Bull. Aust. Math. Soc. **96** (2017), 307–316
doi:10.1017/S0004972717000120

FIBONACCI–MANN ITERATION FOR MONOTONE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

M. R. ALFURAIDAN and M. A. KHAMSI✉

(Received 22 December 2016; accepted 24 December 2016; first published online 13 March 2017)

Abstract

We extend the results of Schu ['Iterative construction of fixed points of asymptotically nonexpansive mappings', *J. Math. Anal. Appl.* **158** (1991), 407–413] to monotone asymptotically nonexpansive mappings by means of the Fibonacci–Mann iteration process

$$x_{n+1} = t_n T^{f(n)}(x_n) + (1 - t_n)x_n, \quad n \in \mathbb{N},$$

where T is a monotone asymptotically nonexpansive self-mapping defined on a closed bounded and nonempty convex subset of a uniformly convex Banach space and $\{f(n)\}$ is the Fibonacci integer sequence. We obtain a weak convergence result in $L_p([0, 1])$, with $1 < p < +\infty$, using a property similar to the weak Opial condition satisfied by monotone sequences.

2010 *Mathematics subject classification*: primary 47H09; secondary 46B20, 47H10.

Keywords and phrases: asymptotically nonexpansive mapping, Fibonacci sequence, fixed point, Mann iteration process, monotone Lipschitzian mapping, Opial condition, uniformly convex Banach space.

1. Introduction

The notion of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972. In particular, they proved that such mappings defined on a nonempty closed convex and bounded subset of a uniformly convex Banach space always have a fixed point. Their proof was not constructive. Schu [10] used a modified Mann iteration to generate an approximate fixed point sequence for such mappings and his approach has proved very useful for computational purposes.

In this work, we investigate Schu's modified Mann iteration sequence associated to a monotone asymptotically nonexpansive mapping. The modified Mann iteration sequence introduced by Schu is defined by

$$x_{n+1} = t_n T^n(x_n) + (1 - t_n)x_n, \tag{1.1}$$

for $t_n \in [0, 1]$ and $n \in \mathbb{N}$, where T is an asymptotically nonexpansive mapping. His reason for introducing the iterate of T in the original Mann iteration sequence is

The authors acknowledge the support provided by the deanship of scientific research at King Fahd University of Petroleum and Minerals in funding this work through project no. IN141040.

based on the good behaviour of the Lipschitz constants associated to the iterates of T. For monotone mappings, the iteration sequence (1.1) does not generate a sequence which is monotone, which is crucial when proving the existence of a fixed point. We therefore modify the iteration (1.1) by using the Fibonacci sequence $\{f(n)\}$ defined by

$$f(0) = f(1) = 1 \quad \text{and} \quad f(n+1) = f(n) + f(n-1) \quad \text{for } n \geq 1.$$

The new iteration scheme, which we call Fibonacci–Mann iteration, is defined by

$$x_{n+1} = t_n T^{f(n)}(x_n) + (1 - t_n)x_n,$$

for $t_n \in [0, 1]$ and $n \in \mathbb{N}$. This new iteration scheme allows us to prove the main results of Schu [10] for monotone asymptotically nonexpansive mappings. This result is surprising since the class of mappings may fail to be continuous.

Monotone Lipschitzian mappings attracted interest after the Banach contraction principle was extended to partially ordered metric spaces by Ran and Reurings [9]. There are many applications of the fixed point theory of monotone mappings in Carl and Heikkilä [4]. For more on metric fixed point theory, the reader may consult Khamsi and Kirk [7], and, for the geometry of Banach spaces, we recommend Beauzamy [3].

2. Monotone asymptotically nonexpansive mappings

The concept of monotone Lipschitzian mappings involves two structures: a partial order and a metric distance. Most of the spaces involved in applications have these two natural structures, with interesting natural intertwining properties.

