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SOME REMARKS ON THE RIEMANN ZETA FUNCTION AND PRIME FACTORS OF NUMERATORS OF BERNOULLI NUMBERS

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Abstract

We prove that the sequence $\{\log \zeta(n)\}_{n\geq 2}$ is not holonomic, that is, does not satisfy a finite recurrence relation with polynomial coefficients. A similar result holds for L-functions. We then prove a result concerning the number of distinct prime factors of the sequence of numerators of even indexed Bernoulli numbers.

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A sequence $\{u_n\}_{n\geq 1}$ is called holonomic if there exist $k \geq 1$ and $k+1$ polynomials $p_0(X), \ldots, p_k(X) \in \mathbb{C}[X]$ not all zero such that the relation

$$\sum_{j=0}^{k} p_j(n) u_{n+j} = 0$$

holds for all $n \geq 0$. Let $\zeta(s)$ be the Riemann zeta function defined as

$$\zeta(s) = \sum_{m\geq 1} \frac{1}{m^s} \quad \text{for all real } s > 1.$$

In [2], it is proved that the sequence $\{\zeta(n)\}_{n\geq 2}$ is not holonomic. The method is very general and extends to other sequences such as the sequence of values at positive integers of an L-function associated to a character χ. Let us state this result.

Theorem 1. *Let $N \geq 2$ be a positive integer and let χ be a character modulo N. Let $a \geq 1$ and $b \geq 0$ be integers. Then the sequence $\{L(\chi, an+b)\}_{n\geq 2}$ is not holonomic.*

Let us go quickly through this proof. We follow the proof of Theorem 15 in [2].

Proof. For typographical convenience, we assume that $(a, b) = (1, 0)$. Suppose that there exist $k \geq 1$ and polynomials $p_j(X)$ for $j = 0, \ldots, k$ with real coefficients such

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that the relation

$$\sum_{j=0}^{k} p_j(n)L(\chi, n+j) = 0$$

holds for all integers $n \geq 2$. Let D be an upper bound for all the degrees of $p_j(X)$ for $j = 0, \ldots, k$. We show that $p_j(X)$ is the zero polynomial for all $j = 0, \ldots, k$. By the estimate

$$\Big|\sum_{\ell \geq L} \frac{\chi(\ell)}{\ell^s}\Big| \leq \sum_{l \geq L} \frac{1}{\ell^s} \leq \frac{1}{L^s} + \int_L^\infty \frac{dt}{t^s} = O\Big(\frac{1}{L^s}\Big),$$

we find

$$\sum_{j=0}^{k} p_j(n) = -\sum_{j=0}^{k} p_j(n) \sum_{\ell \geq 2} \frac{\chi(\ell)}{\ell^{n+j}} = O\Big(\frac{n^D}{2^n}\Big) = o(1)$$

as $n \to \infty$, which implies that $\sum_{j=0}^{k} p_j(X) = 0$. We iterate this argument as follows. Let $1 = m_0 < m_1 < \cdots$ be all the positive integers which are coprime to N. Then $\chi(m) \neq 0$ if and only if $m = m_u$ for some nonnegative integer u. From

$$\sum_{j=0}^{k} p_j(n)\Big(1 + \frac{\chi(m_1)}{m_1^{n+j}}\Big) = \chi(m_1) \sum_{j=0}^{k} \frac{p_j(n)}{m_1^{n+j}} = -\sum_{j=0}^{k} p_j(n) \sum_{\ell \geq m_2} \frac{\chi(\ell)}{\ell^{n+j}},$$

together with the fact that $|\chi(m_1)| = 1$, we obtain

$$\Big|\sum_{j=0}^{k} \frac{p_j(n)}{m_1^j}\Big| = -m_1^n \Big|\sum_{j=0}^{k} p_j(n) \sum_{\ell \geq m_2} \frac{\chi(\ell)}{\ell^{n+j}}\Big| = O\Big(n^D \Big(\frac{m_1}{m_2}\Big)^n\Big) = o(1)$$

as $n \to \infty$, so that $\sum_{j=0}^{k} p_j(X)/m_1^j = 0$. Continuing this argument, we get that

$$\sum_{j=0}^{k} \frac{p_j(X)}{m_u^j} = 0 \quad \text{for all } u \geq 0.$$

Taking $u = 0, 1, \ldots, k$, we arrive at the conclusion that $(p_0(X), \ldots, p_k(X))^T$ is in the kernel of the linear map with associated matrix $(1/m_u^j)_{0 \leq u, j \leq k}$ whose determinant is Vandermonde (hence, nonzero), so $p_j(X) = 0$ for all $j = 0, \ldots, k$, a contradiction. □

REMARK 2. As in [2, Theorem 15], the same argument gives that $\{L(\chi, a_n)\}_{n \geq 0}$ is not holonomic for any increasing sequence $\{a_n\}_{n \geq 1}$ of integers greater than or equal to 2 having *bounded gaps*, that is, for which the estimate $a_{n+1} - a_n = O(1)$ holds.

