < \frac{1}{2}$ and $d_E > 0\}$ belongs to $\mathfrak{U}$ and cannot be empty. Pick some element E in there to get

$$\Big\|\frac{\phi|_E - L_E}{d_E} - g_E\Big\| < \frac{1}{2} \qquad \Longrightarrow \qquad \big\|\phi|_E - L_E - d_E g_E\big\| < \frac{d_E}{2},$$

in clear contradiction of the definition of d_E. □

The wonderful consequence we get is that all twisted sums $X(\varphi)$ are isomorphic to Banach spaces when X is a superreflexive sequence space, in particular if $X = \ell_p$ for $p \in (1, \infty)$. In Proposition 3.11.3, it will be proved that B-convex spaces are also $\mathscr{K}$-spaces. We can now record the *really deep* result of Kalton and Roberts [285, Theorem 6.3] mentioned previously.

3.4.6 Kalton–Roberts theorem *$\mathscr{L}_\infty$-spaces are $\mathscr{K}$-spaces.*

And what about non-$\mathscr{K}$-spaces? Producing non-$\mathscr{K}$-spaces is much easier when working with quasi-Banach spaces because, as we already said, the quotient of any space with trivial dual by a line fails to be a $\mathscr{K}$-space. In a sense, this is the only way there is to not be a $\mathscr{K}$-space:

Proposition 3.4.7 *Let X be quasi-Banach space. The following are equivalent:*

(i) *X is a $\mathscr{K}$-space.*
(ii) *Whenever $Q\colon Z \longrightarrow X$ is a quotient map,* $\ker Q$ *has the HBEP in Z.*
(iii) *Whenever F is a finite-dimensional subspace of a quasi-Banach space Z, every operator $X \longrightarrow Z/F$ lifts to Z.*

Proof To prove (i) $\implies$ (ii), consider a quotient map $Q\colon Z \longrightarrow X$ and a bounded linear functional f on $\ker Q$ and form the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker Q \arrow[r] \arrow[d, "f"'] & Z \arrow[r, "Q"] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[r] & \mathrm{PO} \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

If X is a $\mathscr{K}$-space, the lower sequence splits and f extends to a bounded linear functional on Z; i.e. $\ker Q$ has the HBEP in Z.

(ii) $\implies$ (iii) Let F be a finite-dimensional subspace of Z and $u\colon X \longrightarrow Z/F$ an operator. By (ii), F must have the HBEP in PB. Since finite-dimensional subspaces with the HBEP are complemented (just extend the coordinate functionals of a basis), the lower sequence in the diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] & Z \arrow[r] & Z/F \arrow[r] & 0 \\ 0 \arrow[r] & F \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "u"'] & 0 \end{tikzcd}$$

splits, and by the splitting criterion for pullbacks, u has a lifting to Z, which proves (iii). The implication (iii) $\implies$ (i) is trivial: every exact sequence $0 \longrightarrow \mathbb{K} \longrightarrow Z \longrightarrow X \longrightarrow 0$ splits since the hypothesis allows the identity on X to be lifted to an operator $X \longrightarrow Z$. □

Proposition 3.4.8 *Let Z be a quasi-Banach space and let $Q\colon Z \longrightarrow X$ be a quotient map. If Z is a $\mathscr{K}$-space and* $\ker Q$ *has the HBEP in Z then X is a $\mathscr{K}$-space. In particular, quotients of Banach $\mathscr{K}$-spaces are $\mathscr{K}$-spaces.*

Proof Consider a minimal extension of X and form the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{K} \arrow[r] & \cdot \arrow[r, "\rho"] & X \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u, "Q"'] & 0 \end{tikzcd}$$

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Since Z is a $\mathcal{K}$-space the lower sequence splits, and so Q lifts to a map $\widetilde{Q}\colon Z \longrightarrow \cdot$ such that $Q = \rho\,\widetilde{Q}$. There is therefore a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K} \arrow[r] & \cdot \arrow[r, "\rho"] & X \arrow[r] & 0 \\
0 \arrow[r] & \ker Q \arrow[r] \arrow[u, "\widetilde{Q}|_{\ker Q}"] & Z \arrow[r, "Q"] \arrow[u, "\widetilde{Q}"'] & X \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Since $\ker Q$ has the HBEP in Z, the functional $\widetilde{Q}|_{\ker Q}$ extends to Z and the upper (pushout) sequence splits. □

We are ready to conclude the analysis of the 3-space problem for local convexity, which we initiated with Dierolf's theorem 3.4.4, by explaining, as promised, why counterexamples for the locally convex 3-space problem were obtained as minimal extensions of ℓ_1. If X is a Banach space and some non-trivial sequence

$$0 \longrightarrow \mathbb{K} \xrightarrow{\ \jmath\ } Z \longrightarrow X \longrightarrow 0 \tag{3.14}$$

exists then Z is non-locally convex. Pick a non-locally convex separable subspace $\widetilde{Z}$ of Z containing $\jmath[\mathbb{K}]$ and form the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K} \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[u, equal] & \widetilde{Z} \arrow[r] \arrow[u] & \widetilde{X} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The lower sequence of the diagram does not split, because $\widetilde{Z}$ is non-locally convex, and $\widetilde{X} = \widetilde{Z}/\jmath[\mathbb{K}]$ is isomorphic to a separable subspace of X. Pick any quotient map $Q\colon \ell_1 \longrightarrow \tilde{X}$ and form the commutative pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K} \arrow[r] & \widetilde{Z} \arrow[r] & \widetilde{X} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{K} \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & \ell_1 \arrow[r] \arrow[u, "Q"'] & 0
\end{tikzcd}$$

The space PB cannot be locally convex since the map $\mathrm{PB} \longrightarrow \widetilde{Z}$ is surjective. Thus, any exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ in which X and Y, but not Z, are Banach spaces leads, after a pushout and pullback, to a minimal extension of ℓ_1.

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3.5 The Pullback and Pushout in Quasilinear Terms

As is only natural in a book with the word *homological* in the title, we now study the functorial properties of the assignment $(X, Y) \rightsquigarrow \mathsf{Q}(X, Y)$. In practice, this means studying how quasilinear maps and operators compose:

3.5.1 *If* $\Phi\colon X \longrightarrow Y$ *is quasilinear and* $T : Y \longrightarrow Y'$ *and* $S : X' \longrightarrow X$ *are operators, then* $T \circ \Phi \circ S : X' \longrightarrow Y'$ *is quasilinear with* $Q(T \circ \Phi \circ S) \leq \|T\| Q(\Phi) \|S\|$.

Can we identify the exact sequences induced by $T \circ \Phi$ and $\Phi \circ S$? Of course we can: they are, respectively, the pushout and the pullback sequences. To check this, simply contemplate the commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "T"'] & Y \oplus_\Phi X \arrow[r] \arrow[d, "T\times\mathbf{1}_X"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Y' \oplus_{T\Phi} X \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \tag{3.15}$$

where $(T \times \mathbf{1}_X)(y, x) = (Ty, x)$ and

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & Y \oplus_\Phi X \arrow[r] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Y \oplus_{\Phi S} X \arrow[r] \arrow[u, "\mathbf{1}_Y\times S"'] & X' \arrow[r] \arrow[u, "S"'] & 0 \end{tikzcd}$$

where $(\mathbf{1}_X \times S)(y, x') = (y, Sx')$. The operators in the middle are clearly bounded:

$$\|(Ty, x)\|_{T\Phi} = \|Ty - T\Phi(x)\| + \|x\| \leq \max(\|T\|, 1)\|(y, x)\|_\Phi,$$
$$\|(y, Sx')\|_\Phi = \|y - \Phi S x'\| + \|Sx'\| \leq \max(1, \|S\|)\|(y, x')\|_{\Phi S}.$$

Thus, the quasilinear representation of the pushout and pullback constructions is much simpler than the original: plain left and right composition! And, better yet, the same will happen for the rest of homological constructions studied in Chapter 2 and for others that will be introduced in Chapter 4. Let us give the suspicious reader a taste: the fact that pushout and pullback operations commute required a proof in Chapter 2, but it is completely obvious now – in both cases, we arrive at the exact sequence generated by $T \circ \Phi \circ S$ (if we are being pedantic, in the first case, we arrive at $(T \circ \Phi) \circ S$ and in the second, at $T \circ (\Phi \circ S)$). Another more formal example: remember the diagonal sequence in a pushout diagram? Assertion 2.10.2 reformulates now as:

3.5.2 *The diagonal pushout sequence is induced by* $\Phi \circ \overline{\rho}$.

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The proof consists of observing that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "\jmath"] \arrow[d, equal] & Z(\Phi) \arrow[r, "\rho"] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r, "(-\tau, \bar{\jmath})"] & Y' \oplus Z \arrow[r, "\mathbf{1}_{Y'} \oplus \tau"] \arrow[u, "T"] & \mathrm{PO} \arrow[r] \arrow[u, "\bar{\rho}"'] & 0
\end{tikzcd}$$

simply putting $T(y', z) = z$. Analogously, assertion 2.10.1 becomes

3.5.3 *The diagonal pullback sequence is induced by $\underline{\jmath} \circ \Phi$.*

The proof consists of observing that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "\jmath"] \arrow[d, "\underline{\jmath}"] & Z(\Phi) \arrow[r, "\rho"] \arrow[d, "T"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{PB} \arrow[r, "(\underline{\tau}, \underline{\rho})"] & Z \oplus X' \arrow[r, "\rho \oplus (-\tau)"] & X \arrow[r] & 0
\end{tikzcd}$$

where $T(z) = (z, 0)$.

We conclude quantifying the knot between triviality, extension and lifting: its proof clearly follows from Lemma 3.3.4, taking into account that for $\|T\| \leq 1$, the map $T \times \mathbf{1}_X$ in Diagram (3.15) is contractive; and similarly with S.

Lemma 3.5.4 *Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map, $T\colon Y \longrightarrow Y'$ and $S\colon X' \longrightarrow X$ contractive operators and $\mu \geq 0$.*

- *If $T \circ \Phi$ is μ-trivial, T has an extension to $Y \oplus_\Phi X$ bounded by* $\max(\Delta_{Y'}, \mu)$. *If T has a μ-extension then $T \circ \Phi$ is μ-trivial.*
- *$\Phi \circ S$ is μ-trivial if and only if S has a lifting $L\colon X' \longrightarrow Y \oplus_\Phi X$ such that $\|L\| \leq 1 + \mu$.*

3.6 Spaces of Quasilinear Maps

Spaces of quasilinear maps deserve to be studied on their own, and that is what we will do now. Given a pair of quasinormed spaces X, Y, we can consider the following vector spaces:

- the space of quasilinear maps $\mathsf{Q}(X, Y)$,
- the space of bounded homogeneous maps $\mathsf{B}(X, Y)$,
- the space of linear maps $\mathsf{L}(X, Y)$.

When the spaces X, Y are fixed and there is no possibility of confusion, the spaces will be omitted and we will just write Q, L and B. Recalling from Lemma 3.3.2 that two quasilinear maps acting between the same spaces

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induce equivalent exact sequences if and only if their difference belongs to $\mathsf{B}(X,Y)+\mathsf{L}(X,Y)$, we are especially interested in the quotient space

$$\mathsf{Q}_{\mathsf{LB}}(X,Y)=\frac{\mathsf{Q}(X,Y)}{\mathsf{L}(X,Y)+\mathsf{B}(X,Y)},$$

which appeared, without a name, inside Definition 3.3.3 and Proposition 3.3.7. If we denote the class of Φ in the quotient space $\mathsf{Q}_{\mathsf{LB}}(X,Y)$ by $[\Phi]$ then the quasilinearity constant naturally induces the semi-quasinorm

$$Q[\Phi]=\inf\{Q(\Phi+L+B): L\in\mathsf{L}, B\in\mathsf{B}\}$$

so that 3.5.1 can be upgraded to:

Proposition 3.6.1 *If X, X', Y, Y' are quasinormed spaces then the mapping $\mathfrak{L}(X',X)\times\mathsf{Q}_{\mathsf{LB}}(X,Y)\times\mathfrak{L}(Y,Y')\longrightarrow\mathsf{Q}_{\mathsf{LB}}(X',Y')$ given by $(u,[\Phi],v)\mapsto[v\circ\Phi\circ u]$ is well defined, trilinear and contractive.*

Proof The map is well defined: if $[\Phi]=[\Psi]$ then $v\circ\Phi\circ u-v\circ\Psi\circ u=v\circ(\Phi-\Psi)\circ u$ is in $\mathsf{B}+\mathsf{L}$. The only other not entirely obvious point is that $[\Phi\circ(u+u')]=[\Phi\circ u+\Phi\circ u']$, which follows from the quasilinearity of Φ since $\Phi\circ(u+u')-\Phi\circ u-\Phi\circ u'$ is bounded from X' to Y. □

Completeness

We now consider in some detail $\mathsf{Q}(X,Y)$ as a topological space under the semi-quasinorm defined by the quasilinearity constant. Readers who have not yet looked at Note 1.8.1 are advised to do so. Observe that when Y is a p-normed space, the quasilinearity constant is p-subadditive: if $\Phi,\Psi\in\mathsf{Q}(X,Y)$ then $Q(\Phi+\Psi)^p\le Q(\Phi)^p+Q(\Psi)^p$, as clearly follows from the formula

$$Q(\Phi)=\sup_{x,y\in X}\frac{\|\Phi(x+y)-\Phi(x)-\Phi(y)\|}{\|x\|+\|y\|}.$$

It is obvious that $Q(\Phi)=0$ if and only if Φ is linear. Hence the main properties of $(\mathsf{Q}(X,Y),Q(\cdot))$ depend only on the Hausdorff quotient

$$\mathsf{Q}_{\mathsf{L}}(X,Y)=\frac{\mathsf{Q}(X,Y)}{\ker Q(\cdot)}=\frac{\mathsf{Q}(X,Y)}{\mathsf{L}(X,Y)},$$

where $Q(\cdot)$ becomes a genuine quasinorm since $Q(\Phi)=Q(\Phi+L)$ when L is a linear map. We are going to prove that $\mathsf{Q}(X,Y)$ is complete when Y is. We need the following estimate, whose proof is a straightforward induction argument, taking into account that $(s+t)^p\le s^p+t^p$ for all $s,t\ge 0$ and $p\in(0,1]$.

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Lemma 3.6.2 *Let X and Y be p-normed spaces. Then, for every $\Phi \in \mathsf{Q}(X,Y)$ and every N one has*

$$\left\|\Phi\left(\sum_{n=1}^{N} x_n\right) - \sum_{n=1}^{N} \Phi x_n\right\|^p \leq Q(\Phi)^p \left(\sum_{n=1}^{N} n\|x_n\|^p\right).$$

Theorem 3.6.3 *If X is a quasinormed space and Y is a quasi-Banach space, then $(\mathsf{Q}(X,Y), Q(\cdot))$ is complete.*

Proof We must prove that if $(\Phi_n)_{n\geq 1}$ is a Cauchy sequence in Q then there is a quasilinear map $\Phi\colon X \longrightarrow Y$ such that $\lim_n Q(\Phi_n - \Phi) = 0$. There is no loss of generality if we assume that X and Y are p-normed spaces, so that the preceding lemma applies. Let $\mathscr{H}$ be a normalised Hamel basis for X, so that each $x \in X$ can be written as a finite sum $x = \sum_{h\in\mathscr{H}} x_h h$. This allows us to introduce a 'control function' $\varrho\colon X \longrightarrow \mathbb{R}^+$ given by $\varrho(x) = \sum_n n(x^*(n))^p$, where $x^*(n)$ is the decreasing rearrangement of the coefficients of x with respect to the basis $\mathscr{H}$. It follows from Lemma 3.6.2 that for *any* $\Phi \in \mathsf{Q}$, one has the estimate

$$\left\|\Phi(x) - \sum_{h\in\mathscr{H}} x_h \Phi(h)\right\|^p \leq Q(\Phi)^p \varrho(x). \tag{3.16}$$

For each n, we consider the linear map $L_n\colon X \longrightarrow Y$ defined by $L_n(h) = \Phi_n(h)$ for all $h \in \mathscr{H}$ so that $\Phi_n - L_n$ vanishes on $\mathscr{H}$. Applying (3.16) to $(\Phi_n - L_n) - (\Phi_k - L_k)$, one gets

$$\|(\Phi_n - L_n)(x) - (\Phi_k - L_k)(x)\|^p \leq Q((\Phi_n - L_n) - (\Phi_k - L_k))^p \varrho(x) = Q(\Phi_n - \Phi_k)^p \varrho(x),$$

which means that $((\Phi_n - L_n)(x))_{n\geq 1}$ is a Cauchy sequence in Y for every $x \in X$. Set $\Phi(x) = \lim_n (\Phi_n - L_n)(x)$ and let us check that Φ is quasilinear and $Q(\Phi - \Phi_n) \longrightarrow 0$. It is quite straightforward that the constants $Q(\Phi_n)$ are uniformly bounded. Indeed, there is a k such that $Q(\Phi_k - \Phi_n) \leq 1$ for every $n \geq k$, so for these n, we have $Q(\Phi_n - L_n) = Q(\Phi_n) \leq (1 + Q(\Phi_k)^p)^{1/p}$. Hence Φ is quasilinear since it is obvious that

$$Q(\Phi) \leq \liminf_{n\to\infty} Q(\Phi_n - L_n) = \liminf_{n\to\infty} Q(\Phi_n) < \infty.$$

To prove that $Q(\Phi - \Phi_n) = Q(\Phi - (\Phi_n - L_n)) \to 0$, pick $\varepsilon > 0$ and let k be large enough that $Q(\Phi_m - \Phi_n) \leq \varepsilon$ for $n, m \geq k$. Suppose $n \geq k$ and take $x, y \in X$ such that

$$\frac{\|(\Phi - \Phi_n)(x+y) - (\Phi - \Phi_n)(x) - (\Phi - \Phi_n)(y)\|}{\|x\| + \|y\|} > \frac{Q(\Phi - \Phi_n)}{2}.$$

