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COMPACT SETS IN $C_p(X)$ AND CALIBERS

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ABSTRACT. This presentation concerns the relation of chain conditions on a space X, with the weights of compact sets in $C_p(X)$, generalizing up to the class of $d\sigma$-bounded spaces, or stable spaces. In the last case, stronger results are obtained for Corson compact subsets of $C_p(X)$.

1. **Introduction.** All the spaces under consideration are assumed to be Tychonoff. Notations, terminology and cardinal inequalities left unexplained, could be found in [1] and [6]. If X is a space, then $C_p(X)$ is the space of all continuous real-valued functions with the topology of pointwise convergence and $C_p^*(X) = \{f \in C_p(X) : f \text{ is bounded}\}$. It is clear, that the family of sets $V(x; G) = \{f \in C_p(X) : f(x) \in G\}$ where G is open in $\mathbb{R}$, is an open subbase of $C_p(X)$.

For any cardinal function φ we put $h\varphi = \sup\{\varphi(Y) : Y \text{ is a subspace of } X\}$ and $h\varphi$ is called the *hereditary version* of φ.

Let A be an index set and $\mathbb{R}^A$ the usual product of $|A|$ real lines. We set $\Sigma_*(|A|) = \big\{f \in \mathbb{R}^A : \{a \in A : |f(a)| \geq \varepsilon\} \text{ is finite for every } \varepsilon > 0\big\}$ and $\Sigma(|A|) = \big\{f \in \mathbb{R}^A : |\{a \in A : f(a) \neq 0\}| \leq \omega\big\}$.

A compact space X is *Eberlein (Corson) compact* if and only if X is homeomorphic to a compact subspace of $\Sigma_*(|A|)$ $\big(\Sigma(|A|)\big)$. It is apparent, that every Eberlein compact space is Corson compact.

A supersequence is the one-point compactification of any infinite discrete space. We put $\alpha(X) = \sup\{\tau : \text{there is a supersequence } Y \text{ in } X, \text{ such that } |Y| = \tau\}$. It is known (see [5]) that $\Sigma_*(\tau)$ is homeomorphic to $C_p(A)$ for every supersequence A, $|A| = \tau$, where $\Sigma_*(\tau) = \Sigma_*(|A|)$.

The cardinal $\min\{\tau : \tau^+ \text{ is a caliber of } X\}$ is denoted by $\mathrm{sh}(X)$ and the point finite cellularity of X, by $p(X)$.

A space X is σ-pseudocompact (σ-bounded), if X is the union of countably many pseudocompact (bounded) subsets. ∎

It is well known the fact proved by Arkhangel'skii (see [3]), that the Suslin number of any compact space X is the least upper bound of the weights of compact sets lying in $C_p(X)$. But when F is a compact subset of $C_p(X)$, where X is pseudocompact, F can be considered, using arguments of [3], as a subset of $C_p(\beta X)$ where $c(X) = c(\beta X)$, obtaining this way the following:

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PROPOSITION 1. *For every pseudocompact space X, $c(X) = \sup\{w(F) : F$ is compact set in $C_p(X)\}$.* ■

REMARK 1. Let X be a non-metrizable Eberlein compact space. Then, after Proposition 7.1 of [3], $C_p(X)$ contains a dense and obviously with countable cellularity σ-compact subspace Y. Since X embeds in $C_p(Y)$, if the above proposition was valid for σ-compact spaces, the space X would be metrizable contradicting the hypothesis. Below, other "stronger" cardinal functions appear as upper bounds for the weights of compact sets in $C_p(X)$, when X is $d\sigma$-pseudocompact ($d\sigma$-bounded), *i.e.* contains a dense σ-pseudocompact (σ-bounded) subspace. ■

REMARK 2. We cannot extend Proposition 1 to pseudocompact subsets of $C_p(X)$. Indeed, let X be a Šakhmatov space (X is infinite), *i.e.* a pseudocompact space where all countable subspaces are closed and C^*-embedded. Then $C_p(X, I)$ is pseudocompact, where I is the closed unit interval of the real line, has a countable cellularity and does not have a G_δ diagonal ([8]). In view of the fact that X is embedded in $C_pC_p(X, I)$, if $w(X) \leq c\big(C_p(X, I)\big)$, then X would be compact and metrizable. But, if X is (infinite) compact and metrizable, then X cannot be a Šakhmatov space. ■

COROLLARY 1.1. *For every pseudocompact space X, $p(X) = c(X)$.*

PROOF. It is known ([2]) that for every space X, $p(X) = \alpha\big(C_p(X)\big)$. Let now $p(X) = \tau$. It is immediate from Proposition 1, that $c(X) \geq \tau$. The reverse inequality is obvious. ■

COROLLARY 1.2. *Consider the pseudocompact spaces X, Y and a continuous, 1-1, function θ from $C_p(X)$ into $C_p(Y)$. If Y satisfies τ. c. c, where $\tau > \omega$, then so does X.* ■

We may return now, to the promise given in Remark 1. Let $s(Y) = \sup\{|Z| : Z$ is a discrete subspace of $Y\}$, the "spread" of the space Y. It is known (see [7]), that for every space Y, $c(Y) \leq s(Y)$. Then, the following is valid.