Next, we prove that the same conclusion holds for $\{\log L(\chi, an + b)\}_{n \geq 2}$.

THEOREM 3. *Let $N \geq 2$ be a positive integer and let χ be a character modulo N. Let $a \geq 1$ and $b \geq 0$ be integers. Then the sequence $\{\log L(\chi, an + b)\}_{n \geq 2}$ is not holonomic.*

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PROOF. Again, for notational simplicity, we assume that $(a, b) = (1, 0)$. Suppose that there exist $k \geq 1$ and polynomials $p_j(X)$ for $j = 0, \dots, k$ with real coefficients such that the relation

$$\sum_{j=0}^{k} p_j(n) \log L(\chi, n+j) = 0$$

holds for all integers $n \geq 2$. Let D be an upper bound for all the degrees of $p_j(X)$ for $j = 0, \dots, k$. We show that $p_j(X)$ is the zero polynomial for all $j = 0, \dots, k$. Using the Euler product representation of $L(\chi, n)$, we have

$$\log L(\chi, n) = -\sum_{p \geq 2} \log\Big(1 - \frac{\chi(p)}{p^n}\Big) = \sum_{a \geq 1, p \geq 2} \Big(\frac{\chi(p)^a}{a}\Big) \frac{1}{p^{an}}.$$

Let $p_1 < p_2 < \cdots$ be all the primes that do not divide N and let $m_1 < m_2 < \cdots$ be the increasing sequence of all the numbers of the form p_i^a for some $i \geq 1, a \geq 1$. If $m_u = p_i^a$, we then put $c_u := \chi(p_i)^a/a$. Note that $c_u \neq 0$ for all $u \geq 1$. We then have

$$\sum_{j=0}^{k} p_j(n) \frac{c_1}{m_1^{n+j}} = -\sum_{j=0}^{k} p_j(n) \sum_{u \geq 2} \frac{c_u}{m_u^{n+j}},$$

so

$$|c_1| \bigg| \sum_{j=0}^{k} p_j(n) \frac{1}{m_1^j} \bigg| = m_1^n \bigg| \sum_{j=0}^{k} p_j(n) \sum_{u \geq 2} \frac{c_u}{m_u^{n+j}} \bigg| = O\Big(n^D \Big(\frac{m_1}{m_2}\Big)^n\Big) = o(1)$$

as $n \to \infty$, which implies that $\sum_{j=0}^{k} p_j(X)/m_1^j = 0$. Continuing in this way, we get, as in the proof of Theorem 1, that

$$\sum_{j=0}^{k} \frac{p_j(X)}{m_u^j} = 0 \quad \text{for all } u \geq 1.$$

Taking $u = 1, 2, \dots, k+1$, we get again that $(p_0(X), \dots, p_k(X))$ is a zero of a nondegenerate linear system of $k+1$ equations, so $p_j(X) = 0$ for all $j = 0, \dots, k$, which is a contradiction. □

Theorem 3 shows that $\{\log \zeta(2n)\}_{n \geq 1}$ does not satisfy any finite-order linear recurrence. In particular, there are no $k \geq 1$ and integer exponents $a_0, \dots, a_k$ not all zero such that the multiplicative relation

$$\prod_{j=0}^{k} \zeta(2n + 2j)^{a_j} = 1 \tag{1}$$

holds for all sufficiently large n. While we have shown that a nontrivial relation of the form (1) cannot hold for all sufficiently large n, this does not exclude the possibility that some relations of the form (1) hold for some particular values of n, k

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and $a_0, \dots, a_k$. We could not find any such multiplicative combinations, but, allowing some special values of L-functions, we did find the relation

$$L(\chi_3, 1)^4 L(\chi_4, 1)^{-2} \zeta(2)^4 \zeta(4)^{-5} \zeta(6)^{-5} \zeta(8)^5 = 1,$$

where χ_3 and χ_4 are the only nonprincipal characters modulo 3 and 4, respectively. Via the formula

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}, \tag{2}$$

where B_{2n} is the Bernoulli number, we get that the existence of multiplicative relations of the form (1) is driven by the number of distinct prime factors of the numerators and denominators of the Bernoulli numbers B_{2n}. We write $B_{2n} = (-1)^{n+1} C_n / D_n$, with coprime positive integers C_n and D_n. The prime factors of D_n are well understood by the von Staudt–Clausen theorem, which asserts that D_n is squarefree and its prime factors p are precisely the ones for which $p - 1 \mid 2n$. In what follows, we give a result about the prime factors of the numerators C_n.