Then, for $m \geq k$ large enough, we still have

$$\frac{\|(\Phi_m - \Phi_n)(x+y) - (\Phi_m - \Phi_n)(x) - (\Phi_m - \Phi_n)(y)\|}{\|x\| + \|y\|} > \frac{Q(\Phi - \Phi_n)}{2},$$

whence $\frac{1}{2}Q(\Phi - \Phi_n) < Q(\Phi_m - \Phi_n) \leq \varepsilon$. □

Nice, isn't it? Unlike $\mathsf{Q_L}(X, Y)$, which is always Hausdorff, $\mathsf{Q_{LB}}(X, Y)$ is very often not Hausdorff (find full details in Section 4.5). Since the spaces $\mathsf{Q_L}(X, Y)$ and $\mathsf{Q_{LB}}(X, Y)$ are quotients of $\mathsf{Q}(X, Y)$ one has:

Corollary 3.6.4 *If X is a quasinormed space and Y is a quasi-Banach space, then* $(\mathsf{Q_L}(X, Y), Q(\cdot))$ *and* $(\mathsf{Q_{LB}}(X, Y), Q[\cdot])$ *are complete.*

This completeness result carries important consequences with it in the form of uniform boundedness principles for quasilinear maps, a harvest of ideas that will be carefully reaped in Chapter 5. The bare facts behind those ideas can be formulated as:

Theorem 3.6.5 *Let X be a quasinormed space and let Y be a quasi-Banach space. Assume that every quasilinear map $X \longrightarrow Y$ is trivial. Then, there is a constant K such that for each quasilinear map $\Phi\colon X \longrightarrow Y$ there exists a linear map $L\colon X \to Y$ satisfying $\|\Phi - L\| \leq KQ(\Phi)$.*

Proof The hypothesis implies that the function $D(\Phi) = \operatorname{dist}(\Phi, \mathsf{L})$ is a semi-quasinorm on Q. Note that if L is linear and $\|\Phi - L\| < \infty$ then

$$\begin{aligned}\|\Phi(x+y) - \Phi(x) - \Phi(y)\| &= \|\Phi(x+y) - L(x+y) - \Phi(x) + Lx - \Phi(y) + Ly\|\\ &\leq M\|\Phi - L\|\left(\|x\| + \|y\|\right),\end{aligned}$$

where M is a constant depending only on the moduli of concavity of the quasinorms of X and Y. Therefore $Q(\Phi) \leq MD(\Phi)$ and so $D(\cdot)$ has the same kernel as $Q(\cdot)$. Thus $D(\cdot)$ and $Q(\cdot)$ define genuine quasinorms on the quotient space $\mathsf{Q_L}$. Since $D(\Phi)$ and $Q(\Phi)$ depend only on the class of Φ in $\mathsf{Q_L}$, there is no need to change names. Thus, the proof will immediately follow from the open mapping theorem once we are guaranteed that $(\mathsf{Q_L}, Q(\cdot))$ is complete (as has been already proved) and also that $(\mathsf{Q_L}, D(\cdot))$ is complete, which is true under the additional hypothesis of the theorem: indeed, if every quasilinear map $X \longrightarrow Y$ is trivial then $(\mathsf{Q_L}(X, Y), D(\cdot))$ is complete. To prove this and conclude the proof, just observe that the hypothesis is $\mathsf{Q} = \mathsf{B} + \mathsf{L}$, and since $\mathsf{B} \cap \mathsf{L} = \mathfrak{L}(X, Y)$, the Diamond lemma applied to

$$\begin{tikzcd} & \mathsf{Q} & \\ \mathsf{B} \arrow[ur] & & \mathsf{L} \arrow[ul] \\ & \mathfrak{L} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

yields $\mathsf{Q_L} = \mathsf{Q}/\mathsf{L} = \mathsf{B}/\mathfrak{L}$. □

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The result just proved allows us to attach a parameter to each pair of quasi-Banach spaces X and Y:

$$K[X,Y] = \sup\left\{\frac{D(\Phi)}{Q(\Phi)} : \Phi \in \mathsf{L}(X,Y)\right\}. \tag{3.17}$$

We then have $\operatorname{Ext}(X,Y) = 0$ if and only if $K[X,Y] < \infty$, which quantifies the fact that every extension of X by Y splits.

Exact Sequences of p-Banach Spaces and p-Linear Maps

For each $0 < p \leq 1$ there are exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ in which both X and Y are p-Banach spaces but Z is not, and we have already encountered some of them: the $\ell_p(\varphi)$ spaces in Proposition 3.2.7. Since short exact sequences correspond to quasilinear maps, it is natural to seek some condition on a quasilinear map Φ defined between p-Banach spaces that ensures that the twisted sum $Y \oplus_\Phi Z$ is isomorphic to a p-Banach space. Here it is:

Definition 3.6.6 A homogeneous map $\Phi\colon X \longrightarrow Y$ is said to be p-linear if there is $K > 0$ such that for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in X$, we have

$$\left\|\Phi\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n \Phi(x_i)\right\| \leq K\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}. \tag{3.18}$$

The least possible constant K above shall be referred to as the p-linearity constant of Φ and denoted by $Q^{(p)}(\Phi)$. It is clear that each p-linear map is quasilinear (isn't it?), and indeed, p-linearity can be seen as a stronger form of quasilinearity involving an arbitrary number of variables instead of two. The choice $p = 1$ in Banach spaces is, by far, the most interesting one, and Section 3.8 explains why. Before going any further, let us denote by $\mathsf{Q}^{(p)}(X,Y)$ the space of p-linear maps from X to Y and observe that if X, Y are p-Banach spaces and $0 < r < p \leq 1$, we have the following containments:

$$\begin{tikzcd}[column sep=small] & \mathsf{B}(X,Y) \arrow[dr] & & & \\ \mathfrak{L}(X,Y) = \mathsf{B}\cap\mathsf{L} \arrow[ur] \arrow[dr] & & \mathsf{Q}^{(p)}(X,Y) \arrow[r] & \mathsf{Q}^{(r)}(X,Y) \arrow[r] & \mathsf{Q}(X,Y) \\ & \mathsf{L}(X,Y) \arrow[ur] & & & \end{tikzcd}$$

And here is the promised characterisation:

Proposition 3.6.7 *Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map acting between p-normed spaces. Then Φ is p-linear if and only if $Y \oplus_\Phi X$ is isomorphic to a p-normed space.*

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Proof Recall from Lemma 1.1.2 that a quasinormed space Z is locally p-convex if and only if there is a constant M such that

$$\Big\|\sum_{i=1}^{n} z_i\Big\| \leq M\Big(\sum_{i=1}^{n}\|z_i\|^p\Big)^{1/p} \tag{3.19}$$

for every finite set $z_1,\ldots,z_n \in Z$. If Φ is p-linear and we momentarily write $K = Q^{(p)}(\Phi)$ to ease notation then, since $s^p + t^p \leq 2^{1-p}(s+t)^p$ for $0 < p \leq 1$, we have

$$\begin{aligned}
\Big\|\sum_i (y_i, x_i)\Big\|_\Phi &= \Big\|\sum_i y_i - \Phi\Big(\sum_i x_i\Big)\Big\| + \Big\|\sum_i x_i\Big\| \\
&\leq \left(\Big\|\sum_i y_i - \sum_i \Phi x_i\Big\|^p + \Big\|\Phi\Big(\sum_i x_i\Big) - \sum_i \Phi x_i\Big\|^p + \Big\|\sum_i x_i\Big\|^p\right)^{1/p} \\
&\leq \left(\sum_i (\|y_i - \Phi x_i\|^p + \|x_i\|^p) + K^p \sum_i \|x_i\|^p\right)^{1/p} \\
&\leq (1+K^p)^{1/p}\left(\sum_i (\|y_i - \Phi x_i\|^p + \|x_i\|^p)\right)^{1/p} \\
&\leq (1+K^p)^{1/p}\left(\sum_i 2^{1-p}(\|y_i - \Phi x_i\| + \|x_i\|)^p\right)^{1/p} \\
&\leq (1+K^p)^{1/p} 2^{1/p-1}\left(\sum_i \|(y_i, x_i)\|_\Phi^p\right)^{1/p}.
\end{aligned}$$

Suppose now that $Y \oplus_\Phi X$ is locally p-convex. Then

$$\Big\|\sum_i (\Phi x_i, x_i)\Big\|_\Phi \leq M\left(\sum_i \|(\Phi x_i, x_i)\|^p\right)^{1/p}$$

for every finite set $x_1,\ldots,x_n \in X$, and we have

$$\left\|\sum_i \Phi x_i - \Phi\left(\sum_i x_i\right)\right\| \leq M\left(\sum_i \|x_i\|^p\right)^{1/p}. \qquad \square$$

The reader might feel some trepidation at yet another new class of maps. We can all, however, sigh in relief: all that has been done for quasilinear maps translates verbatim to the p-world, just replacing everywhere ‘quasi’ by ‘p’, including Q by $\mathsf{Q}^{(p)}$. In particular, $Q^{(p)}(\cdot)$ is a semi-p-norm on $\mathsf{Q}^{(p)}(X,Y)$ whose kernel is the subspace of linear maps. Also, if we set

$$\mathsf{Q}^{(p)}_{\mathsf{LB}}(X,Y) = \frac{\mathsf{Q}^{(p)}(X,Y)}{\mathsf{L}(X,Y)+\mathsf{B}(X,Y)}$$

endowed with the semi-p-norm $Q^{(p)}[\Phi] = \inf\{Q^{(p)}(\Phi + L + B)\colon L \in \mathsf{L}, B \in \mathsf{B}\}$, it turns out that the following p-versions of Theorems 3.6.3 and 3.6.5 are true and even have simpler proofs:

Theorem 3.6.8 *Let X be a p-normed space and Y be a p-Banach space.*

(a) $(\mathsf{Q}^{(p)}(X,Y), Q^{(p)}(\cdot))$ *is complete, as are its quotients* $(\mathsf{Q}_{\mathsf{L}}^{(p)}(X,Y), Q^{(p)}(\cdot))$ *and* $(\mathsf{Q}_{\mathsf{LB}}^{(p)}(X,Y), Q^{(p)}(\cdot))$.
(b) *If every p-linear map $X \longrightarrow Y$ is trivial then there is a constant K such that every p-linear map $\Phi\colon X \longrightarrow Y$ admits a linear map $L\colon X \longrightarrow Y$ such that $\|\Phi - L\| \leq KQ^{(p)}(\Phi)$.*

The road to quantifying the splitting is paved: given p-Banach spaces X, Y, set

$$K^{(p)}[X,Y] = \sup\left\{\frac{D(\Phi)}{Q^{(p)}(\Phi)} : \Phi \in \mathsf{L}(X,Y)\right\}. \tag{3.20}$$

Then, if we denote the set of classes of exact sequences of p-Banach spaces $0 \longrightarrow Y \longrightarrow \cdot \longrightarrow X \longrightarrow 0$ modulo equivalence (full details will be given in Section 4.1) by $\mathrm{Ext}_{p\mathbf{B}}(X,Y)$, we have:

Corollary 3.6.9 $\mathrm{Ext}_{p\mathbf{B}}(X,Y) = 0$ *if and only if* $K^{(p)}[X,Y] < \infty$.

3.7 Homological Properties of ℓ_p and L_p When $0 < p \leq 1$

As we said in 2.7.1, the spaces $\ell_p(I)$ are the only projective p-Banach spaces. Thus, any other p-Banach space X, in particular L_p, can be placed in a non-trivial sequence of p-Banach spaces $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$; in fact, any projective presentation $0 \longrightarrow \kappa(X) \longrightarrow \ell_p \longrightarrow X \longrightarrow 0$ serves this purpose. However, having the same local structure as ℓ_p, the spaces L_p exhibit a partially projective character that we now describe.

Theorem 3.7.1 *Let $p \in (0,1]$. Assume Y is a p-Banach ultrasummand and that X has a directed set of finite-dimensional subspaces uniformly isomorphic to the ℓ_p space of the corresponding dimension whose union is dense in X. Then every exact sequence of p-Banach spaces $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ splits.*

Proof It is clear that if Y is a p-normed space then for every p-linear map $\Psi\colon \ell_p^n \longrightarrow Y$, there is a linear map $L\colon \ell_p^n \longrightarrow Y$ such that $\|\Psi - L\| \leq Q^{(p)}(\Psi)$, namely, the linear map that agrees with Ψ on the unit basis. It follows that, if E is λ-isomorphic to ℓ_p^n, then for every p-linear map $\Psi\colon E \longrightarrow Y$, there is a

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linear map $L\colon E \longrightarrow Y$ such that $\|\Psi - L\| \le \lambda Q^{(p)}(\Psi)$. Now let $I \subset \mathscr{F}(X)$ be the hypothesised set of subspaces for which we assume

- $\sup\{d(E, \ell_p^{\dim E}) : \; E \in I\} = \lambda < \infty$
- $X_0 = \bigcup_{E\in I} E$ is dense in X.

According to Proposition 3.6.7, it suffices to check that every p-linear map $\Phi\colon X_0 \longrightarrow Y$ is trivial. To this end, let $\mathfrak{U}$ be an ultrafilter refining the order filter on I, and, since Y is an ultrasummand, let $P\colon Y_{\mathfrak{U}} \longrightarrow Y$ be a bounded projection along the diagonal embedding. For each $E \in I$, pick a linear map $L_E\colon E \longrightarrow Y$ such that $\|\Phi|_E - L_E\| \le \lambda Q^{(p)}(\Phi)$. Form a mapping $L\colon X_0 \longrightarrow Y$ as follows: given $x \in X_0$, we consider the bounded family $(L_E(x))_{E\in I}$ (understood to take the value 0 when $x \notin E$) and set $L(x) = P[(L_E(x))_{E\in I}]$. We have to show two things: that L is linear, for which it obviously suffices to observe that $x \in X_0 \longmapsto [(L_E(x))] \in Y_{\mathfrak{U}}$ is linear, and that $\|\Phi - L\| \le \lambda\|P\|Q^{(p)}(\Phi)$. For the second statement, note that for normalised $x \in X_0$, we have

$$\begin{aligned}\|\Phi x - Lx\| &= \|P([(\Phi|_E(x) - L_E(x))_E])\| \le \|P\| \lim_{\mathfrak{U}(E)} \|\Phi(x) - L_E(x)\| \\ &\le \|P\| \sup_{E\in I} \sup_{x\in E} \|\Phi(x) - L_E(x)\| \le \lambda\|P\|Q^{(p)}(\Phi). \qquad \square\end{aligned}$$

The proof actually gives $K^{(p)}[X_0, Y] \le \lambda\|P\|$ and, as we announced just before Theorem 3.7.1, it establishes a homological property of L_p-spaces in $p\mathbf{B}$. The hypotheses of the theorem are satisfied by L_p taking $I = (E_n)_{n\ge1}$, where E_n is the subspace spanned by the characteristic functions of the intervals $[(k-1)/2^n, k/2^n]$ for $1 \le k \le 2^n$, which is isometric to $\ell_p^{2^n}$. Thus

Corollary 3.7.2 $\operatorname{Ext}_{p\mathbf{B}}(L_p, Y) = 0$ *whenever Y is a p-Banach ultrasummand.*

The case $p = 1$ is the popular

3.7.3 Lindenstrauss' lifting *If X is an $\mathscr{L}_1$-space and Y is an ultrasummand then every exact sequence of Banach spaces* $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ *splits.*

A number of homological properties of the spaces ℓ_p and L_p hold not only in $p\mathbf{B}$ but even in $\mathbf{Q}$:

Theorem 3.7.4 *Let I be any index set,* $0 < p < q \le 1$ *and let Y be a q-Banach space. Then* $\operatorname{Ext}(\ell_p(I), Y) = 0$. *In particular, ℓ_p is a $\mathscr{K}$-space for all* $0 < p < 1$.