PROPOSITION 2. *Let X be a $d\sigma$-pseudocompact space. Then, $s(X) \geq \sup\{w(F) : F$ is a compact subset of $C_p(X)\}$.*

PROOF. The statement in question, trivially reduces to the case when $X = \oplus\{D_n : n \in \omega\}$ with each D_n pseudocompact. As $C_p(X) = \Pi\{C_p(D_n) : n \in \omega\}$ it is immediate that $\sup\{w(F) : F \subset C_p(X)$ and F is compact$\} = \sup_{n<\omega} \sup\{w(F) : F \subset C_p(D_n)$ and F is compact$\}$ and this finishes the proof. ■

NOTE. We wish to thank the referee who suggested the above proof.

Let F be a subset of $C_p(X)$. Obviously the induced function e_F from X to $C_p(F)$, such that for every x in X and f in F, $e_F(x)(f) = f(x)$, is continuous. If F separates the points of X, then e_F is also 1-1.

The next lemma is easy to prove. The basic idea comes from [9].

LEMMA 3. *Let X be a space. If $A \subset C_p(X)$ separates points in X then the algebra generated by A is dense in $C_p(X)$.* ∎

PROPOSITION 3.1. *Let X be a space such that there exists a set $F \subset C_p(X)$ with $t\big(C_p(F)\big) = \omega$ and $d\big(C_p(F)\big) = \tau$ where $\mathrm{cf}\tau > \omega$. Then X has no $\mathrm{cf}\tau$ caliber.*

PROOF. Consider $\{\mu_j : j < \tau\}$, a dense subset of $C_p(F)$. Lemma 3 implies that for every $i < \tau$, there are $f_i, g_i \in F, f_i \neq g_i$, such that $\mu_j(f_i) = \mu_j(g_i)$ for all $j < i$. Thus, for every $i < \tau$ there exist $r_i \in Q$, $\delta_i > 0$, such that

$$f_i^{-1}(-\infty, r_i) \cap g_i^{-1}(r_i + \delta_i, +\infty) \neq \emptyset, \text{ or}$$
$$g_i^{-1}(-\infty, r_i) \cap f_i^{-1}(r_i + \delta_i, +\infty) \neq \emptyset.$$

Since $\mathrm{cf}\tau > \omega$, we may suppose without loss of generality, that there are $A \subset \tau$, $|A| = \tau$, and $r \in Q$, $\delta > 0$ such that

$$V_i = f_i^{-1}(-\infty, r) \cap g_i^{-1}(r + \delta, +\infty) \neq \emptyset, \text{ for every } i \in A.$$

Let $\{i_n : n < \mathrm{cf}\tau\} \subset A$ where $i_n < i_{n'}$, if $n < n' < \mathrm{cf}\tau$ and $\sup_{n<\mathrm{cf}\tau} i_n = \tau$.

Suppose that X has $\mathrm{cf}\tau$ caliber. Then, there is a cofinal set $B \subset \{i_n : n < \mathrm{cf}\tau\}$ with $|B| = \mathrm{cf}\tau$, such that $\bigcap\{V_i : i \in B\} \neq \emptyset$. Let $x \in \bigcap\{V_i : i \in B\}$. Since $t\big(C_p(F)\big) = \omega$ there exist $i_0 \in B$ such that $e_F(x) \in \overline{\{\mu_i : i < i_0\}}$. Choose $i_1 < i_0$ such that $|f_{i_0}(x) - \mu_{i_1}(f_{i_0})| < \delta/4$ and $|g_{i_0}(x) - \mu_{i_1}(g_{i_0})| < \delta/4$. We have $\mu_{i_1}(f_{i_0}) = \mu_{i_1}(g_{i_0})$ and therefore $|f_{i_0}(x) - g_{i_0}(x)| < \delta/2$ contradicting the fact that $i_0 \in B$. ∎

COROLLARY 3.2 ([2]). *Let X be a compact space and $w(X) = \tau$. If $\lambda = \mathrm{cf}\tau > \omega$, then λ is not a caliber of $C_p(X)$.* ∎

COROLLARY 3.3 ([2]). *Suppose that $2^{\omega_1} = \omega_2$. Then the following are valid:*

(a) If X has ω_1 and ω_2 calibers, then every compact subset of $C_p(X)$ is metrizable.