For a positive integer m, let $\omega(m)$ be the number of distinct prime factors of m.

Theorem 4. *The estimate*

$$\omega\Big(\prod_{n=1}^{N} C_n\Big) \geq (1 + o(1)) \frac{\log N}{\log \log N} \tag{3}$$

holds as $N \to \infty$.

Proof. We shall use the formula (2) under the form

$$C_n = 2D_n (2n)! (2\pi)^{-2n} \zeta(2n) = 2D_n (2n)! (2\pi)^{-2n} \Big(1 + O\Big(\frac{1}{2^{2n}}\Big)\Big).$$

Taking logarithms, we get

$$\log C_n = \log(2D_n) + \log(2n)! - 2n \log(2\pi) + O\Big(\frac{1}{2^{2n}}\Big).$$

We evaluate the above formula in $n, n + 1, n + 2$ for some $n \in (N/2 + 2, N - 6)$, where N is large, and take the second difference of the resulting relations, getting

$$\log\Big(\frac{C_n C_{n+2}}{C_{n+1}^2}\Big) - \log\Big(\frac{D_n D_{n+2}}{D_{n+1}^2}\Big) - \log\Big(\frac{(2n + 3)(2n + 4)}{(2n + 1)(2n + 2)}\Big) = O\Big(\frac{1}{2^N}\Big). \tag{4}$$

We take $n = p - 2$ in the relation (4). Since $p \| D_{n+1}$, p does not divide $D_n D_{n+2}$ and

$$\frac{(2n + 3)(2n + 4)}{(2n + 1)(2n + 2)} = \frac{p(2p - 1)}{(2p - 3)(p - 1)},$$

it follows that the rational number

$$\frac{D_n D_{n+2} (2n + 3)(2n + 4)}{D_{n+1}^2 (2n + 1)(2n + 2)} \tag{5}$$

has the prime p appearing in its denominator.

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We put $K := \omega(\prod_{n\le N} C_n)$ and assume that $K \le \log N$, for if not there is nothing to prove. By sieve methods, there exist positive constants c_1, c_2 and N_0 such that for $N > N_0$ there are at least $c_1 N/(\log N)^3$ primes $p \in (N/2, N-8)$ which are congruent to 1987 (mod 2310) such that the smallest prime factor of both $(p-1)/6$ and $(p-2)/5$ exceeds N^{c_2} (see [4, Theorem 2.6′, p. 87]). We take N_0 so large such that $c_1 N/(\log N)^3 > 2\log N > 2K$ for $N > N_0$. Then there exist $K+1$ distinct primes $p_1, \dots, p_{K+1}$ in $(N/2, N-8)$ which do not divide any of the numbers C_n for $n \le N$ and such that for each one of these primes p we have that the smallest prime factor of both $(p-1)/6$ and $(p-2)/5$ exceeds N^{c_2}. We evaluate the relation (4) in $n = p_i - 2$ for $i = 1, \dots, K+1$. Since

$$\max\{\Omega(2n), \Omega(2n+2), \Omega(2n+4)\} \le 3 + c_2^{-1} =: c_3$$

for all $n = p_i - 2$ with $i = 1, \dots, K+1$, it follows that each of the numbers $2n, 2n+2, 2n+4$ can have at most $c_4 := 2^{c_3}$ divisors of the form $p-1$ for some prime p. This shows, via the von Staudt–Clausen theorem, that

$$\max\{\log D_n, \log D_{n+1},\ \ \log D_{n+2}\} = O(\log N)$$

for all $n = p_i - 2$ and $i = 1, \dots, K+1$. Hence, putting E_i for the rational number shown in (5) for $n = p_i - 2$, we get that its logarithmic height, which for a nonzero rational number $r = a/b$ with coprime integers a and b is defined as $h(r) := \max\{\log |a|, \log |b|\}$, satisfies

$$h(E_i) \le \max\{\log(D_n D_{n+2}(2n+4)^2), \log(D_{n+1}^2(2n+2)^2)\} < c_5 \log N \tag{6}$$

for some suitable constant c_5. Now let us assume that $Q = \{q_1, \dots, q_K\}$ is the set of all the prime factors of $\prod_{m\le N} C_m$. Write

$$\frac{C_{p_i-2} C_{p_i}}{C_{p_i-1}^2} = \prod_{j=1}^{K} q_j^{a_{i,j}}.$$

Then the relation (4) for $n = p_i - 2$ is

$$\left|\sum_{j=1}^{K} a_{i,j} \log q_j - \log E_i\right| = O\Big(\frac{1}{2^N}\Big). \tag{7}$$