Proof Actually, if $\Omega\colon \ell_p^0(I) \longrightarrow Y$ is quasilinear, then the linear map $L\left(\sum_{i\in I} \lambda_i e_i\right) = \sum_i \lambda_i \Omega(e_i)$ is at finite distance from Ω on $\ell_p^0(I)$, as it follows from the next lemma, taking into account that the series $\sum_{n=1}^{\infty} n^r$ converges for $r < -1$. $\square$

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Lemma 3.7.5 *Let Y be a q-normed space and let $\Omega\colon \ell_p^0(I) \longrightarrow Y$ be a quasilinear map. If f_i have disjoint supports then*

$$\left\| \Omega\left(\sum_{i=1}^n f_i\right) - \sum_{i=1}^n \Omega(f_i) \right\| \le Q(\Omega) \cdot \left(\sum_{i=1}^n \left(\frac{2}{i}\right)^{q/p} \right)^{1/q} \left\| \sum_{i=1}^n f_i \right\|_p . \tag{3.21}$$

Proof The proof is by induction on n, the number of summands in (3.21). The initial case is less than a tautology. Assume that (3.21) holds for $n-1$ summands, and let us check it for n summands. Let $f = \sum_{i=1}^n f_i$. Since $\|f\|^p = \sum_{i=1}^n \|f_i\|^p$, there exist k, l such that $\|f_k\|^p + \|f_l\|^p \le 2\|f\|^p/n$. Now, as desired,

$$\begin{aligned}
&\left\| \Omega(f) - \sum_{i=1}^n \Omega(f_i) \right\|^q \\
&= \left\| \Omega(f) - \Omega(f_k + f_l) - \sum_{i \ne k,l} \Omega(f_i) + \Omega(f_k + f_l) - \Omega(f_k) - \Omega(f_l) \right\|^q \\
&\le \underbrace{\left\| \Omega(f) - \Omega(f_k + f_l) - \sum_{i \ne k,l} \Omega(f_i) \right\|^q}_{n-1 \text{ summands}} + \|\Omega(f_k + f_l) - \Omega(f_k) - \Omega(f_l)\|^q \\
&\le Q(\Omega)^q \cdot \left(\sum_{i=1}^{n-1} \left(\frac{2}{i}\right)^{q/p} \right) \cdot \|f\|^q + Q(\Omega)^q \left(\|f_k\| + \|f_l\|\right)^q \\
&\le Q(\Omega)^q \cdot \left[\left(\sum_{i=1}^{n-1} \left(\frac{2}{i}\right)^{q/p} \right) \cdot \|f\|^q + \left(\|f_k\|^p + \|f_l\|^p\right)^{q/p} \right] \\
&\le Q(\Omega)^q \cdot \left[\left(\sum_{i=1}^{n-1} \left(\frac{2}{i}\right)^{q/p} \right) \cdot \|f\|^q + \left(\frac{2}{n}\right)^{q/p} \|f\|^q \right].
\end{aligned}$$

□

Proposition 3.7.6 *Let X be a p-Banach space and let Y be a q-Banach space. Every twisted sum of Y and X is (isomorphic to) an r-Banach space for all $r < \min(p, q)$. If, moreover, $0 < p < q \le 1$ then every twisted sum of Y and X is p-convex.*

Proof Assume Z is a twisted sum of Y and X. Let $Q\colon \ell_r(I) \longrightarrow X$ be any quotient map. By Theorem 3.7.4, the lower row in the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & \ell_r(I) \arrow[r] \arrow[u, "Q"'] & 0
\end{tikzcd}$$

splits. Therefore, Z is a quotient of $Y \times \ell_r(I)$, and thus it is r-convex. As for the second part, just take $r = p$ and proceed. □

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The raison d'être of the following application is withheld until Chapter 4, but we can enjoy it now:

Corollary 3.7.7 *Let $0 < r < p \leq 1$. There is $K(p,r) > 0$ such that if X, Y are p-normed spaces and $\Phi : X \longrightarrow Y$ is quasilinear then Φ is r-linear and $Q^{(r)}(\Phi) \leq K(p,r)Q(\Phi)$.*

Proof Proposition 3.7.6 gives that every quasilinear map between p-normed spaces is r-linear for $r < p$. If the qualitative conclusion that every quasilinear map is r-linear fails, there must be a sequence of quasilinear maps $\Phi_n \colon X_n \longrightarrow Y_n$, where X_n and Y_n are p-Banach spaces, such that $Q(\Phi_n) \leq 1$ and $Q^{(r)}(\Phi_n) \longrightarrow \infty$. These maps can be amalgamated into a single quasilinear map $\Phi \colon c_0^0(\mathbb{N}, X_n) \longrightarrow c_0^0(\mathbb{N}, Y_n)$ given by $\Phi((x_n)_n) = (\Phi_n(x_n))_n$ with $Q(\Phi) \leq 1$ and $Q^{(r)}(\Phi) = \infty$, a contradiction. □

Theorem 3.7.8 $\mathrm{Ext}(L_p, Y) = 0$ *whenever Y is a q-Banach space with $0 < p < q \leq 1$. In particular, L_p is a $\mathscr{K}$-space when $0 < p < 1$.*

Proof Let $0 \longrightarrow Y \longrightarrow Z \xrightarrow{\rho} L_p \longrightarrow 0$ be an extension, with Y a q-Banach space. The second part of Proposition 3.7.6 shows that Z is p-convex. For ease of notation, for each $n \in \mathbb{N}$ and $1 \leq k \leq 2^n$, let χ_k^n be the characteristic function of the interval $[(k-1)/2^n, k/2^n]$ and let E_n be the subspace of L_p spanned by χ_k^n, $1 \leq k \leq 2^n$. Since E_n is isometric to $\ell_p^{2^n}$, there is a linear map $s_n \colon E_n \longrightarrow Z$ such that $\rho s_n = \mathbf{1}_{E_n}$ with $\|s_n\| \leq C$ for some constant C independent of n. Let us check that $s(f) = \lim_n s_n(f)$ exists for $f \in \bigcup_n E_n$, in which case it extends to an operator $S : L_p \to Z$ which is a section of the quotient map $\rho \colon Z \longrightarrow L_p$. By linearity, it suffices to verify that $(s_n \chi_k^j)_n$ is a Cauchy sequence in Z. Suppose $j \leq m < n$ and $1 \leq k \leq 2^j$. The difference $s_m \chi_k^j - s_n \chi_k^j$ lies in Y and

$$\begin{aligned}
\|s_m\chi_k^j - s_n\chi_k^j\| &= \left\|(s_m - s_n)\left(\sum_{i=1}^{2^{m-j}} \chi_{2^{m-j}k+i}^m\right)\right\| = \left\|\sum_{i=1}^{2^{m-j}} (s_m - s_n)\chi_{2^{m-j}k+i}^m\right\| \\
&\leq C\left(\sum_{i=1}^{2^{m-j}} \left\|(s_m - s_n)\chi_{2^{m-j}k+i}^m\right\|^q\right)^{1/q} \leq C\left(\sum_{i=1}^{2^{m-j}} \|\chi_{2^{m-j}k+i}^m\|_p^q\right)^{1/q} \\
&= C2^{-j/q}2^{m(1/q-1/p)}.
\end{aligned}$$

□

The section S appearing in the above proof is unique: if $\tilde{S}$ is another right inverse for ρ in $\mathfrak{L}(L_p, Z)$ then $\tilde{S} = S$ as the difference $\tilde{S} - S$ is an operator from L_p to Y. In a sense, S attracts the local sections s_n despite the arbitrariness of their choice. The full force of the proof is unnecessary to establish that L_p is a $\mathscr{K}$-space since this follows from the p-convexity of Z, Proposition 3.7.6 and Theorem 3.7.1, taking into account that $\mathbb{K}$ is, obviously, an ultrasummand.

Our next result is a link in the chain formed by Corollary 4.2.8, Corollary 4.5.3 and Proposition 7.2.16.

Theorem 3.7.9 *Let $0 < p < 1$ and let A, A' be closed subspaces of L_p such that $L_p/A \simeq L_p/A'$. Assume that each of the spaces A, A' is either q-normable for some $0 < p < q \leq 1$ or an ultrasummand. Then there is an automorphism U of L_p such that $U[A] = A'$.*

Proof The proof depends on the fact that if A is either a q-Banach space for some $0 < p < q \leq 1$ or a p-Banach ultrasummand then $\mathfrak{L}(L_p, A) = 0$ by 1.1.5 and the corollary in Section 1.8.3, and also that $\operatorname{Ext}_{p\mathbf{B}}(L_p, Y) = 0$ by the preceding theorem. Let $u\colon L_p/A \longrightarrow L_p/A'$ be an isomorphism and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] & L_p \arrow[r, "\pi'"] & L_p/A' \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & L_p \arrow[r, "\pi"] & L_p/A \arrow[r] \arrow[u, "u"'] & 0
\end{tikzcd} \tag{3.22}$$

Since the lower sequence in the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] & L_p \arrow[r, "\pi'"] & L_p/A' \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & L_p \arrow[r] \arrow[u, "u\pi"'] & 0
\end{tikzcd}$$

splits because $\operatorname{Ext}(L_p, A') = 0$, the splitting criterion for pullback sequences yields a lifting $U\colon L_p \longrightarrow L_p$ for $u\pi$ through π'. This lifting is unique since the difference of two liftings should map L_p to A' and $\mathfrak{L}(L_p, A') = 0$. Thus, U maps A to A and we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] & L_p \arrow[r, "\pi'"] & L_p/A' \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "U|_A"'] & L_p \arrow[r, "\pi"] \arrow[u, "U"'] & L_p/A \arrow[r] \arrow[u, "u"'] & 0
\end{tikzcd}$$

The same argument applies to u^{-1}, thus showing that there is exactly one operator $V\colon L_p \longrightarrow L_p$ such that $\pi V = u^{-1}\pi'$ sitting in a commutative diagram

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$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & L_p \arrow[r, "\pi"] & L_p/A \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u, "V|_{A'}"] & L_p \arrow[r, "\pi'"] \arrow[u, "V"] & L_p/A' \arrow[r] \arrow[u, "u^{-1}"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "U|_A"] & L_p \arrow[r, "\pi"] \arrow[u, "U"] & L_p/A \arrow[r] \arrow[u, "u"] & 0
\end{tikzcd}$$

Clearly, VU is the identity of L_p since $\mathbf{1}_{L_p} - VU = 0$ because it maps L_p to A and, for the same reason, $UV = \mathbf{1}_{L_p}$. □

Theorem 3.7.9 shows that taking quotients of L_p when $0 < p < 1$ by q-normable or ultrasummand subspaces is extremely sensitive to the choice of subspace. Thus, not only is the quotient of L_p by a line not isomorphic to L_p; but in fact L_p/A and L_p/A' are not isomorphic for any A and A' with finite but different dimensions (the converse is also true; see Proposition 7.2.16), or if there are $p < q < r \leq 1$ such that A is q-normable but not r-normable and A' is r-normable. It is perhaps worth recalling that L_p contains isometric copies of each L_q for $p < q \leq 2$. Beyond that, the lines are blurry: L_p is isomorphic to its quotient by the subspace $L_p[0, \frac{1}{2}]$. A more interesting example springs from the interaction between the Hardy class H_p and $L_p(\mathbb{T})$. Recall that H_p can be seen as a subspace of $L_p(\mathbb{T})$ by means of the boundary values. Put $\overline{H}_p = \{f \in L_p(\mathbb{T})\colon \overline{f} \in H_p\}$. An important result of Aleksandrov establishes that $L_p(\mathbb{T}) = H_p + \overline{H}_p$ for $0 < p < 1$ (combine Theorems 2.4 and 3.2 of [6] for a much more general result). Set $J_p = H_p \cap \overline{H}_p$ and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & J_p \arrow[r, "\text{diag}"] \arrow[d, "\text{inclusion}"] & H_p \times \overline{H}_p \arrow[r, "\text{diff}"] \arrow[d] & L_p(\mathbb{T}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_p(\mathbb{T}) \arrow[r] \arrow[d] & \mathrm{PO} \arrow[r] \arrow[d] & L_p(\mathbb{T}) \arrow[r] & 0 \\
 & L_p/J_p \arrow[r, equal] \arrow[d] & \mathrm{PO}/(H_p \times \overline{H}_p) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

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where $\operatorname{diag}(f) = (f, f)$ and $\operatorname{diff}(f, g) = f - g$. The upper exact sequence does not split because $H_p \times \overline{H}_p$ has separating dual, and so every operator $L_p(\mathbb{T}) \longrightarrow H_p \times \overline{H}_p$ is zero. This implies that J_p is not an ultrasummand since $\operatorname{Ext}_{p\mathbf{B}}(L_p, J_p) \neq 0$; in particular, J_p is not isomorphic to the ultrasummand $H_p \times \overline{H}_p$. The middle horizontal sequence splits because the inclusion of J_p into $L_p(\mathbb{T})$ factors through H_p and thus $\mathrm{PO} \simeq L_p(\mathbb{T})^2 \approx L_p$. Therefore, L_p contains two subspaces, one isomorphic to J_p and the other to $H_p \times \overline{H}_p$, whose corresponding quotients are isomorphic. Moreover, the Diamond lemma in combination with Aleksandrov's equality yields

$$\begin{tikzcd}
 & L_p(\mathbb{T}) & \\
H_p \arrow[ur] & & \overline{H}_p \arrow[ul] \\
 & J_p \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

such that $L_p(\mathbb{T})/H_p \simeq \overline{H}_p/J_p$. In fact, the three quotient spaces $L_p(\mathbb{T})/J_p$, $L_p(\mathbb{T})/H_p$, H_p/J_p are isomorphic; see [255, Section 9].

3.8 Exact Sequences of Banach Spaces and Duality

Proposition 3.6.7 says that exact sequences of Banach spaces correspond to 1-linear maps. Since exact sequences of Banach spaces have dual sequences, 1-linear maps should admit dual 1-linear maps. This, by the way, is one of the main differences between p-linear maps for $0 < p < 1$ and 1-linear maps. About the question of how to find that dual 1-linear map, the standard method of taking differences between a bounded homogeneous and a linear selection for the quotient map works as well here as elsewhere. But if the question is how exactly to construct that dual 1-linear map, recall that virtually every non-trivial fact about Banach space duality ultimately depends upon the Hahn–Banach theorem, and this construction, even if it is non-linear, is not an exception.

A Non-Linear Hahn–Banach Theorem

Applying Proposition 3.6.7, the Hahn–Banach theorem and Lemma 3.3.2, in that order, shows that each 1-linear map $\phi\colon X \longrightarrow \mathbb{K}$ admits a linear functional $\ell\colon X \longrightarrow \mathbb{K}$ at finite distance. But ultimately, we should not need to use a sledgehammer to crack a nut. So, let us present a direct proof for that fact, yielding, as a bonus, the optimal distance from a quasilinear map to the space of linear functionals. Given a homogeneous mapping $\Phi\colon X \longrightarrow Y$ acting between Banach spaces, define

$$Q_0^{(1)}(\Phi) = \sup \frac{\|\sum_i \Phi(x_i)\|}{\sum_i \|x_i\|},$$

where the sup is taken over all $x_1, \ldots, x_n \in X$ whose sum is zero. Clearly, $Q_0^{(1)}(\Phi) \le Q^{(1)}(\Phi) \le 2Q_0^{(1)}(\Phi)$.

Lemma 3.8.1 *Let $\phi\colon X \longrightarrow \mathbb{K}$ be a homogeneous function. Then there is a linear function $\ell\colon X \longrightarrow \mathbb{K}$ such that $\|\phi - \ell\| = \operatorname{dist}(\phi, \mathsf{L}(X, \mathbb{K})) = Q_0^{(1)}(\phi)$.*

Proof We first observe that if ℓ is any linear functional and $\sum_i x_i = 0$, then

$$\left|\sum_i \phi(x_i)\right| = \left|\sum_i \phi(x_i) - \sum_i \ell(x_i)\right| \le \|\phi - \ell\| \left(\sum_i \|x_i\|\right),$$

so we certainly have $Q_0^{(1)}(\phi) \le \operatorname{dist}(\phi, \mathsf{L}(X, \mathbb{K}))$. To complete the proof we must find a linear map whose distance to ϕ is exactly $Q_0^{(1)}(\phi)$. The proof of this part depends on which ground we are working over.

Real case. This proof goes as in the classical proof via Zorn's lemma of the Hahn–Banach theorem. The main difficulty to overcome is that an induction hypothesis such as 'there is a linear functional ℓ defined on a subspace U of X such that $|\phi(x) - \ell(x)| \le Q_0^{(1)}(\phi)\|x\|$ holds for all $x \in U$' is not strong enough to ensure that ℓ can be extended to a larger subspace, say $W = [w] \oplus U$, in such a way that the previous estimate still holds for $x \in W$. Our strategy, then, is to use as the induction hypothesis that there is a linear mapping $\ell\colon U \longrightarrow \mathbb{R}$ satisfying

$$\left|\sum_i \phi(x_i) - \ell\left(\sum_i x_i\right)\right| \le Q_0^{(1)}(\phi)\left(\sum_i \|x_i\|\right) \tag{3.23}$$

for every finite set $\{x_i\} \subset X$ such that $\sum x_i \in U$. This plainly implies that $\|\phi - \ell\| \le Q_0^{(1)}(\phi)$ when $U = X$. Assume that a linear mapping ℓ has been defined on a subspace $U \subset X$ such that (3.23) holds when $x = \sum x_i \in U$. Fixing $w \notin U$, we want to see that it is possible to define $\ell(w) = \lambda \in \mathbb{R}$ in such a way that (3.23) holds for $x = \sum x_i \in [w] \oplus U$. Since U is a linear subspace, one can assume that $x = w - u$, with $u \in U$. In this case, it suffices to prove that there exists a number λ satisfying

$$\left|\lambda - \ell(u) - \sum_i \phi(z_i)\right| \le Q_0^{(1)}(\phi)\left(\sum_i \|x_i\|\right)$$

when $w - u = \sum x_i$ and $u \in U$. This is equivalent to

$$\ell(u) + \sum_i \phi(x_i) - Q_0^{(1)}(\phi)\left(\sum_i \|x_i\|\right) \le \lambda \le \ell(u) + \sum_i \phi(x_i) + Q_0^{(1)}(\phi)\left(\sum_i \|x_i\|\right).$$

So, the question is whether

$$\ell(u) + \sum_i \phi(x_i) - Q_0^{(1)}(\phi)\left(\sum_i \|x_i\|\right) \le \ell(v) + \sum_j \phi(s_j) + Q_0^{(1)}(\phi)\left(\sum_j \|s_j\|\right)$$

whenever $w - u = \sum x_i$, $w - v = \sum s_j$, $u, v \in U$ and $x_i, s_j \in X$. The preceding inequality can be written as

$$\ell(u) - \ell(v) + \sum_i \phi(x_i) - \sum_j \phi(s_j) \le Q_0^{(1)}(\phi)\left(\sum_i \|x_i\| + \sum_j \|s_j\|\right)$$

and follows from the induction hypothesis, which yields

$$\left|\ell(u - v) - \left(\sum_i \phi(-x_i) + \sum_j \phi(s_j)\right)\right| \le Q_0^{(1)}(\phi)\left(\sum_i \|-x_i\| + \sum_j \|s_j\|\right),$$

since $u - v = -(w - u) + (w - u) = \sum_i -x_i + \sum_j s_j$ and $u - v \in U$. Observe that so far we have only used the homogeneity of ϕ. In fact, the induction step holds independently of the value or meaning of $Q_0^{(1)}(\phi)$. The 1-linearity of ϕ only appears as the condition we need to start the induction: when $U = 0$ the inequality of the proof states that whenever (x_i) is a finite collection of points of X with $\sum x_i = 0$,

$$\left|\sum \phi(x_i)\right| \le Q_0^{(1)}(\phi)\left(\sum \|x_i\|\right).$$

The rest of the proof is a typical application of Zorn's lemma. Of course, there is no need for Zorn when the space X is finite-dimensional.

