Second Meeting, December 10th, 1897.

J. B. Clark, Esq., M.A., F.R.S.E., President, in the Chair.

On some questions in Arithmetic.

By Prof. Steggall.

Note on the Transformations of the Equations of Hydrodynamics.

By H. S. Carslaw, M.A., Glasgow and Cambridge.

This summer there came into my hands a copy of the spring issue of the *Mittheilungen der Math. Gesellschaft in Hamburg* containing a paper on the "Transformationen der hydrodynamischen Gleichungen mit Berücksichtigung der Reibung."

On examination, I found embodied in the somewhat lengthy communication practically the following method, which I had entered in my notes three years ago when working at the subject. Thinking that it was bound to have been used earlier, I simply preserved it, as likely to prove useful if I were ever called upon to teach Hydrodynamics.

The fact of the aforesaid paper being afforded a prominent place in that German journal prompts me to submit this note to the Society. If the method is new, its publication in English seems not uncalled for.

§ 1.

The equations of motion in a viscous liquid, with regard to fixed rectangular axes, are accepted as

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u,$$

and two others,

where $\dfrac{Df}{Dt}$ stands for $\left(\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z}\right)f.$

Now in this case $\nabla^2 u = 2\left\{\dfrac{\partial\eta}{\partial z} - \dfrac{\partial\zeta}{\partial y}\right\}$

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ξ, η, ζ being the components of molecular rotation at the point (x, y, z).

Thus we have our equations in the form,

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + 2\nu\left\{\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}\right\},$$

in which, since (ξ, η, ζ) are the components of a vector, we are able easily to transform to any system of orthogonal axes, and as special cases we have the spherical polar and the cylindrical systems.

§ 2.

Taking the system defined by

$$ds^2 = h_1{}^2 d\alpha^2 + h_2{}^2 d\beta^2 + h_3{}^2 d\gamma^2,$$

let the axes of (x, y, z) coincide with those of (α, β, γ) for the point (α, β, γ).

Then we have merely to consider the alteration in our equations when the (u, v, w) at $(\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$ are measured along the (α, β, γ) axes there.

If we are dealing with vectors this only involves the determination of the infinitesimal rotations of the axes.

As usual, we have

$$d\theta_3 = \frac{1}{h_1}\frac{\partial h_2}{\partial \alpha} \cdot d\beta - \frac{1}{h_2}\frac{\partial h_1}{\partial \beta} \cdot d\alpha,$$

and two other equations.

Then for any vector with components U, V, W, along the axes,

$$\delta U = dU - V d\theta_3 + W d\theta_2,*$$

giving

$$\left(\frac{\partial U}{\partial x}\right) = \frac{1}{h_1}\frac{\partial U}{\partial \alpha} + \frac{V}{h_1 h_2}\frac{\partial h_1}{\partial \beta} + \frac{W}{h_1 h_2}\frac{\partial h_1}{\partial \gamma},$$

$$\left(\frac{\partial U}{dy}\right) = \frac{1}{h_2}\frac{\partial U}{\partial \beta} - \frac{V}{h_1 h_2}\frac{\partial h_2}{\partial \alpha},$$

$$\left(\frac{\partial U}{\partial z}\right) = \frac{1}{h_3}\frac{\partial U}{\partial \gamma} - \frac{W}{h_1 h_3}\frac{\partial h_3}{\partial \alpha}.$$

§ 3.

We have now to apply this to the case in question.

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* For this and the first results in the next paragraph, see § 121 of *Love's Elasticity*, Vol. I.

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Considering (u, v, w),

$$2\zeta = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial\alpha}\cdot(h_2 v) - \frac{\partial}{\partial\beta}(h_1 u)\right\}.$$

Likewise from (ξ, η, ζ)

$$\frac{\partial\eta}{\partial z} - \frac{\partial\zeta}{\partial y} = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial\gamma}(\eta h_2) - \frac{\partial}{\partial\beta}(\zeta h_3)\right\}$$

Remembering the meaning of $\frac{\mathrm{D}u}{\mathrm{D}t}$, we find

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + \frac{1}{h_1}u\frac{\partial u}{\partial\alpha} + \frac{1}{h_2}v\frac{\partial u}{\partial\beta} + \frac{1}{h_3}w\frac{\partial u}{\partial\gamma}$$

$$-\frac{v}{h_1 h_2}\left\{v\frac{\partial h_2}{\partial\alpha} - u\frac{\partial h_1}{\partial\beta}\right\} + \frac{w}{h_1 h_3}\left\{u\frac{\partial h_1}{\partial\gamma} - w\frac{\partial h_3}{\partial\alpha}\right\},$$

and our first equation of motion becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \mathrm{X} - \frac{1}{\rho h_1}\frac{\partial p}{\partial\alpha} + \frac{2\nu}{h_2 h_3}\left\{\frac{\partial}{\partial\gamma}(\eta h_2) - \frac{\partial}{\partial\beta}(\zeta h_3)\right\}.$$

The other two follow in the same way.

§ 4.

The advantage of this method seems to consist in its straightforwardness and also in the fact that we are almost bound to have had already before us the expression for (ξ, η, ζ) in the general coordinates. In the spherical polar coordinates they are very easily calculated, and the transformation to the (r, θ, ϕ) system is thus a simple one.

To deduce the equations for that case we have to take

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta.$$

This gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial\phi} - \frac{v^2 + w^2}{r}$$

$$= \mathrm{R} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{2\nu}{r^2\sin\theta}\left\{\frac{\partial}{\partial\phi}(\eta r) - \frac{\partial}{\partial\theta}(\zeta r\sin\theta)\right\}$$

$$= \mathrm{R} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{2\nu}{r\sin\theta}\left\{\frac{\partial\eta}{\partial\phi} - \frac{\partial}{\partial\theta}(\zeta\sin\theta)\right\}$$

Similar equations result for the other two directions, and these can be reduced to depend only on (u, v, w) by substituting for (ξ, η, ζ) their values as given above.