(b) Every compact space X such that ω_1 and ω_2 are calibers of $C_p(X)$ is metrizable.

COROLLARY 3.4 (GCH). *If B is a Banach space such that (B, w) has ω_1 and ω_2 calibers, then B is separable.*

PROOF. It is well known that (S_{B^*}, w^*), the unit ball of B^* with the w^*-topology, is contained homeomorphically into $C_p(B, w)$. Since B is contained isometrically into $C(S_{B^*}, w^*)$, the proof is completed using Corollary 3.3. ∎

Recall that a space X is τ-monolithic if $nw(A) \leq \tau$ for every $A \subset X$ with $|A| \leq \tau$. X is called *monolithic* when it is τ-monolithic, for every cardinal τ.

We can avoid the set theoretic assumptions in Corollary 3.3 enriching X or F properly. Indeed if X is stable, meaning that $iw(Y) = nw(Y)$ for each continuous image Y of X, keeping also in mind that this happens if and only if $C_p(X)$ is monolithic ([1]), we obtain the following results.

PROPOSITION 4. *For every space X,* $\mathrm{sh}(X) \geq \sup\{w(F) : F$ *is a monolithic compact subset of* $C_pX)\}$. ∎

PROOF. Let F be a compact subset of $C_p(X)$. If $d(F) > \tau$, where $\tau = \mathrm{sh}(X)$ then there is a left separated subset A of F, such that $|A| = \tau^+$. But $w(A) = d\big(C_p(A)\big) = \tau^+$ contradicting the hypothesis since Proposition 3.1 is valid. Hence $d(F) = w(F) \leq \tau$. ∎

COROLLARY 4.1. *Let X be a dσ-bounded space. Then,* $\mathrm{sh}(X) \geq \sup\{w(F) : F$ *is a compact subset of* $C_p(X)\}$.

PROOF. Let F be a compact subset of $C_p(X)$. Then, according to Theorem 9.23 of [3], F is Eberlein compact and the proof is completed. ∎

PROPOSITION 4.2. *For every stable space X,* $\mathrm{sh}(X) \geq \sup\{w(F) : F$ *is a compact subset of* $C_p(X)\}$. ∎

LEMMA 4.3. *For every compact space X,* $w(X) = \sup\{w(F) : F$ *is a compact subset of* $C_pC_p(X)\}$.

PROOF. It is known (see [1]) that $w(X) = d\big(C_p(X)\big) = iw\big(C_pC_p(X)\big)$. But $iw(F) = w(F) \leq iw\big(C_pC_p(X)\big)$ for every compact subset F of $C_pC_p(X)$. Since X embeds in $C_pC_p(X)$, the proof is completed. ∎

COROLLARY 4.4. *If X is a monolithic compact space, then* $\mathrm{sh}\big(C_p(X)\big) = w(X)$.

PROOF. Since $C_p(X)$ is stable, it is immediate from Lemma 4.3 and Proposition 4 that $\mathrm{sh}\big(C_p(X)\big) \geq w(X)$. The reverse inequality comes true since $w(X) = d\big(C_p(X)\big)$. ∎

COROLLARY 4.5. *For every monolithic compact space X, the cardinal* τ^+*, where* $\tau \geq t(X)$*, is a caliber of X if and only if it is a caliber of* $C_p(X)$.

PROOF. In view of Corollary 4.4 sufficiency is obvious. However, Šapirovskii has proved (see [7]) that for every compact space X the condition: $(*)$ τ^+ caliber and $\tau \geq t(X)$ means that $\pi w(X) < \tau^+$ and the necessity comes true. ∎

Baturov has proved (see [1]), that $l(Y) = e(Y)$ for $Y \subset C_p(X)$, where $e(Y) = \sup\{|A| : A$ is a closed discrete subspace of $Y\}$. Therefore, $s(Y) \geq l(Y)$. Hence, $s\big(C_p(X)\big) \geq hl\big(C_p(X)\big)$. But, $d(X) \leq hl\big(C_p(X)\big)$ (see [1]). Since X is monolithic compact, $w(X) \leq hl\big(C_p(X)\big)$. Keeping in mind that $w(X) = nw(X) = nw\big(C_p(X)\big) \geq s\big(C_p(X)\big)$ the following is valid.

PROPOSITION 5. *If X is a monolithic compact space, then a)* $w(X) = s\big(C_p(X)\big)$ *and b)* $\mathrm{sh}\big(C_p(X)\big) = s\big(C_p(X)\big)$. ∎

Arkhangel'skii proves in [4] that for a space X, $C_p(X)$ is $2^{l(X)}$ monolithic where $l(X)$ is the Lindelöf degree of X. Hence, under GCH we can state the following.