Complex case. The proof for complex-valued functions depends on the familiar decomposition of a complex function into real and imaginary parts. Given a complex function f on X, we define the real part $f_r\colon X \longrightarrow \mathbb{R}$ by $f_r(x) = \frac{1}{2}(f(x) + \overline{f(x)})$. It is clear that if f is (complex) homogeneous, then f_r is (real) homogeneous and $\|f\| = \|f_r\|$. Also, if $g\colon X \longrightarrow \mathbb{R}$ is (real) homogeneous, then there is a unique complex homogeneous $g_c\colon X \longrightarrow \mathbb{C}$ whose real part is g, namely $g_c(x) = g(x) - ig(ix)$. Needless to say, g_c is complex linear if and only if g is real linear. Now, let $\phi\colon X \longrightarrow \mathbb{C}$ be complex 1-linear. Then the real part of ϕ is (real) 1-linear, with $Q_0^{(1)}(\phi_r) \le Q_0^{(1)}(\phi)$, so there is a (real) linear map $\ell\colon X \longrightarrow \mathbb{R}$ such that $\|\phi_r - \ell\| = Q_0^{(1)}(\phi_r)$. The map ℓ_c is a complex linear functional on X, and since the assignment $g \longmapsto g_c$ is real linear and $\phi = (\phi_r)_c$ we have

$$\|\phi - \ell_c\| = \|(\phi_r - \ell)_c\| = \|\phi_r - \ell\| = Q_0^{(1)}(\phi_r) \leq Q_0^{(1)}(\phi).$$

This completes the proof and shows that $Q_0^{(1)}(\phi) = Q_0^{(1)}(\phi_r)$. □

Dual Sequence and Dual 1-Linear Map

If $\Phi\colon X \longrightarrow Y$ is a 1-linear map acting between Banach spaces then $0 \longrightarrow Y \longrightarrow Z(\Phi) \longrightarrow X \longrightarrow 0$ is an exact sequence of Banach spaces (after renorming) with a dual sequence $0 \longrightarrow X^* \longrightarrow Z(\Phi)^* \longrightarrow Y^* \longrightarrow 0$ by virtue of the Hahn–Banach theorem. This sequence can be generated by some 1-linear map $\Psi\colon Y^* \longrightarrow X^*$ that could quite judiciously be called the dual of Φ. Its explicit construction is implicit in the proof of Lemma 3.8.1 and will be made still more explicit in this section. The following lemma explains how a map Φ and its dual Ψ yoke together:

Lemma 3.8.2 *Let X, Y be Banach spaces and let X_0, Y_0^* be dense subspaces of X and Y^*, respectively. Let $\Phi\colon X_0 \longrightarrow Y$ and $\Psi\colon Y_0^* \longrightarrow X^*$ be quasilinear maps. The following statements are equivalent:*

(i) *There is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & X^* \arrow[r] \arrow[d, equal] & Z(\Psi) \arrow[r] \arrow[d, "u"] & Y^* \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X^* \arrow[r] & Z(\Phi)^* \arrow[r] & Y^* \arrow[r] & 0 \end{tikzcd} \tag{3.24}$$

(ii) *There is a bilinear form $\beta\colon Y_0^* \times X_0 \longrightarrow \mathbb{K}$ such that*

$$|\langle \Psi(y^*), x\rangle + \langle y^*, \Phi(x)\rangle - \beta(y^*, x)| \leq C\|y^*\|\,\|x\| \tag{3.25}$$

for some constant C, all $y^* \in Y_0^*$ and all $x \in X_0$.

Proof Before the alert reader panics, let us note that the inequality in (ii) already implies that both Φ and Ψ are 1-linear. Indeed, pick $x_1, \ldots, x_n \in X$ and choose a normalised $y^* \in Y^*$ almost norming $\Phi(\sum_i x_i) - \sum_i \Phi(x_i)$. Since

$$\left|\left\langle \Psi(y^*), \sum_i x_i \right\rangle + \left\langle y^*, \Phi\Big(\sum_i x_i\Big)\right\rangle - \beta\left(y^*, \sum_i x_i\right)\right| \leq \left\|\sum_i x_i\right\|$$

and $|\langle \Psi(y^*), x_i\rangle + \langle y^*, \Phi(x_i)\rangle - \beta(y^*, x_i)| \leq C\|x_i\|$ for each i, we have

$$\left\|\Phi\left(\sum_i x_i\right) - \sum_i \Phi x_i\right\| \leq (1+\varepsilon)\left|\left\langle y^*, \Phi\left(\sum_i x_i\right) - \sum_i \Phi x_i\right\rangle\right| \leq 2(1+\varepsilon)C \sum_i \|x_i\|,$$

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so Φ is 1-linear, with $Q^{(1)}(\Phi) \leq 2C$. Interchanging the roles of $\sum_i x_i$ and y^*, we get that also $Q^{(1)}(\Psi) \leq 2C$. We now prove the implication (ii) $\Longrightarrow$ (i). Since $Y \oplus_\Phi X_0$ is dense in $Z(\Phi)$ by Lemma 3.3.1, both spaces have the same dual. Assuming that (ii) holds, define $u\colon X^* \oplus_\Psi Y_0^* \longrightarrow (Y \oplus_\Phi X_0)^*$ by

$$u(x^*, y^*)(y, x) = \langle x^*, x\rangle + \langle y^*, y\rangle - \beta(y^*, x).$$

This pairing is correctly defined. We check now that it is also bounded and makes Diagram (3.24) commute.

- u is bounded:

$$\begin{aligned} |u(x^*, y^*)(y, x)| &= |\langle x^*, x\rangle + \langle y^*, y\rangle - \beta(y^*, x)| \\ &\leq |\langle x^*, x\rangle + \langle y^*, y\rangle - \langle \Psi(y^*), x\rangle - \langle (y^*, \Phi(x)\rangle| + C\|y^*\|\,\|x\| \\ &\leq |\langle x^* - \Psi(y^*), x\rangle| + |\langle y^*, y - \Phi(x)\rangle| + C\|y^*\|\,\|x\| \\ &\leq \|x^* - \Psi(y^*)\|\,\|x\| + \|y^*\|\,\|y - \Phi(x)\| + C\|y^*\|\,\|x\| \\ &\leq \max(1, C)\,\|(x^*, y^*)\|_\Psi\,\|(y, x)\|_\Phi. \end{aligned}$$

- The left-hand square of (3.24) is commutative since $u(x^*, 0)(y, x) = \langle x^*, x\rangle$.
- The right-hand square is commutative since $u(x^*, y^*)(y, 0) = \langle y^*, y\rangle$.

The implication (i) $\Longrightarrow$ (ii) is also easy. Indeed, assume that u is linear and makes (3.24) commute, and let us take a look at the action of $u(x^*, y^*)$ on $Y \oplus_\Phi X_0$. We have $u(x^*, 0)(y, x) = \langle x^*, x\rangle$ and $u(x^*, y^*)(y, 0) = \langle y^*, y\rangle$. The function $\beta\colon Y_0^* \times X_0 \longrightarrow \mathbb{K}$ defined by $\beta(y^*, x) = -u(0, y^*)(0, x)$ is bilinear and it is obvious that

$$u(x^*, y^*)(y, x) = \langle x^*, x\rangle + \langle y^*, y\rangle - \beta(y^*, x).$$

Finally, let us assume u is bounded: since $\|(\Psi y^*, y^*)\|_\Psi = \|y^*\|$ and $\|(\Phi x, x)\|_\Phi = \|x\|$, from

$$u(\Psi y^*, y^*)(\Phi x, x) = \langle \Psi(y^*), x\rangle + \langle y^*, \Phi(x)\rangle - \beta(y^*, x)$$

we get the estimate in (ii) with $C = \|u\|$. □

We now obtain the dual map of a given 1-linear map and the dual pairing. Let X be a normed space, Y a Banach space and $\Phi\colon X \longrightarrow Y$ a 1-linear map. Glancing out the corner of our eye at (3.25), it is clear that we need to assign a linear functional $\Lambda(y^*)$ to each $y^* \in Y^*$ such that $\langle \Lambda(y^*), x\rangle$ 'almost cancels' $\langle y^*, \Phi(x)\rangle$. And to do that, we form the 1-linear composition $y^* \circ \Phi$ for which $Q^{(1)}(y^* \circ \Phi) \leq \|y^*\| Q^{(1)}(\Phi)$ and $Q_0^{(1)}(y^* \circ \Phi) \leq \|y^*\| Q_0^{(1)}(\Phi)$. Lemma 3.8.1 yields

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a linear map $\Lambda(y^*)\colon X \longrightarrow \mathbb{K}$ such that $\|\Lambda(y^*) + y^* \circ \Phi\| \leq \|y^*\| Q_0^{(1)}(\Phi)$. We thus have

$$|\langle \Lambda(y^*), x\rangle + \langle y^*, \Phi(x)\rangle| \leq Q_0^{(1)}(\Phi)\|y^*\|\|x\|,$$

which is quite close to, if not exactly, what we wanted. Part of the problem is that the map $\Lambda\colon Y^* \longrightarrow \mathsf{L}(X,\mathbb{K})$ takes values in the wrong space. To push its values down into X^* what we will do is to make it vanish on a Hamel basis of Y^*. Let us assume that $\Lambda(y^*)$ depends homogeneously on y^*. Let $\mathscr{H}$ be a Hamel basis of Y^* and let $L\colon Y^* \longrightarrow \mathsf{L}(X,\mathbb{K})$ be the linear map that agrees with Λ on $\mathscr{H}$. Define $\Phi^*\colon Y^* \longrightarrow \mathsf{L}(X,\mathbb{K})$ by $\Phi^*(y^*) = \Lambda(y^*) - L(y^*)$ to get:

Theorem 3.8.3 *The map $\Phi^*\colon Y^* \longrightarrow X^*$ is correctly defined and 1-linear, with $Q^{(1)}(\Phi^*) \leq Q^{(1)}(\Phi)$ and $Q_0^{(1)}(\Phi^*) \leq Q_0^{(1)}(\Phi)$. Moreover, there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & X^* \arrow[r] \arrow[d, equal] & X^* \oplus_{\Phi^*} Y^* \arrow[r] \arrow[d] & Y^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^* \arrow[r] & (Y \oplus_{\Phi} X)^* \arrow[r] & Y^* \arrow[r] & 0
\end{tikzcd} \tag{3.26}$$

In particular, the extension generated by Φ^* is equivalent to the dual of that generated by Φ.

Proof First of all, we have to check that the linear functional $\Phi^*(y^*)$ is bounded for every $y^* \in Y^*$, which is obvious once we notice that

- $\Phi^*(y^*) = 0$ when $y^* \in \mathscr{H}$.
- The set $\{y^* \in Y^* : \|\Phi^*(y^*)\| < \infty\}$ is a linear subspace of Y^*.

Since Φ^* is homogeneous, in order to check the second point, it suffices to see that if $\Phi^*(y_1^*)$ and $\Phi^*(y_2^*)$ are bounded then $\Phi^*(y_1^* + y_2^*)$ is bounded as well:

$$\begin{aligned}
\|\Phi^*(y_1^* + y_2^*) - \Phi^*(y_1^*) - \Phi^*(y_2^*)\| &= \|\Lambda(y_1^* + y_2^*) - \Lambda(y_1^*) - \Lambda(y_2^*)\| \\
&= \|\Lambda(y_1^* + y_2^*) + (y_1^* + y_2^*) \circ \Phi - y_1^* \circ \Phi - \Lambda(y_1^*) - y_2^* \circ \Phi - \Lambda(y_2^*)\| \\
&\leq \|\Lambda(y_1^* + y_2^*) + (y_1^* + y_2^*) \circ \Phi\| + \|y_1^* \circ \Phi + \Lambda(y_1^*)\| + \|y_2^* \circ \Phi + \Lambda(y_2^*)\| \\
&< \infty.
\end{aligned}$$

That Φ^* is 1-linear is also straightforward: pick finitely many $y_i^* \in Y^*$ such that $\sum_i y_i^* = 0$; then

$$\left\|\sum_i \Phi^* y_i^*\right\| = \left\|\sum_i \Lambda(y_i^*)\right\| = \left\|\sum_i \Lambda(y_i^*) + \sum_i y_i^* \circ \Phi\right\| \leq Q_0^{(1)}(\Phi)\left(\sum_i \|y_i^*\|\right).$$

It remains to determine the form of the duality between $Y \oplus_\Phi X$ and $X^* \oplus_{\Phi^*} Y^*$. By the preceding lemma, it suffices to identify a bilinear map $\beta\colon Y^* \times X \longrightarrow \mathbb{K}$ satisfying an estimate of the form

$$|\langle \Phi^*(y^*), x\rangle + \langle y^*, \Phi(x)\rangle - \beta(y^*, x)| \leq C\|y^*\|\,\|x\|.$$

The linear map $L\colon Y^* \longrightarrow \mathsf{L}(X, \mathbb{K})$ can be also regarded as a bilinear map $\beta\colon Y^* \times X \longrightarrow \mathbb{K}$ simply by letting $\beta(y^*, x) = \langle L(y^*), x\rangle$. We have

$$\begin{aligned} |\langle \Phi^*(y^*), x\rangle + \langle y^*, \Phi(x)\rangle - \beta(y^*, x)| &= |\langle \Phi^*(y^*) - L(y^*), x\rangle + \langle y^*, \Phi(x)\rangle| \\ &\leq \|\Lambda(y^*) + y^* \circ \Phi\|\,\|x\| \\ &\leq C\|y^*\|\,\|x\|. \end{aligned}$$

This completes the proof and shows, in passing, that the map defined by

$$u(x^*, y^*)(y, x) = \langle x^*, x\rangle + \langle y^*, y\rangle - L(y^*)(x)$$

is a linear homeomorphism making Diagram (3.26) commute. □

'The' 1-linear map $\Phi^*\colon Y^* \longrightarrow X^*$ dual to Φ provided by Theorem 3.8.3 can be written as $\Phi^* = \Lambda - L$, where $\Lambda\colon Y^* \longrightarrow \mathsf{L}(X, \mathbb{K})$ satisfies $\|\Lambda(y^*) + y^* \circ \Phi\| \leq M\|y^*\|$ and $L\colon Y^* \longrightarrow \mathsf{L}(X, \mathbb{K})$ is a linear map such that $\Lambda - L$ takes values in X^*. Actually, every version of Φ^* can be written in this way since if $\Psi = \Phi^* + L' + B$ with $L'\colon Y^* \longrightarrow X^*$ linear and $B\colon Y^* \longrightarrow X^*$ bounded then $\Psi = (\Lambda + B) - (L - L')$ is the desired decomposition.

The Dual of a Kalton–Peck Map

In this section, $1 < p, q < \infty$ are always conjugate exponents ($p^{-1} + q^{-1} = 1$), and we identify the dual of ℓ_p with ℓ_q in the usual way: if $g \in \ell_q, f \in \ell_p$ then $\langle g, f\rangle = \sum_n g(n)f(n)$. We proceed to compute the duals of the Kalton–Peck maps when acting on ℓ_p. This allows us to describe the dual of the twisted sum space $\ell_p(\varphi)$ and to establish that it is locally convex without needing to invoke Proposition 3.4.5. We will find that the dual of each Kalton–Peck map KP_φ is just a 'scaled' version of KP_φ acting on the dual space. To achieve an optimal matching of the maps, throughout this section we will consider the map $\mathsf{KP}_{p,\varphi}\colon \ell_p^0 \longrightarrow \ell_p$ defined as

$$\mathsf{KP}_{p,\varphi}(f) = f\varphi\left(p \log \frac{\|f\|_p}{|f|}\right). \tag{3.27}$$

If $\varphi(t) = t$, we omit it. It is clear that the twisted sum generated by $\mathsf{KP}_{p,\varphi}$ is just $\ell_p(\varphi_p)$, where $\varphi_p(t) = \varphi(pt)$. The main estimate that we need is a consequence of $|t \log t| \leq e^{-1}$ for $0 < t \leq 1$.

Lemma 3.8.4 *For every $s, t \in \mathbb{C}$, we have* $\left|st \log\left(|s|^q/|t|^p\right)\right| \leq e^{-1}(p|s|^q + q|t|^p)$.