From the remark following (5), p_i divides the denominator of E_i and $p_i \notin Q$, so the expressions appearing on the left-hand side of (7) are nonzero for $i = 1, \dots, K+1$. Moreover, for varying $i = 1, \dots, K+1$, the expressions appearing on the left-hand side of (7) are linear forms in $\{\log q_j : j = 1, \dots, K\} \cup \{\log E_i : i = 1, \dots, K+1\}$, which are linearly independent. To see why, we claim that the number p_i, which divides the denominator of E_i, divides neither the numerator nor the denominator of any other E_k for $k \ne i$ in $\{1, \dots, K+1\}$. Indeed, assume that this were not true.

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First, since $4p_i > 2N > 2n+4$ for all $n < N-8$, we get that if one of the numbers $2n+1, 2n+2, 2n+3, 2n+4$ is a multiple of p_i, then it must be p_i, $2p_i$ or $3p_i$. Hence, we get equations of the form

$$2n+\delta = \lambda p_i \quad \text{with } \lambda \in \{1,2,3\} \text{ and } \delta \in \{1,2,3,4\}.$$

There are 12 possible pairs (λ, δ) leading to 12 possible equations. Only six of them can actually occur, since by parity reasons we must have $\delta \equiv \lambda \pmod 2$ and, of the six possible equations, one of them is the trivial one with $(\lambda, \delta) = (2, 4)$ for which $n = p_i - 2$. The remaining ones are

$$n = \frac{p_i-1}{2}, \quad \frac{p_i-3}{2}, \quad p_i-1, \quad \frac{3p_i-1}{2}, \quad \frac{3p_i-3}{2}.$$

Putting $n = p_k - 2$ for some $k \neq i$, we get

$$p_k = \frac{p_i+3}{2}, \quad \frac{p_i+1}{2}, \quad p_i+1, \quad \frac{3p_i+3}{2}, \quad \frac{3p_i+1}{2}.$$

None of these is possible, since by the way we have chosen the primes p_i, the numbers from the above list are, from left to right, multiples of 5, 7, 2, 3 and 11, respectively. However, it could still be the case that p_i divides one of D_n, D_{n+1} or D_{n+2} for some $n \neq p_i - 2$. This is possible only if $p_i - 1$ divides one of $2n, 2n+2, 2n+4$. Since $4(p_i-1) > 2N-4 > 2n+4$ for all $n < N-8$, it follows that if one of $2n, 2n+2, 2n+4$ is a multiple of $p_i - 1$, then it must be one of p_i-1, $2(p_i-1)$ or $3(p_i-1)$. So, again we get equations of the form

$$2n+\delta = \lambda(p_i-1) \quad \text{with } \delta \in \{0,2,4\} \text{ and } \lambda \in \{1,2,3\}.$$

This leads to a totality of nine equations of which one is the trivial one corresponding to $(\lambda,\delta) = (2,2)$ for which $n = p_i - 2$. Of the remaining ones, we must have $n = p_k - 2$ for some $k \neq i$. The options $(\lambda, \delta) = (2, 0)$ or $(2, 4)$ are not possible by parity reasons, while the other six lead to

$$p_k = \frac{p_i+3}{2}, \quad \frac{p_i+1}{2}, \quad \frac{p_i-1}{2}, \quad \frac{3p_i+1}{2}, \quad \frac{3p_i-1}{2}, \quad \frac{3p_i-3}{2}.$$

Again, none of the above relations is possible, since from the way we have chosen the primes p_i, in the above list, the numbers from left to right are divisible by 5, 7, 3, 11, 5 and 3, respectively. Hence, the forms appearing on the left-hand sides of (7) are linearly independent for $i = 1, \ldots, K+1$. Since

$$C_n < 2\zeta(2)D_n(2n)! < 4n^{c_4}(2n)! < n^{2n} < N^{2N}$$

for all sufficiently large N, it follows that $a_{i,j} = O(N \log N)$ for all $i = 1, \ldots, K+1$ and $j = 1, \ldots, K$.