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PROPOSITION 6 (GCH). *Let X be a space such that $l(X) = \tau$. If τ^+ is a caliber of X, then $w(F) \leq \tau$ for every compact subset F of $C_p(X)$.* ∎

LEMMA 7. *Let F be a compact set in $C_p(X)$. Then $d\big(e_F(X)\big) = w(F)$.*

PROOF. Since $e_F(X)$ separates the points of F, the induced function e^* from F to $C_p\big(e_F(X)\big)$ such that for every f in F and g in $e_F(X)$, $e^*(f)(g) = g(f)$, is a homeomorphic embedding. Thus, $w(F) = nw(F) \leq nw\Big(C_p\big(e_F(X)\big)\Big) = nw\big(e_F(X)\big)$, provided that for every space Y the equality $nw(Y) = nw\big(C_p(Y)\big)$ is valid (see [1]. Theorem 1, p. 14). But $e_F(X)$ is monolithic ([3]). Hence, $d\big(e_F(X)\big) = nw\big(e_F(X)\big) \leq nw\big(C_p(F)\big) = nw(F) = w(F)$. ∎

PROPOSITION 7.1. *Let X be stable. Then $p(X) = \sup\{w(F) : F$ is a Corson compact subset of $C_p(X)\}$.*

PROOF. Since every supersequence is a Corson compact space, $p(X) \leq \sup\{w(F) : F$ is a Corson compact subset of $C_p(X)\}$. Now, let F be a Corson compact subset of $C_p(X)$, such that $w(F) = \lambda$. Then, there is a function θ from $C_p(F)$ to a $\Sigma_*(\tau)$ continuous, linear and 1-1, ([5]). Thus, there is a supersequence A in $C_pC_p(F)$ which separates the points of $C_p(F)$ ([2], Proposition 2.9). Therefore, A separates the points of $Y = e_F(X)$. Hence $B = \pi_Y(A)$, where π_Y is the natural projection from $C_pC_p(F)$ to $C_p(Y)$ such that $\pi_Y(g) = g|Y$, is a supersequence in $C_p(Y)$ separating the points of Y. Thus, $nw(Y) \geq nw(B) = w(B)$ and $iw(Y) \leq w\big(C_p(B)\big) = |B| = w(B)$, since e_B from Y to $C_p(B)$ is continuous and 1-1. From the stability of Y, we get that $nw(Y) = w(B)$. But, Lemma 4.3 implies that $nw(Y) = w(F)$. Hence, $w(B) = |B| = \lambda$, meaning that Y and accordingly X, has no (λ, ω) caliber. ∎

COROLLARY 7.2. *If X is a Corson compact space, then (a) $w(X) = p\big(C_p(X)\big)$ and (b) $\mathrm{sh}\big(C_p(X)\big) = p\big(C_p(X)\big) = s\big(C_p(X)\big)$.*

PROOF. (a) Since X is monolithic, then $C_p(X)$ is stable. Thus $w(X) \leq p\big(C_p(X)\big)$. However, in view of Proposition 7.1, Lemma 4.3 gives $w(X) \geq p\big(C_p(X)\big)$. ∎

REFERENCES

1. A. V. Arkhangel'skii, *A survey on C_p-theory*, Q and A in General Topology, **5**(1987), Special Issue.
2. A. V. Arkhangel'skii and V. V. Tkachuk, *Calibers and point finite cellularity of the space $C_p(X)$ and some questions of S. Gul'ko and M. Husek*, Topology and its applications **23**(1986), 63–73.
3. A. V. Arkhangel'skii, *Function spaces in the topology of pointwise convergence, and compact sets*, Russian Math. Surveys (5) **39**(1984), 9–56.
4. ———, *Topological properties of function spaces: Duality theorems*, Soviet Math. Dokl. (2) **27**(1983), 470–473.
5. S. P. Gul'ko, *On the structure of spaces of continuous functions and their complete paracompactness*, Russian Math. Surveys (6) **34**(1979), 36–44.
6. W. W. Comfort and S. Negrepontis, *Chain Conditions in Topology*, Cambridge University Press, 1982.
7. I. Juhasz, *Cardinal functions in Topology—Ten years later*, Math. Centrum, Amsterdam, 1980.

8. V. V. Tkachuk, *Calibers of spaces of functions and the metrization problem for compact subsets of* $C_p(X)$, Vestnik Moskovskogo Universiteta Matematika (3) **43**(1988), 21–24.

9. ———, *The smallest subring of the ring* $C_p(C_p(X))$ *containing* $X \cup \{1\}$ *is everywhere dense in* $C_p(C_p(X))$, Vestnik Moskovskogo Universiteta Matematika (1) **42**(1987), 20–23.

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