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Proof We may assume that s and t are real and positive. Now, if $s^q \le t^p$, then

$$\left|st\log\frac{s^q}{t^p}\right| = qst\left|\log\frac{s}{t^{p/q}}\right| = q\frac{s}{t^{p/q}}t\left|\log\frac{s}{t^{p/q}}\right|t^{p/q} \le \frac{q}{e}t^{1+p/q} \le \frac{q}{e}t^p.$$

Otherwise $s^q \ge t^p$ and, reversing the roles of s, q and t, p, we get

$$\left|st\log\frac{s^q}{t^p}\right| \le \frac{p}{e}s^q. \qquad \square$$

To illustrate how the inequality we just proved will produce $\mathsf{KP}^*_{p,\varphi}$, let us rewrite it as

$$\left|stp\log\frac{1}{|t|} - stq\log\frac{1}{|s|}\right| \le \frac{p|s|^q + q|t|^p}{e},$$

which tell us that if f, g are finitely supported and $\|f\|_p = \|g\|_q = 1$ then

$$\left|\langle g, \mathsf{KP}_p(f)\rangle - \langle \mathsf{KP}_q(g), f\rangle\right| \le \frac{p+q}{e}.$$

Since the left-hand side of the preceding inequality is positively homogeneous in f and g, we actually have

$$\left|\langle g, \mathsf{KP}_p(f)\rangle - \langle \mathsf{KP}_q(g), f\rangle\right| \le \frac{p+q}{e}\|f\|_p\|g\|_q$$

for finitely supported f, g, and Lemma 3.8.2 shows that $-\mathsf{KP}_q$ can be used as the dual of KP_p. The general case is similar:

Proposition 3.8.5 *Let $p, q > 1$ be such that $p^{-1} + q^{-1} = 1$ and $\varphi \in \mathrm{Lip}_0(\mathbb{R}^+)$. Then*

(a) *there is $C > 0$ such that, for every finitely supported f and g, we have*

$$\left|\langle g, \mathsf{KP}_{p,\varphi}(f)\rangle - \langle \mathsf{KP}_{q,\varphi}(g), f\rangle\right| \le C\|g\|_q\|f\|_p,$$

(b) *there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \ell_q \arrow[r] \arrow[d, equal] & \ell_q(-\varphi_q) \arrow[r] \arrow[d, "u"] & \ell_q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell_p^* \arrow[r] & \ell_p(\varphi_p)^* \arrow[r] & \ell_p^* \arrow[r] & 0
\end{tikzcd}$$

(c) *in particular, $\ell_q(-\varphi_q)$ and $\ell_q(\varphi_q)$ are isomorphic to $\ell_p(\varphi_p)^*$.*

Proof Since $-\mathsf{KP}_{q,\varphi} = \mathsf{KP}_{q,-\varphi}$ and $Z(\Phi)$ and $Z(-\Phi)$ are always isometric via $(y, x) \leftrightarrow (-y, x)$ or $(y, x) \leftrightarrow (y, -x)$, it suffices to check (a) because this already implies (b) and (c). Pick f, g with $\|f\|_p = \|g\|_q = 1$. If $I = \{n\colon f(n)g(n) \neq 0\}$, we have

$$\begin{aligned}\left|\left\langle g, \mathsf{KP}_{p,\varphi}(f)\right\rangle - \langle \mathsf{KP}_{q,\varphi}(g), f\rangle\right| &= \left|\left\langle g, f\varphi\left(p\log\frac{1}{|f|}\right)\right\rangle - \left\langle g\varphi\left(q\log\frac{1}{|g|}\right), f\right\rangle\right|\\ &\leq \sum_{n\in I} |g(n)f(n)| \left|\varphi\left(p\log\frac{1}{|f(n)|}\right) - \varphi\left(q\log\frac{1}{|g(n)|}\right)\right|\\ &\leq \sum_{n\in I} |g(n)|\cdot|f(n)|\cdot \mathrm{Lip}(\varphi)\cdot\left|\log\frac{|g(n)|^q}{|f(n)|^p}\right|\\ &\leq \frac{1}{e}\sum_n (p|g(n)|^q + q|f(n)|^p)\\ &= \frac{p+q}{e}. \qquad \square\end{aligned}$$

When $p = 2$, we have the further simplification that $\ell_2^* = \ell_2$, so $\varphi_q = \varphi_p$, and since $\varphi = (\varphi_{1/2})_2$, the manifest consequence is:

Corollary 3.8.6 *For every $\varphi \in \mathrm{Lip}_0(\mathbb{R}^+)$, the space $\ell_2(\varphi_2)^*$ is isomorphic to $\ell_2(-\varphi_2)$ and, therefore, the space $\ell_2(\varphi)$ is isomorphic to its dual via the map $D\colon \ell_2(\varphi_2) \longrightarrow \ell_2(\varphi_2)^*$ given by $D(y, x)(y', x') = \langle y, x'\rangle - \langle x, y'\rangle$.*

The appearance of one minus sign is unavoidable: if we want an isomorphism between $\ell_2(\varphi)$ and $\ell_2(\varphi)^*$ then the form of the duality has to be $\langle y, x'\rangle - \langle x, y'\rangle$ (or $\langle x, y'\rangle - \langle y, x'\rangle$, of course); if we insist on keeping the 'straight' duality $\langle y, x'\rangle + \langle x, y'\rangle$ then $\ell_2(\varphi)^*$ must be represented as $\ell_2(-\varphi)$.

We leave the duality issues with a remark on the intertwining bilinear form β in Lemma 3.8.2. Note that for the Kalton–Peck map we have $\beta(g, f) = 0$ for $g \in \ell_q^0$; even so:

3.8.7 *Let X, Y be Banach spaces and let $\Phi\colon X \longrightarrow Y, \Psi\colon Y^* \longrightarrow X^*$ be quasilinear maps for which there is a bounded bilinear map $\beta\colon Y^* \times X \longrightarrow \mathbb{K}$ such that for some constant C independent on $y^* \in Y^*$ and $x \in X$, we have*

$$|\langle \Psi(y^*), x\rangle + \langle y^*, \Phi(x)\rangle - \beta(y^*, x)| \leq C\|y^*\|\,\|x\|.$$

Then Φ and Ψ are both bounded.

Indeed, if β is continuous then we have $|\langle \Psi(y^*), x\rangle + \langle y^*, \Phi(x)\rangle| \leq C'\|y^*\|\,\|x\|$, which implies that the composition $y^* \circ \Phi$ is bounded for each $y^* \in Y^*$. Hence

$$Y^* = \bigcup_{n\in\mathbb{N}} \{y^* \in Y^*\colon \|y^* \circ \Phi\| \leq n\}.$$

The Baire category theorem and the homogeneity of Φ yield that the closure of $\{y^* \in Y^*\colon \|y^* \circ \Phi\| \leq 1\}$ must contain an open set of Y^* and therefore an open ball centered at the origin. If this ball has radius r then $\|y^* \circ \Phi\| \leq 1$ whenever $\|y^*\| \leq r$, and thus $\|\Phi\| \leq 1/r$.

3.9 Different Versions of a Quasilinear Map

A *version* of a quasilinear map is another equivalent quasilinear map. Quasilinear maps have many different versions. What is not immediate at this stage is that some versions are better than others depending on the purpose we have in mind. Here we present three versions, each with some additional useful property.

Killing a Quasilinear Map on a Hamel Basis

If $\Phi\colon X \longrightarrow Y$ is a quasilinear map and $\mathscr{H}$ is a Hamel basis of X then there is exactly one version of Φ modulo L vanishing on $\mathscr{H}$: let $L\colon X \longrightarrow Y$ be the only linear map such that $La = \Phi a$ for all $a \in \mathscr{H}$ and set $\Phi - L$. We will denote by $\mathsf{Q}_{\mathscr{H}}(X, Y)$ the space of all quasilinear maps vanishing on $\mathscr{H}$. We have that $\mathsf{Q}_{\mathscr{H}}(X, Y)$ is isometric to $\mathsf{Q}_{\mathsf{L}}(X, Y)$, and there is a decomposition $\mathsf{Q}(X, Y) = \mathsf{L}(X, Y) \oplus \mathsf{Q}_{\mathscr{H}}(X, Y)$. These versions have already appeared: during the proof of Theorem 3.6.3, for instance.

Quasilinear Maps on Finite-Dimensional Spaces

It is easy to give examples of quasilinear maps on finite-dimensional spaces having infinite-dimensional range: consider $\mathbb{C}$ as a real space of dimension 2 and define $\Phi\colon \mathbb{C} \to C[0, 1]$ by sending $e^{i\theta}$ to the function x^θ for $0 \leq \theta < \pi$, and by homogeneity on the rest. Sometimes we need to guarantee that the image of a given finite-dimensional subspace spans a finite-dimensional subspace in the target space. We present here a construction of 'finite-dimensional versions' of a quasilinear map suitable for applications.

Lemma 3.9.1 *Let F be a finite-dimensional p-Banach space, let Y be a p-normed space and let $\Phi\colon F \longrightarrow Y$ be a p-linear map. For each $\varepsilon > 0$, there is a p-linear map $\Phi'\colon F \longrightarrow Y$ such that*

- $\|\Phi - \Phi'\| \leq (1 + \varepsilon) Q^{(p)}(\Phi)$,
- $\Phi'[F]$ *spans a finite-dimensional subspace of Y,*
- $Q^{(p)}(\Phi') \leq 3^{1/p}(1 + \varepsilon) Q^{(p)}(\Phi)$.

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Proof The key point is to associate with every $f \in F$ a good-natured p-convex decomposition. To this end, let S be the unit sphere of F. As S is compact, for fixed $\delta > 0$, select a δ-net $f_1, \ldots, f_n \in S$. For $f \in S$, pick f_i in the net such that $\|f - f_i\| \le \delta$ (take the smallest i if the minimun is attained on several elements of the net). Let us set $f_0 = 0$ and consider the new point $g = (f - f_i)/\|f - f_i\|$. Now, choose $0 \le j \le n$ minimising $\|g - f_j\|$, with the same tie-break rule as before, to get $\big\|f - f_i - (\|f - f_i\|)f_j\big\| < \delta^2$. If $f = f_i$, then we take $j = 0$. In any case, continuing in this way we can select sequences $i(k)$ and λ_k such that

- $0 \le i(k) \le n$ and $0 \le \lambda_k < \delta^k$ for all $k = 0, 1, 2, \ldots$,
- for every $m \in \mathbb{N}$, we have $\|f - \sum_{0 \le k \le m} \lambda_k f_{i(k)}\| \le \delta^m$.

Grouping the terms in the obvious way and taking into account that the ℓ_1-norm is dominated by the ℓ_p-quasinorm, we can write $f = \sum_{i=1}^n c_i f_i$ with $\left(\sum_{i=1}^n |c_i|^p\right)^{1/p} \le \left(\sum_{k=0}^\infty \lambda_k^p\right)^{1/p} \le (1 - \delta^p)^{-1/p} < 1 + \varepsilon$ for δ sufficiently small. This 'greedy algorithm' specifies a unique decomposition for each f in the unit sphere of F. However, it does not guarantee any kind of homogeneity in these decompositions. To amend that, let S_0 be a subset of the unit sphere of F that is maximal with respect to the property that any two points of S_0 are linearly independent (of course, when the ground field is $\mathbb{R}$, this just means that S_0 does not contain 'antipodal' points). Equivalently, S_0 is a subset of the sphere such that every non-zero $f \in F$ can be written in a unique way as $f = cx$, with $c \in \mathbb{K}$ and $x \in S_0$. Now we define $\Phi' \colon F \longrightarrow Y$ as follows: if $f \in S_0$, we set

$$\Phi'(f) = \sum_{i=1}^n c_i \Phi(f_i),$$

where $f = \sum_{i=1}^n c_i f_i$ is the decomposition provided by the algorithm. We extend the map to the whole of F by homogeneity; that is, for arbitrary $f \in F$, we write $x = cf$, with $c \in \mathbb{K}$ and $f \in S_0$, in the only way that this can be done, and we set $\Phi'(f) = c\Phi'(f)$. It is clear that the resulting map is homogeneous. Let us check that Φ' does the job. Let $Q^{(p)}$ denote the p-linearity constant of the starting map Φ. For $f \in S_0$, we have

$$\|\Phi(f) - \Phi'(f)\| = \left\|\Phi(f) - \sum_{i=1}^n c_i \Phi(f_i)\right\| \le Q^{(p)} \left(\sum_{i=1}^n |c_i|^p\right)^{1/p} < (1 + \varepsilon) Q^{(p)},$$

and for arbitrary f, just use the homogeneity of both maps. It is obvious that the range of Φ' is contained in $[\Phi(f_1), \ldots, \Phi(f_n)]$, which is a finite-dimensional subspace of Y. Finally, to estimate the new quasilinearity constant $Q^{(p)}(\Phi')$, note that

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$$\begin{aligned}
&\left\|\Phi'\left(\sum_{i=1}^k x_i\right) - \sum_{i=1}^k \Phi'(x_i)\right\|^p \\
&= \left\|\Phi'\left(\sum_{i=1}^k x_i\right) - \Phi\left(\sum_{i=1}^k x_i\right) + \Phi\left(\sum_{i=1}^k x_i\right) - \sum_{i=1}^k \Phi x_i + \sum_{i=1}^k \Phi x_i - \sum_{i=1}^k \Phi' x_i\right\|^p \\
&\le \left((1+\varepsilon)Q^{(p)}\right)^p \left\|\sum_{i=1}^k x_i\right\|^p + \left(Q^{(p)}\right)^p \sum_{i=1}^k \|x_i\|^p + \left((1+\varepsilon)Q^{(p)}\right)^p \sum_{i=1}^k \|x_i\|^p \\
&\le 3(1+\varepsilon)^p \left(Q^{(p)}\right)^p \sum_{i=1}^k \|x_i\|^p
\end{aligned}$$

for all $x_1, \ldots, x_k \in F$, hence $Q^{(p)}(\Phi') \le 3^{1/p}(1+\varepsilon)Q^{(p)}$. □

A subtler version is still possible:

Lemma 3.9.2 *Let $\Phi\colon X \longrightarrow Y$ be a p-linear map, where X and Y are p-normed spaces. Let F be a finite-dimensional subspace of X, and let $x_1, \ldots, x_n$ be points in the unit sphere of F. Then, for each $\varepsilon > 0$, there is a p-linear map $\Phi_F\colon X \longrightarrow Y$ such that*

- $\|\Phi - \Phi_F\| \le (1+\varepsilon)Q^{(p)}(\Phi)$,
- $\Phi_F[F]$ *spans a finite-dimensional subspace of Y,*
- $\Phi_F(x_i) = \Phi(x_i)$ *for* $1 \le i \le n$,
- $Q^{(p)}(\Phi_F) \le 3^{1/p}(1+\varepsilon)Q^{(p)}(\Phi)$.

Proof Fix $\varepsilon > 0$, and let us consider the map $\Phi|_F$ as a p-linear map from F to Y. Let us apply Lemma 3.9.1 to $\Phi|_F$. We ensure that the δ-net $f_1, \ldots, f_n$ appearing in the third line of the previous proof contains the set $\{x_1, \ldots, x_m\}$ simply by adding these points, if necessary, and then observe that the resulting map $\Phi'\colon F \longrightarrow Y$ agrees with Φ on every f_i. We define $\Phi_F\colon X \longrightarrow Y$ by

$$\Phi_F(x) = \begin{cases} \Phi'(x) & \text{if } x \in F \\ \Phi(x) & \text{otherwise.} \end{cases}$$

Observe that $\|\Phi - \Phi_F\| \le (1+\varepsilon)Q^{(p)}(\Phi)$, and then repeat the arguments in the proof of Lemma 3.9.1 to check the remaining properties of Φ_F. □

Corollary 3.9.3 *Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map acting between quasinormed spaces. There is a constant M such that if F is a finite-dimensional subspace of X then there is a quasilinear map $\Phi_F\colon F \longrightarrow Y$ such that*

- $\|\Phi|_F - \Phi_F\| \le M$,

- $Q(\Phi_F) \leq M$,
- $\Phi_F[F]$ *is contained in some finite-dimensional subspace of* Y.

Proof Consider the twisted sum $Y \oplus_\Phi X$. By the Aoki–Rolewicz theorem, there is $0 < p \leq 1$ such that $Y \oplus_\Phi X$ is p-normable and, according to Proposition 3.6.7, the constant $Q^{(p)}(\Phi)$ is finite for this p. Fix some ε and use Lemma 3.9.1. □

Quasilinear Maps on Separable Spaces

Let $\Phi\colon X \longrightarrow Y$ be a quasilinear map on a separable space X. There is no guarantee that $\Phi[X]$ is separable (unless Y itself is separable). In fact, unless X has dimension 1, each quasilinear map $\Phi\colon X \longrightarrow Y$ into a non-separable space Y has a version with non-separable range (it should be clear why). To discipline this possibly bad behaviour, observe that given a quotient map $\rho\colon Z \longrightarrow X$ between quasi-Banach spaces, there is a closed subspace $Z' \subset Z$ with the same dimension as X such that $\rho\colon Z' \longrightarrow X$ is still a quotient map (it should be even clearer why). Thus, if $0 \longrightarrow Y \xrightarrow{\jmath} Z \xrightarrow{\rho} X \longrightarrow 0$ is an exact sequence of quasi-Banach spaces, taking Z' as before and then forming $Y' = \jmath^{-1}[\ker \rho \cap Z']$, we obtain the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "\jmath"] & Z \arrow[r, "\rho"] & X \arrow[r] & 0
\end{tikzcd}$$

from which the desired version emerges:

Lemma 3.9.4 *Let* $\Phi\colon X \longrightarrow Y$ *be a quasilinear map. There is a version of* Φ *whose range is contained in a subspace of* Y *having dimension at most* $\dim(X)$.

3.10 Linearisation of Quasilinear Maps

What we present here is the construction of 'non-linear envelopes', which linearise quasilinear functions in a similar way as tensor products linearise bilinear mappings. A passing glance at Note 4.6.1, 'Linearisation constructions', will probably clarify the general schema. In this section, we will work with p-Banach spaces and p-linear maps for some fixed p. Let I be the set of all closed subspaces Y of $\ell_p(\kappa)$, and form the ℓ_p-amalgam $U_\kappa = \ell_p(I, \ell_p(\kappa)/Y)$.

Lemma 3.10.1 *For every cardinal* κ*, there is a* p*-Banach space* U_κ *containing an isometric copy of every* p*-Banach space whose cardinality is at most* κ.

Now, let X be a p-Banach space, for which we fix a Hamel basis $\mathscr{H}$. Let κ be the cardinality (not dimension) of X. Consider the spaces $\mathsf{Q}^{(p)}(X, U_\kappa)$ and let $\mathsf{Q}^{(p)}_{\mathscr{H}}(X, U_\kappa)$. We can take as $\mathrm{co}^{(p)}(X)$ the closed subspace of $\mathfrak{L}\left(\mathsf{Q}^{(p)}_{\mathscr{H}}(X, U_\kappa), U_\kappa\right)$ generated by the evaluation maps $(\delta_x)_{x\in X}$, once we have checked that the maps δ_x are bounded: since Φ vanishes on $\mathscr{H}$ then, writing $x = \sum_{h\in\mathscr{H}} x_h h$, we have

$$\|\delta_x(\Phi)\| = \Big\|\Phi(x) - \sum_h x_h\Phi(h)\Big\| \le Q^{(p)}(\Phi)\Big(\sum_h |x_h|^p\|h\|^p\Big)^{1/p}.$$

The universal property behind this construction is uncovered now:

Theorem 3.10.2 *Let $\mho\colon X \longrightarrow \mathrm{co}^{(p)}(X)$ be the map $\mho(x) = \delta_x$.*

(a) *$\mho$ is p-linear, with $Q^{(p)}(\mho) \le 1$.*
(b) *For every p-Banach space Y and every p-linear map $\Phi\colon X \longrightarrow Y$ vanishing on $\mathscr{H}$, there exists a unique operator $\phi\colon \mathrm{co}^{(p)}(X) \longrightarrow Y$ such that $\Phi = \phi\circ\mho$.*
(c) $\|\phi\| = Q^{(p)}(\Phi)$.