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Let $(\Delta_1, \ldots, \Delta_{K+1})$ be a nonzero vector in the null-space of the $K \times (K+1)$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{K+1,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{K+1,2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,K} & a_{2,K} & \cdots & a_{K+1,K} \end{pmatrix}.$$

One such nonzero vector can be computed with Cramer's rule and its size satisfies

$$\max\{|\Delta_i| : i = 1, \ldots, K+1\} \leq (K+1)! \max\{|a_{i,j}|\}^K < N^{2K} \tag{8}$$

for $N > N_0$. More precisely, let $r \leq K$ be the rank of A and, up to rearranging some of its rows and columns, assume that the $r \times r$-subdeterminant appearing in the upper-left corner of A is nonzero and has the value $\boldsymbol{\Delta}$. Then by Cramer's rule, $\Delta_1, \ldots, \Delta_r$ are linear combinations of $\Delta_{r+1}, \ldots, \Delta_{K+1}$ with rational coefficients the denominators of which are $\boldsymbol{\Delta}$. Thus, taking say $\Delta_{r+1} = \cdots = \Delta_{K+1} = \boldsymbol{\Delta}$, we get that $\Delta_1, \ldots, \Delta_r$ are integers and the inequality (8) is satisfied. As the referee observed, we may invoke some result from the geometry of numbers, such as Minkowski's convex body theory or Siegel's lemma, to conclude that an estimate of the shape of (8) holds, but, as we have just explained above, classical linear algebra suffices.

Then taking the linear combination of the relations (7) with coefficients Δ_i for $i = 1, \ldots, K+1$, we get

$$\left|\sum_{i=1}^{K+1} \Delta_i \log E_i\right| = O\left(\frac{(K+1)\max\{|\Delta_i|\}}{2^N}\right) = O\left(\frac{1}{2^{N/2}}\right). \tag{9}$$

The linear form on the left-hand side of (9) above is nonzero. We apply a result of Matveev (see [5] or [3, Theorem 9.4]) to bound from below the expression appearing on the left-hand side of the estimate (9) above by

$$\exp(-1.4 \times 30^{K+4}(K+1)^{4.5}(1+\log B)A_1 \cdots A_{K+1}),$$

where we can take $B \geq \max\{|\Delta_i| : i = 1, \ldots, K+1\}$ and $A_i \geq h(E_i)$ for all $i = 1, \ldots, K+1$. Thus, we can take $A_i := c_5 \log N$ for all $i = 1, \ldots, K+1$ (see (6)) and $B := N^{2K}$ (see (8)) and now the inequality (9) gives

$$c_6 N - c_7 < 1.4 \times 30^{K+4}(K+1)^{4.5}(1 + 2K\log N)(c_5 \log N)^{K+1}$$

with $c_6 := (\log 2)/2$ and some suitable constant c_7, which implies immediately the estimate (3). □

Unfortunately, our inequality (3) is too weak to yield any meaningful conclusion regarding multiplicative independence among the values of $\zeta(2n)$ for $n = 1, 2, \ldots$. As for the values $\{\zeta(2n+1)\}_{n\geq 1}$, the situation is even less understood. As far as

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linear independence relations over $\mathbb{Q}$ among the values of $\zeta(2n+1)$ for varying n are concerned, by [1], it is known that if $N > N_0$, then

$$\dim_{\mathbb{Q}}(\mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(5) + \cdots + \mathbb{Q}\zeta(2N+1)) > c_8 \log N,$$

where one can take $c_8 := 1/8$. However, we are not aware of any result regarding the multiplicative independence of $\zeta(2n+1)$ for $n = 1, 2, \ldots$. We leave the following problem to the reader.

PROBLEM 5. Prove that the $\mathbb{Q}$-linear space

$$\mathbb{Q} \log \zeta(3) + \mathbb{Q} \log \zeta(5) + \cdots + \mathbb{Q} \log \zeta(2N+1) + \cdots$$

is infinite dimensional.

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References

[1] K. Ball and T. Rivoal, ‘Irrationalité d’une infinité de valeurs de la fonction zeta aux entiers impairs’, *Invent. Math.* **146** (2001), 193–207.

[2] J. P. Bell, S. Gerhold, M. Klazar and F. Luca, ‘Non-holonomicity of sequences defined via elementary functions’, *Ann. Comb.* **12** (2008), 1–16.

[3] Y. Bugeaud, M. Mignotte and S. Siksek, ‘Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers’, *Ann. of Math.* (2) **163** (2006), 969–1018.

[4] H. Halberstam and H.-E. Richert, *Sieve Methods* (Academic Press, London, 1974).

[5] E. M. Matveev, ‘An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II’, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180 English translation *Izv. Math.* **64** (2000), 1217–1269.

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