Proof A proof is only needed because the definition of $\mathrm{co}^{(p)}(X)$ is kind of a tongue-twister. It is clear that $\mho$ is homogeneous since

$$\mho(\lambda x)(\Phi) = \delta_{\lambda x}\Phi = \Phi(\lambda x) = \lambda\Phi(x) = (\lambda\mho x)(\Phi).$$

The map $\mho$ is p-linear because if one picks finitely many $x_i \in X$ then

$$\begin{aligned}\Big\|\mho\Big(\sum_i x_i\Big) - \sum_i \mho(x_i)\Big\| &= \sup_{Q^{(p)}(\Phi)\le 1}\Big\|\Big(\mho\Big(\sum_i x_i\Big) - \sum_i \mho(x_i)\Big)(\Phi)\Big\| \\ &= \sup_{Q^{(p)}(\Phi)\le 1}\Big\|\Phi\Big(\sum_i x_i\Big) - \sum_i \Phi(x_i)\Big\| \le \Big(\sum_i \|x_i\|^p\Big)^{1/p},\end{aligned}$$

which, in particular, yields $Q^{(p)}(\mho) \le 1$. To prove (b), let us first consider the case $Y = U_\kappa$ and assume $\Phi\colon X \longrightarrow U_\kappa$ is a p-linear map vanishing on $\mathscr{H}$. Since $\mathrm{co}^{(p)}(X) \subset \mathfrak{L}\left(\mathsf{Q}^{(p)}_{\mathscr{H}}(X, U_\kappa), U_\kappa\right)$, it is clear that the required operator $\phi\colon \mathrm{co}^{(p)}(X) \longrightarrow U_\kappa$ is just the restriction to $\mathrm{co}^{(p)}(X)$ of the evaluation map δ_Φ given by $\delta_\Phi(u) = u(\Phi)$. It is obvious that $\|\phi\| \le \|\delta_\Phi\| = \sup_{\|u\|\le 1}\|u(\Phi)\| \le Q^{(p)}(\Phi)$. That $\Phi = \phi\circ\mho$ is easy as well: $\phi(\mho(x)) = \phi(\delta_x) = \delta_\Phi(\delta_x) = \delta_x(\Phi) = \Phi(x)$ for every $x \in X$. This also shows that $\|\phi\| = Q^{(p)}(\Phi)$ since $\Phi = \phi\circ\mho$ implies $Q^{(p)}(\Phi) \le \|\phi\|Q^{(p)}(\mho) \le \|\phi\|$. To complete the proof, we treat the general case. Let $\Phi\colon X \longrightarrow Y$ be a p-linear map that vanishes on $\mathscr{H}$. Let Y' be the closed subspace spanned by $\Phi[X]$ in Y, which necessarily has $|Y'| \le \kappa$. Recall that what we are considering here is *cardinality*, not dimension. Fix an isometry $\imath\colon Y' \longrightarrow U_\kappa$ and form the composition $\imath\circ\Phi\colon X \longrightarrow U_\kappa$ to get an operator $\phi\colon \mathrm{co}^{(p)}(X) \longrightarrow U_\kappa$ such that $\phi\circ\mho = \imath\circ\Phi$. Since $\phi[\mathrm{co}^{(p)}(X)] \subset \imath[Y']$, the desired operator is $\imath^{-1}\phi$. □

The exact sequence

$$0 \longrightarrow \mathrm{co}^{(p)}(X) \xrightarrow{\ \imath\ } \mathrm{co}^{(p)}(X) \oplus_{\mho} X \xrightarrow{\ \pi\ } X \longrightarrow 0 \tag{3.28}$$

behaves, in many respects, as a projective presentation of X (except that we do not know that $\mathrm{co}^{(p)}(X) \oplus_{\mho} X$ is projective):

Corollary 3.10.3 *Every exact sequence* $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ *of p-Banach spaces is a pushout of* (3.28). *Therefore* (3.28) *is semi-equivalent to any projective presentation of X in p***B**.

3.11 The Type of Twisted Sums

We already know that the type of a twisted sum of two spaces does not necessarily match that of their direct sum; see Proposition 3.2.7. We now perform a systematic study of the type of twisted sums of spaces – mostly Banach spaces, but quasi-Banach spaces as well. We then immediately derive no fewer than three remarkable consequences; further applications are spread throughout later chapters. We start with the observation that since type passes to subspaces and quotients, the type of a twisted sum is at best the worst type between those of the subspace and quotient. To deal with the type of a twisted sum, let us formulate the randomised version of Proposition 3.6.7, whose (randomised) proof is left to the reader.

Lemma 3.11.1 *Let X and Y be quasi-Banach spaces having type p and* $\Phi\colon X \longrightarrow Y$ *a quasilinear map. Then* $Z(\Phi)$ *has type p if and only if there is a constant K such that for every finite set* $x_1, \dots, x_n \in X$ *we have*

$$\left(\int_0^1 \Big\|\Phi\Big(\sum_{1\le i\le n} r_i(t)x_i\Big) - \sum_{1\le i\le n} r_i(t)\Phi(x_i)\Big\|^p dt\right)^{1/p} \le K\Big(\sum_{1\le i\le n} \|x_i\|^p\Big)^{1/p}, \tag{3.29}$$

where $(r_i)_{i\ge1}$ *is the sequence of Rademacher functions.*

We will need a probabilistic trick to handle randomised sums. Suppose that $k\colon \{1,\dots,n\} \longrightarrow \{1,\dots,m\}$ is any surjection. Note that such a k can be regarded as a partition of $\{1,\dots,n\}$ into m many sets $(A_j)_{1\le j\le m}$, where $A_j = k^{-1}(j)$. Then, if $x_1,\dots,x_n$ are points of X, the X^n-valued functions

$$t \in [0,1] \longmapsto (r_1(t)x_1, \dots, r_i(t)x_i, \dots, r_n(t)x_n)$$
$$(s,t) \in [0,1]^2 \longmapsto (r_{k(1)}(s)r_1(t)x_1, \dots, r_{k(i)}(s)r_i(t)x_i, \dots, r_{k(n)}(s)r_n(t)x_n)$$

have the same distribution. Therefore, for every function $f\colon X^n \longrightarrow \mathbb{C}$,

$$\int_0^1 f(r_1(t)x_1,\dots,r_n(t)x_n)\,dt = \int_0^1\int_0^1 f(r_{k(1)}(s)r_1(t)x_1,\dots,r_{k(n)}(s)r_n(t)x_n)\,dt\,ds.$$

We also need to fix some notation: given a mapping $\Phi\colon X \longrightarrow Y$ and finitely many points $(x_i)_{1\le i\le n}$ of X, we set

$$\nabla\Phi(x_1,\dots,x_n) = \Phi\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n \Phi(x_i).$$

Theorem 3.11.2 *Let $0 < p < q \le 2$. Every twisted sum of a space having type p and a space having type q has type p.*

Proof Let X by a space having type p with type constant $T_p(X)$ and let Y by a space having type q with type constant $T_q(Y)$. We set $T = \max(T_p(X), T_q(Y))$. Let K be the Kahane constant of the 'L_q versus L_p' estimate of X, so that for $(x_i)_{1\le i\le n}$ in X, we have

$$\left(\int_0^1 \Big\|\sum_{1\le i\le n} r_i(t)x_i\Big\|^q dt\right)^{1/q} \le K\left(\int_0^1 \Big\|\sum_{1\le i\le n} r_i(t)x_i\Big\|^p dt\right)^{1/p}. \tag{3.30}$$

Finally, let $\Phi\colon X \longrightarrow Y$ be a quasilinear map, and, for each $n \in \mathbb{N}$, let c_n be the least constant for which

$$\left(\int_0^1 \Big\|\nabla\Phi(r_1(t)x_1,\dots,r_n(t)x_n)\Big\|^q dt\right)^{1/q} \le c_n\left(\sum_{1\le i\le n}\|x_i\|^p\right)^{1/p}$$

for all finite sets $x_1,\dots,x_n \in X$. Our immediate goal is to prove that the sequence $(c_n)_{n\ge 1}$ is bounded. Suppose $\|x_1\|^p + \cdots + \|x_n\|^p = 1$ and that $\|x_n\|^p \ge N^{-1}$. Put $u(t) = \sum_{1\le i<n} r_i(t)x_i$. Then

$$\nabla\Phi(r_1x_1,\dots,r_nx_n) = \nabla\Phi(u, r_nx_n) + \nabla\Phi(r_1x_1,\dots,r_{n-1}x_{n-1}).$$

Now, since the Y-valued functions

$$t\in[0,1] \mapsto \nabla\Phi(u(t), r_n(t)x_n) = \Phi\left(\sum_{i\le n} r_i(t)x_i\right) - \Phi\left(\sum_{i<n} r_i(t)x_i\right) - r_n(t)\Phi(x_n)$$

$$(s,t)\in[0,1]^2 \mapsto \nabla\Phi(r_1(s)u(t), r_2(s)r_n(t)x_n)$$
$$= \Phi\left(\sum_{i<n} r_1(s)r_i(t)x_i + r_2(s)r_n(t)x_n\right) - \Phi\left(\sum_{i<n} r_1(s)r_i(t)x_i\right) - r_2(s)r_n(t)\Phi(x_n)$$

https://doi.org/10.1017/9781108778312.005 Published online by Cambridge University Press

have the same distribution, it follows that

$$\int_0^1 \|\nabla\Phi(u(t), r_n(t)x_n)\|^q dt = \int_0^1\int_0^1 \|\nabla\Phi(r_1(s)u(t), r_2(s)r_n(t)x_n)\|^q ds\, dt$$
$$\leq c_2^q \int_0^1 (\|u(t)\|^p + \|x_n\|^p)^{q/p}\, dt.$$

Hence, by Minkowski's inequality (recall that $q/p > 1$),

$$\begin{aligned}\left(\int_0^1 \|\nabla\Phi(u, r_n x_n)\|^q dt\right)^{p/q} &\leq c_2^p \left(\int_0^1 (\|u(t)\|^p + \|x_n\|^p)^{q/p}\, dt\right)^{p/q}\\ &\leq c_2^p\left(\left(\int_0^1 \|u(t)\|^q dt\right)^{p/q} + \|x_n\|^p\right)\\ &\leq c_2^p\left(K^p\left(\int_0^1 \|u(t)\|^p dt\right)^p + \|x_n\|^p\right)\\ &\leq c_2^p\left(K^p T_p(X)^p\left(\sum_{1\leq i<n} \|x_i\|^p\right) + \|x_n\|^p\right).\end{aligned}$$

Thus, if $\|x_1\|^p + \cdots + \|x_n\|^p = 1$ and $\max_i \|x_i\|^p \geq N^{-1}$, then

$$\left(\int_0^1 \left\|\nabla\Phi(r_1(t)x_1, \ldots, r_n(t)x_n)\right\|^q dt\right)^{1/q} \leq c_{n-1}\left(1 - \frac{1}{N}\right)^{1/p} + Kc_2c. \tag{3.31}$$

Now assume $\|x_1\|^p + \cdots + \|x_n\|^p = 1$ and $\|x_i\|^p < N^{-1}$ for $1 \leq i \leq n$. Then we can partition $\{1, \ldots, n\}$ into N subsets $(A_j)_{1\leq j\leq N}$ such that $\sum_{i\in A_j} \|x_i\|^p \leq 2/N$ for all j. Then, letting $u_j(t) = \sum_{i\in A_j} r_i(t)x_i$, we have

$$\nabla\Phi(r_1x_1, \ldots, r_nx_n) = \nabla\Phi(u_1, \ldots, u_N) + \sum_{j=1}^N \nabla(\Phi)\big((r_ix_i)_{i\in A_j}\big).$$

Using the same argument as before,

$$\left(\int_0^1 \|\nabla\Phi(u_1(t), \ldots, u_N(t))\|^q dt\right)^{p/q} \leq c_N^p\left(\int_0^1\left(\sum_{1\leq j\leq N} \|u_i(t)\|^p\right)^{q/p} dt\right)^{p/q}$$
$$\leq c_N^p \sum_{1\leq j\leq N}\left(\int_0^1 \|u_i(t)\|^q dt\right)^{p/q} \leq c_N^p \sum_{1\leq j\leq N} K^p T_p(X) \sum_{i\in A_j} \|x_i\|^p \leq c_N^p K^p c^p.$$

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We are almost there:

$$\left(\int_0^1 \Big\| \sum_{j=1}^{N} \nabla\Phi((r_i x_i)_{i\in A_j})\Big\|^q dt\right)^{1/q} \le c\left(\sum_{j=1}^{N}\int_0^1 \|\nabla\Phi((r_i x_i)_{i\in A_j})\|^q dt\right)^{1/q}$$

$$\le cc_n\left(\sum_{j=1}^{N}\left(\sum_{i\in A_j}\|x_i\|^q\right)^{p/q}\right)^{1/q} \le cc_n\left(\sum_{j=1}^{N}\left(\frac{2}{N}\right)^{q/p-1}\sum_{i\in A_j}\|x_i\|^p\right)^{1/q} = \left(\frac{2}{N}\right)^{1/p-1/q} cc_n.$$

All together now,

$$\left(\int_0^1 \Big\|\delta(\Phi)(r_1(t)x_1,\ldots,r_n(t)x_n)\Big\|^q dt\right)^{1/q} \le cKc_N + \left(\frac{2}{N}\right)^{1/p-1/q} c\, c_n.$$

Thus, for all N,

$$c_n \le \max\left(c_{n-1}\left(1-\frac{1}{N}\right)^{1/p} + Kc_2c, cKc_N + \left(\frac{2}{N}\right)^{1/p-1/q} c\, c_n\right).$$

Choosing N so that $c\left(\frac{2}{N}\right)^{1/p-1/q} < 1$ yields a bound for (c_n). □

The many consequences of this result make the considerable effort invested in its proof worth it. The first consequence is the extension of Proposition 3.4.5 to B-convex spaces.

Corollary 3.11.3 *Banach spaces having type $p > 1$ are $\mathscr{K}$-spaces.*

Indeed, since the ground field has type 2, every minimal extension of a space having type $p > 1$ also has type strictly greater than 1, thus it has to be a Banach space (Proposition 1.4.4).

Corollary 3.11.4 *Twisted sums of Banach spaces having type p have type $p - \varepsilon$ for every $\varepsilon > 0$. If X and Y have type $p > 1$ and cotype q then every twisted sum of Y and X has cotype $q + \varepsilon$ for every $\varepsilon > 0$. In particular, twisted Hilbert spaces are near-Hilbert.*

Moving to the border of the Banach zone, we encounter:

Proposition 3.11.5 *Minimal extensions of Banach spaces have type* 1.

This is surprising since, for instance, the non-locally convex Ribe space has type 1, despite estimate (3.10) clearly showing that its vector-valued versions $\ell_1(\varphi)$ do not have type 1 when φ is unbounded. It is even possible to derive Theorem 3.7.4 from Theorem 3.11.2: indeed, if $0 < p < q \le 1$, then every twisted sum of a q-Banach space and $\ell_p(I)$ has type p, hence it is a p-Banach space, and the result follows just by lifting the unit basis of $\ell_p(I)$.

3.12 A Glimpse of Centralizers

We introduce centralizers on function spaces. Centralizers combine two seemingly contradictory ideas: a relaxation of the 'approximate additivity' property of quasilinear maps and a strengthening of their homogeneity. To explain how the first idea is used, observe that what really matters about a quasilinear map are the Cauchy differences $\Phi(x + x') - \Phi(x) - \Phi(x')$ rather than the values of Φ themselves. The reader who is sceptical of this point should skip forward to Section 3.13.2, where the argument is taken to its extreme. So, assume that X, Y are quasinormed spaces and W is a (not necessarily topological) linear space containing Y. A homogeneous mapping $\Phi\colon X \longrightarrow W$ is *quasilinear* (no need to change the name) from X to Y if

(a) for every $x, x' \in X$, the difference $\Phi(x + x') - \Phi(x) - \Phi(x')$ lies in Y,

(b) there is a constant Q such that $\|\Phi(x + x') - \Phi(x) - \Phi(x')\| \le Q(\|x\| + \|x'\|)$ for all $x, x' \in X$.

This definition generalises the standard one in which $W = Y$. We have encountered this situation already: when $\Lambda\colon Y^* \longrightarrow \mathsf{L}(X, \mathbb{K})$ appeared during the construction of the dual 1-linear map in Section 3.8 or, more implicitly, when amalgams in the proof of Lindenstrauss p-lifting (Theorem 3.7.1) occurred in an ultrapower. In those cases we hastened to push the values down to the right space. We can now take a more relaxed attitude: if $\Phi\colon X \longrightarrow W$ is quasilinear from X to Y, it still generates a twisted sum space $Y \oplus_\Phi X = \{(w, x) \in W \times X\colon w - \Phi(x) \in Y\}$, which is a linear subspace of $W \times X$ thanks to (a) and, if endowed with the functional $\|(w, x)\|_\Phi = \|w - \Phi(x)\| + \|x\|$, which is a quasinorm by (b), we get an isometrically exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Y \oplus_\Phi X \xrightarrow{\ \pi\ } X \longrightarrow 0$$

in which $\imath(y) = (y, 0)$ and $\pi(w, x) = x$ as always. In particular, $Y \oplus_\Phi X$ is complete when X and Y are. The criterion for triviality is almost the same as Lemma 3.3.2: Φ generates a trivial extension if and only if $\Phi = B + L$, where $L\colon X \longrightarrow W$ is linear and $B\colon X \longrightarrow Y$ is homogeneous bounded. The reader should not have any difficulty in adapting the proof of Lemma 3.3.2 to this situation. In fact, this is a consequence of the comparison criterion:

3.12.1 *Let $\Phi, \Psi\colon X \longrightarrow W$ be quasilinear maps from X to Y. Then Φ and Ψ generate equivalent extensions if and only if $\Phi - \Psi = B + L$, where $L\colon X \longrightarrow W$ is linear and $B\colon X \longrightarrow Y$ is homogeneous bounded.*

There is a loose end to tie up: if, by pure bad luck, $\Phi\colon X \longrightarrow W$ and $\Psi\colon X \longrightarrow W'$ are quasilinear from X to Y but taking values in different spaces (even if $W = W'$ in most practical situations), just form the pushout

$$\begin{tikzcd} & W \arrow[dr] & \\ Y \arrow[ur] \arrow[dr] & & W'' \\ & W' \arrow[ur] & \end{tikzcd}$$

(in the linear category) and consider that both Φ, Ψ take values in W''.

Centralizers on Function Spaces

Centralizers are fussy objects: they require the presence of a Banach algebra and a comfortable ambient space to exist. A measure space (S, μ) provides both: the algebra $L_\infty(\mu)$ and the ambient space $L_0(\mu)$. It is clear that if X is a function space on μ then for every $a \in L_\infty(\mu)$ and every $x \in X$, we have $ax \in X$ and $\|ax\| \leq \|a\|_\infty \|x\|$, that is, X is a module over $L_\infty(\mu)$ under the pointwise operations.

Definition 3.12.2 Let X and Y be function spaces. A homogeneous mapping $\Phi\colon X \longrightarrow L_0(\mu)$ is said to be a centralizer from X to Y if, for every $a \in L_\infty(\mu)$ and every $x \in X$, the difference $\Phi(ax) - a\Phi(x)$ lies in Y and obeys, for some constant C, an estimate of the form $\|\Phi(ax) - a\Phi(x)\| \leq C\|a\|_\infty \|x\|$.

When $Y = X$, we just say that Φ is a centralizer on X.

Lemma 3.12.3 *Every centralizer is quasilinear.*

Proof This is very easy. Let $\Phi\colon X \longrightarrow L_0(\mu)$ be a centralizer from X to Y. Pick $x, y \in X$, and set $z = |x| + |y|$ such that $z \in X$ and $\|z\| \leq \Delta_X(\|x\| + \|y\|)$. As $|x|, |y| \leq z$, there are $a, b \in L_\infty(\mu)$, with $\max\{\|a\|_\infty, \|b\|_\infty, \|a + b\|_\infty\} \leq 1$, such that $x = az$ and $y = bz$ and $x + y = (a + b)z$. We thus have

$$\begin{aligned} &\|\Phi(x + y) - (a + b)\Phi(z)\| \leq C(\Phi)\|a + b\|_\infty \|z\|, \\ &\|\Phi(x) - a\Phi(z)\| \leq C(\Phi)\|a\|_\infty \|z\|, \\ &\|\Phi(y) - b\Phi(z)\| \leq C(\Phi)\|b\|_\infty \|z\| \end{aligned}$$

(and the differences belong to Y). Therefore,

$$\|\Phi(x + y) - \Phi(x) - \Phi(y)\| \leq \Delta_Y^2 C(\Phi) 4\|z\| \leq 4\Delta_Y^2 \Delta_X C(\Phi)(\|x\| + \|y\|). \qquad \square$$

The following remark highlights the main feature of twisted sums generated by centralizers:

Lemma 3.12.4 *If $\Phi \longrightarrow L_0(\mu)$ is a centralizer from X to Y and $a \in L_\infty(\mu)$ then the map $(y, x) \longmapsto (ay, ax)$ is a bounded endomorphism of $Y \oplus_\Phi X$.*

Proof The pair (ay, ax) belongs to $Y \oplus_\Phi X$ since $y - \Phi(x), a(y - \Phi x)$ and $\Phi(ax) - a\Phi(x)$ are all in $Y \oplus_\Phi X$. Moreover,

$$\begin{aligned}\|(ay, ax)\|_\Phi &= \|ay - \Phi(ax)\| + \|ax\|\\ &\le \Delta_Y\big(\|ay - a\Phi x\| + \|\Phi x - \Phi(ax)\|\big) + \|ax\|\\ &\le \max(\Delta_Y C(\Phi), 1)\|a\|_\infty \|(y, x)\|_\Phi.\end{aligned}$$ □

The Kalton–Peck Maps Are Centralizers

The following result provides the continuous version of the Kalton–Peck maps. Note that the Lipschitz function must now be defined on the whole line $\mathbb{R}$.

Proposition 3.12.5 *Let X be a function space and $\varphi \in \mathrm{Lip}_0(\mathbb{R})$. The map $\Phi\colon X \longrightarrow L_0(\mu)$ defined by*

$$\mathsf{KP}_\varphi(x) = x \cdot \varphi\left(\log \frac{\|x\|}{|x|}\right) \tag{3.32}$$

is a centralizer on X, and $C(\Phi)$ depends only on $\mathrm{Lip}(\varphi)$ and Δ_X.

Proof We write KP instead of KP_φ. As in the proof of Proposition 3.2.6, we consider the non-homogeneous map $\mathsf{kp}_\varphi\colon X \longrightarrow L_0(\mu)$ defined by $\mathsf{kp}_\varphi(x) = x\varphi(-\log|x|)$. From

$$\begin{aligned}\mathsf{KP}_\varphi(x) - \mathsf{kp}_\varphi(x) &= x\left(\varphi\left(\log\frac{\|x\|}{|x|}\right) - \varphi\left(\log\frac{1}{|x|}\right)\right),\\ \mathsf{kp}_\varphi(ax) - a\mathsf{kp}_\varphi(x) &= ax\left(\varphi\left(\log\frac{1}{|ax|}\right) - \varphi\left(\log\frac{1}{|x|}\right)\right),\end{aligned}$$

we obtain the pointwise estimates

$$\begin{aligned}|\mathsf{KP}_\varphi(x) - \mathsf{kp}_\varphi(x)| &\le \mathrm{Lip}(\varphi)\big|x\log\|x\|\big|,\\ |\mathsf{kp}_\varphi(ax) - a\mathsf{kp}_\varphi(x)| &\le \mathrm{Lip}(\varphi)\big|ax\log|a|\big|\end{aligned}$$

so that $\mathsf{KP}_\varphi(x) - \mathsf{kp}_\varphi(x)$ and $\mathsf{kp}_\varphi(ax) - a\mathsf{kp}_\varphi(x)$ belong to Y and

$$\begin{aligned}\|\mathsf{KP}_\varphi(x) - \mathsf{kp}_\varphi(x)\| &\le \mathrm{Lip}(\varphi)\,\big|\,\|x\|\log\|x\|\,\big|,\\ \|\mathsf{kp}_\varphi(ax) - a\mathsf{kp}_\varphi(x)\| &\le \mathrm{Lip}(\varphi)\,\big|\,\|a\|_\infty\log|a|_\infty\,\big|\,\|x\|.\end{aligned}$$

To compute the centralizer constant of KP_φ, it suffices to consider the case $\|x\| = 1$, so that $\mathsf{KP}_\varphi x = \mathsf{kp}_\varphi x$ and $\|a\|_\infty \le 1$. We have

$$
\begin{aligned}
\|\mathsf{KP}_\varphi(ax) - a\mathsf{KP}_\varphi(x)\| &= \|\mathsf{KP}_\varphi(ax) - \mathsf{kp}_\varphi(ax) + \mathsf{kp}_\varphi(ax) - a\mathsf{kp}_\varphi(x)\| \\
&\leq \Delta_Y\big(\|\mathsf{KP}_\varphi(ax) - \mathsf{kp}_\varphi(ax)\| + \|\mathsf{kp}_\varphi(ax) - a\mathsf{kp}_\varphi(x)\|\big) \\
&\leq 2e^{-1}\Delta_X \operatorname{Lip}(\varphi).
\end{aligned}
$$

Hence, $C(\mathsf{KP}_\varphi) \leq 2\Delta_X \operatorname{Lip}(\varphi)e^{-1}$ and $Q(\mathsf{KP}_\varphi) \leq 8\Delta_X^4 \operatorname{Lip}(\varphi)e^{-1}$, according to Lemma 3.12.3. □

From now on, we write

$$X(\varphi) = \Big\{(y, x) \in L_0(\mu) \times X : y - \mathsf{KP}_\varphi(x) \in X\Big\}$$

for the twisted sum defined by the centralizer $\mathsf{KP}_\varphi\colon X \longrightarrow L_0(\mu)$ endowed with the quasinorm $\|(y, x)\|_{\mathsf{KP}_\varphi} = \|y - \mathsf{KP}_\varphi(x)\| + \|x\|$. If X is complete then so is $X(\varphi)$, and no further action is required. Recalling that every sequence space is a function space on $\mathbb{N}$, we see that if X is a sequence space then $X(\varphi)$ is a delightfully concrete completion of $X \oplus_{\mathsf{KP}_\varphi} X^0$, and there is no conflict with the notation of Section 3.2. The analysis of the sequences $0 \longrightarrow X \longrightarrow X(\varphi) \longrightarrow X \longrightarrow 0$ in the continuous case is much more involved and requires techniques specific to centralizers. Anyway, we can exhibit some connections between the discrete and the continuous constructions to prove in passing that:

Proposition 3.12.6 *Let* $\varphi \in \operatorname{Lip}_0(\mathbb{R})$ *and* $0 < p < \infty$.

(a) $L_p(\mathbb{R}^+)(\varphi)$ *contains an isometric copy of* $\ell_p(\varphi)$.
(b) KP_φ *is trivial on* L_p *if and only if* φ *is bounded on* $(-\infty, 0]$.
(c) KP_φ *is trivial on* $L_p(\mathbb{R}^+)$ *if and only if* φ *is bounded on* $\mathbb{R}$.

Proof (a) Let $(A_i)_{i\geq 1}$ be a sequence of disjoint measurable subsets of measure 1. We define $\alpha\colon \mathbb{R}^{\mathbb{N}} \longrightarrow L_0(\mathbb{R}^+)$ by $\alpha(x) = \sum_{i=1}^{\infty} x(i)1_{A_i}$. It is clear that α restricts to an isometry from ℓ_p into $L_p(\mathbb{R}^+)$. If $\mathsf{KP}_{p,\varphi}\colon \ell_p \longrightarrow \mathbb{R}^{\mathbb{N}}$ is the Kalton–Peck map on ℓ_p then it is clear that $\alpha(\mathsf{KP}_{p,\varphi}(x)) = \mathsf{KP}_\varphi(\alpha(x))$. This implies that the map $\alpha \times \alpha\colon \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \longrightarrow L_0(\mathbb{R}^+) \times L_0(\mathbb{R}^+)$ restricts to an isometry $u\colon \ell_p(\varphi) \longrightarrow L_p(\mathbb{R}^+)(\varphi)$ fitting in the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ell_p \arrow[r] \arrow[d, "\alpha"] & \ell_p(\varphi) \arrow[r] \arrow[d, "u"] & \ell_p \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & L_p(\mathbb{R}^+) \arrow[r] & L_p(\mathbb{R}^+)(\varphi) \arrow[r] & L_p(\mathbb{R}^+) \arrow[r] & 0
\end{tikzcd}
$$

Indeed, if $(y, x) \in \ell_p(\varphi)$, then $(\alpha y, \alpha x) \in L_p(\mathbb{R}^+)(\varphi)$ since

$$\begin{aligned}\alpha y - \mathsf{KP}_\varphi(\alpha x) &= \alpha y - \alpha\mathsf{KP}_{p,\varphi}x = \alpha(y - \mathsf{KP}_{p,\varphi}x) \in L_p(\mathbb{R}^+),\\ \|(\alpha y, \alpha x)\|_{\mathsf{KP}_\varphi} &= \|\alpha(y - \mathsf{KP}_{p,\varphi}x)\|_{L_p} + \|\alpha x\|_{L_p}\\ &= \|y - \mathsf{KP}_{p,\varphi}x\|_{\ell_p} + \|x\|_{\ell_p}\\ &= \|(y, x)\|_{\ell_p(\varphi)}.\end{aligned}$$

(b) First assume that $|\varphi(s)| \leq M$ for all $s \in (-\infty, 0]$. Let f be normalised in L_p, and consider the sets $A = \{s \in [0,1] : |f(s)| \leq 1\}$ and $A^c = [0,1]\backslash A$. Write $f = g + h$, where $g = 1_A f$ and $h = 1_{A^c}$, and observe that $\|g\|^p + \|h\|^p = 1$. Let us bound $\mathsf{KP}_\varphi(g)$ and $\mathsf{KP}_\varphi(h)$ separately. We have $|\mathsf{KP}_\varphi(h)| = |h\varphi(\log\|h\|/|h|)| \leq M|h|$ as $|h(s)| \geq \|h\|$ for s off A. In particular, $\|\mathsf{KP}_\varphi(h)\| \leq M$. On the other hand, since $\mathsf{KP}_\varphi(1_{[0,1]}) = 0$, we have

$$\|\mathsf{KP}_\varphi(g)\| = \|\mathsf{KP}_\varphi(g) - g\mathsf{KP}_\varphi(1_{[0,1]})\| \leq C(\mathsf{KP}_\varphi)\|g\|_\infty\|1_{[0,1]}\|_p \leq C(\mathsf{KP}_\varphi).$$

Finally, since KP_φ is quasilinear,

$$\|\mathsf{KP}_\varphi(f) - \mathsf{KP}_\varphi(g) - \mathsf{KP}(h)\| \leq Q(\mathsf{KP}_\varphi)(\|g\| + \|h\|) \leq 2Q(\mathsf{KP}_\varphi),$$

and this leads to a bound for $\|\mathsf{KP}_\varphi(f)\|$. To prove the converse we use the same idea as in Proposition 3.2.7 to show that $L_p(\varphi)$ is not isomorphic to L_p if φ is unbounded on the half-line $(-\infty, 0]$. First of all, note that the sequence $(\varphi(\log n^{-1/p}))_{n\geq 1}$ cannot be bounded, that $\mathsf{KP}_\varphi(1_A) = 1_A\varphi(\log|A|^{1/p})$ for every measurable $A \subset [0,1]$ and that KP_φ vanishes at every unitary function. Fix $n \in \mathbb{N}$ and let $(A_i)_{1\leq i\leq n}$ be a partition of $[0,1]$ into sets of equal measure. Clearly, $\|1_{A_i}\| = n^{-1/p}$ for $1 \leq i \leq n$ such that $\sum_{i\leq n}\|1_{A_i}\|^p = 1$. If (r_i) is the sequence of Rademacher functions, then

$$\begin{aligned}\int_0^1 \Big\|\mathsf{KP}_\varphi\Big(\overbrace{\sum_{i\leq n} r_i(t)1_{A_i}}^{\text{unitary}}\Big) - \sum_{i\leq n} r_i(t)\mathsf{KP}_\varphi(1_{A_i})\Big\|^p dt &= \int_0^1 \Big\|\sum_{i\leq n} r_i(t)\mathsf{KP}_\varphi(1_{A_i})\Big\|^p dt\\ = \int_0^1 \Big\|\sum_{i\leq n} r_i(t)\varphi(\log n^{-1/p})1_{A_i}\Big\|^p dt &= \big|\varphi(\log n^{-1/p})\big|^p.\end{aligned}$$

This shows that $L_p(\varphi)$ does not have type p and concludes the proof for $p \in (0,2]$. For $p \in [2,\infty)$, we show that $L_p(\varphi)$ does not have cotype p using the vectors $(0, 1_{A_i})$. To see this, note that

$$\|(0, 1_{A_i})\|_{\mathsf{KP}_\varphi} = \|1_{A_i}\varphi(\log n^{-1/p})\| + \|1_{A_i}\| = (\varphi(\log n^{-1/p}) + 1)n^{-1/p}$$

so that

$$\sum_{i\leq n}\|(0, 1_{A_i})\|^p_{\mathsf{KP}_\varphi} = (\varphi(\log n^{-1/p}) + 1)^p,$$

while

$$\int_0^1 \Big\| \sum_{i\le n} r_i(t)(0, 1_{A_i}) \Big\|_{\mathsf{KP}_\varphi}^p dt = \int_0^1 \Big\| \Big(0, \sum_{i\le n} r_i(t)1_{A_i}\Big) \Big\|_{\mathsf{KP}_\varphi}^p dt$$

$$= \int_0^1 \left(\Big\| \mathsf{KP}_\varphi\Big(\sum_{i\le n} r_i(t)1_{A_i}\Big) \Big\| + \Big\| \sum_{i\le n} r_i(t)1_{A_i} \Big\| \right)^p dt = 1.$$

(c) If φ is bounded on $\mathbb{R}$, then KP_φ is homogenenous bounded. For the converse, if φ is unbounded on $\mathbb{R}^+$ then $L_p(\mathbb{R}^+)(\varphi)$ cannot be isomorphic to $L_p(\mathbb{R}^+)$ in view of what was proved in Proposition 3.2.7 and (a). Otherwise, φ cannot be bounded on $(-\infty, 0]$, and the result follows from (b) since the restriction of KP_φ to L_p produces a complemented copy of $L_p(\varphi)$ in $L_p(\mathbb{R}^+)(\varphi)$. □

3.13 Notes and Remarks

3.13.1 Domański's Work on Quasilinear Maps

This note reports on Domański's [161] work about quasilinear maps on general topological vector spaces (TVS), which we find most interesting. A mapping $\Phi\colon X \longrightarrow Y$, acting between TVS, is *now* called quasilinear if $\Phi(0) = 0$ and satisfies the following properties:

- Quasiadditivity: $\Phi(x + x') - \Phi(x) - \Phi(x') \to 0$ in Y as $(x, x') \to (0, 0)$ in $X \times X$.
- Quasihomogeneity: $\Phi(cx) - c\Phi(x) \to 0$ in Y as $(c, x) \to (0, 0)$ in $\mathbb{K} \times X$.

Such a map can be used to construct a TVS denoted $Y \oplus_\Phi X$ by endowing $Y \times X$ with the linear topology for which the sets $W(V, U) = \{(y, x) \in Y \times X\colon x \in U, y - \Phi(x) \in V\}$ with $U \in \mathcal{O}_X, V \in \mathcal{O}_Y$ form a neighbourhood base at zero. The resulting sequence $0 \longrightarrow Y \longrightarrow Y \oplus_\Phi X \longrightarrow X \longrightarrow 0$ is topologically exact and splits if and only if Φ is approximable, i.e. there is a linear map $L\colon X \longrightarrow Y$ such that $\Phi(x) - L(x) \to 0$ in Y as $x \to 0$ in X. The question of which topologically exact sequences can be generated by quasilinear maps has a simple answer:

Lemma *A topologically exact sequence is equivalent to a sequence generated by a quasilinear map* $\iff$ *its quotient map has a section that is continuous at zero.*

Proof One implication is clear: if $\Phi\colon X \longrightarrow Y$ is quasilinear, the map $X \longrightarrow Y \oplus_\Phi X$ given by $x \longmapsto (\Phi(x), x)$ is the required section. For the converse, let

$$0 \longrightarrow Y \longrightarrow Z \xrightarrow{\rho} X \longrightarrow 0$$

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be a topologically exact sequence in which we may assume that $Y = \ker\rho$. Let $\Gamma\colon X \longrightarrow Z$ be a section of the quotient map that is continuous at 0. Without loss of generality, we may assume that $\Gamma(0) = 0$; otherwise, we can replace Γ by $\Gamma - \Gamma(0)$. Let $\Lambda\colon X \longrightarrow Z$ be any linear section of ρ. The difference $\Gamma - \Lambda$ takes values in Y and is quasilinear, so we can form the space $Y \oplus_\Phi X$, where $\Phi = \Gamma - \Lambda$. The map $u\colon Z \longrightarrow Y \oplus_\Phi X$ given by $u(z) = (z - \Lambda(z), \rho(z))$ is clearly continuous and makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r, "\rho"] \arrow[d, "u"] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & Y \oplus_\Phi X \arrow[r, "\rho"] & X \arrow[r] & 0
\end{tikzcd}$$

commute. In fact, u is a linear homeomorphism, by Roelcke's lemma. □

In [161], Domański exhibits a locally convex space Z with a quotient map $\rho\colon Z \longrightarrow \mathbb{K}^{\mathbb{N}}$ that has no section that is continuous at zero (and another example having continuous sections but no homogeneous section that is continuous at zero). Thus, not all extensions come from quasilinear maps. The situation is more favourable when one works with F-spaces (metrisable and complete TVS) [161, Lemma 2.2 (a)].

Proposition *All short exact sequences of F-spaces are generated by quasilinear maps.*

Proof Exact sequences of F-spaces are topologically exact, by the open mapping theorem. It suffices to show that if Z is an F-space and $\rho\colon Z \longrightarrow X$ is a quotient map then there is a section $\Gamma\colon X \longrightarrow Z$ that is continuous at zero and has $\Gamma(0) = 0$. Let $(U_n)_{n\geq 0}$ be a (decreasing) base of neighbourhoods of the origin in Z, with $U_0 = Z$. If $V_n = \rho[U_n]$, then $(V_n)_{n\geq 0}$ is a (decreasing) base of neighbourhoods of the origin in X, with $V_0 = X$. For each n, let $\Gamma_n\colon V_n \longrightarrow U_n$ be *any* mapping such that $\rho(\Gamma_n(x)) = x$ for all $x \in V_n$. Finally, we define $\Gamma\colon X \longrightarrow Z$ by $\Gamma(0) = 0$ and $\Gamma(x) = \Gamma_n(x)$, where $n = \sup\{k\colon x \in V_k\}$. □

These ideas can be used to winkle out some homological properties of L_0:

Corollary *If Y is a quasi-Banach space then every topologically exact sequence* $0 \longrightarrow Y \longrightarrow Z \longrightarrow L_0 \longrightarrow 0$ *splits. In particular, L_0 is a $\mathscr{K}$-space.*

Proof It suffices to see that every quasilinear map $\Phi\colon L_0 \longrightarrow Y$ is approximable. We treat the real case. The crucial property of L_0 is that if U is a neighbourhood of zero, then there exist subspaces $X_1, \ldots, X_k \subset U$ such that $L_0 = X_1 \oplus \cdots \oplus X_k$. This is so because $|f|_0 \leq |\operatorname{supp} f|$, no matter which values

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f assumes. For instance, if $U = \{f : |f|_0 \leq \delta\}$, we may take a partition of $[0,1]$ into $k = \lceil \delta^{-1} \rceil$ subintervals $I_1, \ldots, I_k$ of measure at most δ and then write $L_0 = L_0(I_1) \oplus \cdots \oplus L_0(I_k)$. If $\Phi : L_0 \longrightarrow Y$ is quasilinear then there is $\delta > 0$ such that $\|\Phi(f+g) - \Phi(f) - \Phi(g)\| \leq 1$ for $|f|_0, |g|_0 \leq \delta$. Taking $I_1, \ldots, I_k$ as before and letting $\Phi_i = \Phi|_{L_0(I_i)}$ for $1 \leq i \leq k$, we now have $\|\Phi_i(f+g) - \Phi_i(f) - \Phi_i(g)\| \leq 1$ for *all* $f, g \in L_0(I_i)$. We shall show that each Φ_i is approximable, from which follows the same for Φ. Let us state and prove this fact separately:

★ Let $\Phi : X \longrightarrow Y$ be a quasilinear map, where X is a TVS and Y is a quasi-Banach space. Assume that there is $U \in \mathcal{O}_X$ such that $\|\Phi(x + x') - \Phi(x) - \Phi(x')\| \leq \varepsilon$ for some (possibly large) $\varepsilon > 0$ and all $x, x' \in U$. Then Φ is approximable.

Where does the approximating linear map come from? The following argument, due to Hyers, is a celebrity in certain circles: assuming that Y is a p-Banach space, given $x \in X$, we consider the sequence $(\Phi(2^n x)/2^n)_{n\geq 1}$. A straightforward induction argument yields $\|\Phi(2^n x) - 2^n \Phi(x)\|^p \leq (2^{pn} - 1)\varepsilon^p$ for all n. Thus, for $n, m \in N$, we have $\|\Phi(2^{n+m} x) - 2^m \Phi(2^n x)\|^p \leq 2^{pm} \varepsilon^p$. Dividing by 2^{n+m}, we obtain the estimate $\|\Phi(2^{n+m} x)/2^{n+m} - \Phi(2^n x)/2^n\| \leq \varepsilon/2^m$ so that $(\Phi(2^n x)/2^n)_{n\geq 1}$ is a Cauchy sequence and $\|\Phi(2^n x)/2^n - \Phi(x)\| \leq \varepsilon$. Put $L(x) = \lim_n \Phi(2^n x)/2^n$. Let us check that L is additive. Pick $x, y \in X$:

$$\|L(x+y) - L(x) - L(y)\| = \lim_n \left\| \frac{\Phi(2^n(x+y))}{2^n} - \frac{\Phi(2^n x)}{2^n} - \frac{\Phi(2^n y)}{2^n} \right\| \leq \lim_n \frac{\varepsilon}{2^n} = 0.$$

Wow! To complete the proof, we must see that L is linear (and not merely additive) and that $\Phi - L$ is continuous at zero. The second assertion is contained in:

★★ Let $\Phi : X \longrightarrow Y$ be a quasiadditive map, where X is a TVS and Y is a quasi-Banach space. If there is M such that $\|\Phi(x)\| \leq M$ for all $x \in X$ then Φ is continuous at zero.

Fix $\varepsilon > 0$ and take $U \in \mathcal{O}_X$ such that $\|\Phi(x + x') - \Phi(x) - \Phi(x')\| < \varepsilon$ for $x, x' \in U$. Choose n such that $M/2^n < \varepsilon$, and let $V \in \mathcal{O}_X$ such that $V + \cdots + V \subset U$ (2^n times). Induction on $k = 1, \ldots, n$ yields $\|\Phi(2^k x) - 2^k \Phi(x)\| \leq 2^k \varepsilon$ for all $x \in V$. In particular, $\|\Phi(2^n x)/2^n - \Phi(x)\| \leq \varepsilon$, and thus $\|\Phi(x)\| \leq 2^{1-1/p}(\|\Phi(x) - \Phi(2^n x)/2^n\| + \|\Phi(2^n x)/2^n\|) \leq 2^{2-1/p}\varepsilon$ provided $x \in V$. This already implies that L is linear since $\Phi - L$ is continuous at zero, L is quasihomogeneous and, in particular, for every $x \in X$, we have $L(tx) \to 0$ as $t \to 0$ in $\mathbb{R}$. Thus, the additive map $t \in \mathbb{R} \longmapsto L(tx) \in Y$ is continuous, and therefore it is real-linear [161, Lemma 3.1]. □

A still open problem, to which Domański was very attached in the 1980s, is whether Dierolf's theorem extends to the locally convex setting. Stated precisely, if X is a locally convex $\mathscr{K}$-space, must every topological extension of X by a locally convex space be locally convex? This problem is discussed in depth in the papers [160; 161], which contain a number of partial results.

3.13.2 A Cohomological Approach to Quasilinearity

The guiding idea for this chapter has been to use quasilinear maps $\Phi\colon X \longrightarrow Y$ to twist the topology of the product space $Y \times X$ while retaining the underlying linear structure. There is a classical procedure in group theory that proceeds the other way around. A homogeneous bounded mapping $\phi\colon X \times X \longrightarrow Y$ is a cocycle if, for every $x, x', x'' \in X$ and $\lambda \in \mathbb{K}$, one has $\phi(\lambda x, x) = 0$, $\phi(x, x') = \phi(x', x)$ and $\phi(x, x') - \phi(x, x' + x'') = \phi(x', x'') - \phi(x + x', x'')$. Keeping the sum quasinorm $\|(y, x)\| = \|y\| + \|x\|$ and the multiplication by scalars on $Y \times X$, we can define a new sum by the formula

$$(y, x) +_\phi (y', x') = \big(y + y' + \phi(x, x'), x + x'\big).$$

Using the cocycle properties of ϕ, we easily verify that this is a true sum (associative, commutative...) and satisfies the weak triangle estimate $\|(y, x) +_\phi (y', x')\| \leq M(\|(y, x)\| + \|(y', x')\|)$. In particular, the resulting quasinormed space $Y \times_\phi X$ is actually quasi-Banach since we have an isometrically exact sequence

$$0 \longrightarrow Y \xrightarrow{\ I\ } Y \times_\phi X \xrightarrow{\ Q\ } X \longrightarrow 0$$

where $I(y) = (y, 0)$ and $Q(y, x) = x$. The only non-trivial point here is to realise that Q is additive with respect to the new sum. Thus, each cocycle induces an extension. All extensions arise in this way: if $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is an extension and $B\colon X \longrightarrow Z$ is a bounded homogeneous section of the quotient map, then $\phi(x, x') = B(x) + B(x') - B(x + x')$ takes values in Y and is a cocycle and the sequence $0 \longrightarrow Y \longrightarrow Y \times_\phi X \longrightarrow X \longrightarrow 0$ is equivalent to the starting extension. Note that a homogeneous mapping $\Phi\colon X \longrightarrow Y$ is quasilinear if and only if $\phi(x, x') = \Phi(x) + \Phi(x') - \Phi(x + x')$ is a cocycle.

Unlike quasilinear maps, cocycles are very sensitive to the 'quality' of the bounded section $B\colon X \longrightarrow Z$ that generates them: for instance, it is clear that if B is continuous, uniformly continuous or Lipschitz, then so is ϕ. It follows from classical results of Michael that if Z is an F-space and $Y \subset Z$ is a locally convex, closed subspace, then the natural quotient map $Z \longrightarrow Z/Y$ admits a continuous section which can moreover be taken to be homogeneous when Z is a quasi-Banach space. At no point in this chapter have we taken advantage of this fact. We do not know if every quotient map between quasi-Banach spaces has a continuous section.

3.13.3 Table of Correspondences between Diagrams and Quasilinear Maps

<table>
<tr><td>Exact sequence $0 \to Y \to Z \to X \to 0$</td><td>Quasilinear map $\Phi\colon X \to Y$</td></tr>
<tr><td>Trivial (equivalent to direct sum sequence)</td><td>Trivial (bounded plus linear)</td></tr>
<tr><td>Equivalent sequences
$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Z_1 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Z_2 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$</td><td>$\Phi_2 - \Phi_1$ is trivial</td></tr>
<tr><td colspan="2">Let $S : X' \to X$ and $T\colon Y \to Y'$ be operators</td></tr>
<tr><td>Pushout
$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "T"'] & Z \arrow[r, "\rho"] \arrow[d, "\overline{T}"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & \mathrm{PO} \arrow[r, "\overline{\rho}"] & X \arrow[r] & 0 \end{tikzcd}$$</td><td>Left composition $T \circ \Phi$
$$\begin{tikzcd} Y \arrow[d, "T"'] & X \arrow[l, "\Phi"'] \\ Y' & \end{tikzcd}$$</td></tr>
<tr><td>Pullback
$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\jmath"] & Z \arrow[r] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\underline{\jmath}"] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u, "S'"] & X' \arrow[r] \arrow[u, "S"] & 0 \end{tikzcd}$$</td><td>Right composition $\Phi \circ S$
$$\begin{tikzcd} Y & X \arrow[l, "\Phi"'] \\ & X' \arrow[u, "S"'] \end{tikzcd}$$</td></tr>
<tr><td>Commutativity of pullback and pushout</td><td>Associativity of composition
$(S \circ \Phi) \circ T = S \circ (\Phi \circ T)$</td></tr>
<tr><td>Baer's sum</td><td>Pointwise sum</td></tr>
<tr><td>Diagonal pushout
$0 \longrightarrow Y \longrightarrow Y' \oplus Z \longrightarrow \mathrm{PO} \longrightarrow 0$</td><td>Composition $\Phi \circ \overline{\rho}$
$Y \leftarrow X \leftarrow \mathrm{PO}$</td></tr>
<tr><td>Diagonal pullback
$0 \longrightarrow \mathrm{PB} \longrightarrow X' \oplus Z \longrightarrow X \longrightarrow 0$</td><td>Composition $\underline{\jmath} \circ \Phi$
$\mathrm{PB} \leftarrow Y \leftarrow X$</td></tr>
<tr><td>Exact sequences of p-Banach spaces</td><td>p-linear maps</td></tr>
<tr><td>Dual exact sequence of Banach spaces</td><td>Dual 1-linear map</td></tr>
<tr><td>
$$\begin{tikzcd} 0 \arrow[r] & \cdot \arrow[r] \arrow[d, "\alpha"'] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & 0 \end{tikzcd}$$</td><td>$\alpha\Omega \sim \Phi\gamma$</td></tr>
</table>

Sources

Quasilinear techniques burst into Banach space theory in Enflo, Lindenstrauss and Pisier's paper [167], where a quasilinear map is used for the first time to construct a twisted Hilbert space, thus solving a problem they attributed to Palais. Incidentally, and according to Pietsch [387], Palais was unaware that a 3-space problem had been associated with his name. It is clear from [388] that the idea of using a non-linear map to construct an extension of Banach spaces is due to Lindenstrauss.

Enough mathematical gossip. The connection between quasilinearity and the twisted sums in [167] is provided by stipulating that the unit ball of the norm of $Y \times X$ has to be the convex hull of the set $\{(y, 0) \colon y \in B_Y\} \bigcup \{(\Phi(x), x) \colon x \in B_X\}$, which yields the formula

$$|(y, x)|_\Phi = \inf \left\{ \sum_i \|y_i\| + \|x_i\| : x = \sum_i x_i, y = \sum_i \Phi(x_i) + \sum_i x_i \right\}.$$

Kalton [251] adapted this construction for p-Banach spaces. Ribe's paper [401], from where the construction in Section 3.2 is taken, was an important advance in the area. In fact, the clean formula $\|y - \Omega x\| + \|x\|$ for the quasinorm appeared there for the first time and was quickly adopted by Kalton and Peck [280] and has been widely used ever since. The Kalton–Peck construction can be exploited with different levels of depth and generality. A complete account of Kalton's findings, first with Peck and subsequently solo, requires much more time and space than these comments to uncover the wonderful connections between centralizers and complex interpolation theory. In this chapter, we dealt with the simplest of those levels. Most of Sections 3.2 and 3.3 are from [280]; Kalton's map [251] that solves the 3-space problem for local convexity also admits a 'centralizer version' on each ℓ_p that provides a non-trivial twisted Hilbert space; but it is only in the theory of centralizers that it finds its home. The first part of Proposition 3.3.5 appeared in [78]. Although Corollary 3.4.4 makes the study of locally convex $\mathscr{K}$-spaces especially rewarding, the notion of a $\mathscr{K}$-space was motivated by non-locally convex considerations. Actually, $\mathscr{K}$-spaces were introduced by Kalton and Peck in [281] to show that L_p is not isomorphic to its quotient by a line when $0 \leq p < 1$. As far as we know, this was the first time that a homological invariant was used to distinguish between two quasi-Banach spaces. In truth, the paper does not contain the full proof that L_p is a $\mathscr{K}$-space when $0 < p < 1$, but instead the fact that every minimal extension $0 \longrightarrow \mathbb{K} \longrightarrow Z \longrightarrow L_p \longrightarrow 0$ in which Z is a p-Banach space splits. Needless to say, the quotient of L_p by a line (or by any subspace Y with non-trivial dual) cannot have that property, as the sequence

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$0 \longrightarrow Y \longrightarrow L_p \longrightarrow L_p/Y \longrightarrow 0$ shows. The material of Section 3.6, and in particular the ‘uniform boundedness principle for quasilinear maps’, is due to Kalton [251] and enhanced with Ribe’s formula. Section 3.9 is an adaptation of [75], from where the construction of the spaces $\mathrm{co}^{(p)}(\cdot)$ was taken too. The first two parts of Section 3.4 are basically as in Kalton–Peck [281]. The third part is from [251]. The treatment of Section 3.8 follows [65]; however, Lemma 3.8.2 appeared in [112], and the computations leading to the identification of the duals of the Kalton–Peck spaces are taken from the paper of their legitimate owners. Theorem 3.11.2 is taken from [252], where Kalton performs a rather complete study of the type of twisted sums: he considers both the case where the subspace has better type than the quotient space, which corresponds to Theorem 3.11.2, and also the reverse situation, with similar conclusions (twisted sums retain the type of the ‘worst’ summand, although its proof is different). He also considers the subtler case in which the summands have the same type. The corollary in Note 3.13.1 is due to Kalton and Peck [281, Theorem 3.6], although the proof we present is taken from [60]. The observations on cocycles are a straightforward adaptation of [436], which is in turn based on a classical construction in group theory that can be found in [53, Chapter IV]. The paper [312] describes the cocycles acting between Banach spaces that produce locally convex extensions.

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