Definition 2.1. Let $(X, \|\cdot\|, \preceq)$ be a partially ordered Banach space. Let K be a nonempty subset of X. A map $T : K \to K$ is said to be monotone if for any $x, y \in K$ such that $x \preceq y$, we have $T(x) \preceq T(y)$. Moreover, T is said to be:

(a) *monotone Lipschitzian*, if T is monotone and there exists $L \geq 0$ such that

$$\|T(x) - T(y)\| \leq L\,\|x - y\| \quad \text{for any } x, y \text{ in } K \text{ with } x \preceq y;$$

(b) *monotone asymptotically nonexpansive* if T is monotone and there exists a sequence of numbers $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n(x) - T^n(y)\| \leq k_n\,\|x - y\| \quad \text{for any } x, y \text{ in } K \text{ with } x \preceq y \text{ and } n \geq 1.$$

A fixed point of T is any element $x \in K$ such that $T(x) = x$.

Monotone Lipschitzian mappings do not have nice topological behaviour like the regular Lipschitzian mappings because the Lipschitzian condition is only satisfied by comparable elements and may fail to hold in the entire space. These mappings may even fail to be continuous. The extension of the Banach contraction principle for such mappings does not necessarily lead to the existence of a unique fixed point. For more on this, see Jachymski [6].

In order to extend Schu's ideas to the monotone case, we need the following fixed point result which is the monotone version of Goebel and Kirk's fixed point theorem [5] for monotone asymptotically nonexpansive mappings.

THEOREM 2.2 [2]. *Let $(X, \|\cdot\|, \preceq)$ be a uniformly convex partially ordered Banach space for which order intervals are convex and closed. Let K be a nonempty convex closed bounded subset of X not reduced to one point. Let $T : K \to K$ be a continuous monotone asymptotically nonexpansive mapping. If there exists $x_0 \in K$ such that $x_0 \preceq T(x_0)$ (respectively, $T(x_0) \preceq x_0$), then T has a fixed point z such that $x_0 \preceq z$ (respectively, $z \preceq x_0$).*

An order interval is any of the subsets $\{x : \alpha \preceq x\}$, $\{x : x \preceq \beta\}$ and $\{x : \alpha \preceq x \preceq \beta\}$, for any $\alpha, \beta \in X$.

REMARK 2.3. In the proof of Theorem 2.2, the fixed point z was obtained as the minimum point of the type function $\varphi : K_\infty \to [0, +\infty)$ defined by

$$\varphi(x) = \limsup_{n\to\infty} \|T^n(x_0) - x\|,$$

where $K_\infty = \{x \in K : T^n(x_0) \preceq x \text{ for any } n \in \mathbb{N}\}$, assuming $x_0 \preceq T(x_0)$.

3. Fibonacci–Mann iteration

In this section, we investigate the properties of the Fibonacci–Mann iteration sequence associated to a monotone asymptotically nonexpansive mapping.

DEFINITION 3.1. Let K be a nonempty convex subset of a Banach space $(X, \|\cdot\|)$. Let $T : K \to K$ be a mapping. Fix $x_0 \in K$ and $\{t_n\} \subset [0, 1]$. The Fibonacci–Mann iteration is the sequence $\{x_n\}$ defined by

$$x_{n+1} = t_n T^{f(n)}(x_n) + (1 - t_n)x_n, \tag{3.1}$$

for any $n \in \mathbb{N}$, where $\{f(n)\}$ is the Fibonacci sequence. We will write $\{t_n\} \sqsubset [0, 1]$ if there exist two real numbers a, b such that $0 < a \le t_n \le b < 1$ for all n.

Throughout, $(X, \|\cdot\|, \preceq)$ is a partially ordered Banach space for which order intervals are closed and convex.

LEMMA 3.2. *Let K be a convex and bounded nonempty subset of X. Assume that the map $T : K \to K$ is monotone. Let $x_0 \in K$ be such that $x_0 \preceq T(x_0)$ (respectively, $T(x_0) \preceq x_0$) and $\{t_n\} \subset [0, 1]$ and consider the sequence $\{x_n\}$ generated by (3.1). Let z be a fixed point of T such that $x_0 \preceq z$ (respectively, $z \preceq x_0$). Then*

(i) $T^n(x_0) \preceq T^{n+1}(x_0)$ *(respectively, $T^{n+1}(x_0) \preceq T^n(x_0)$),*
(ii) $x_0 \preceq x_n \preceq z$ *(respectively, $z \preceq x_n \preceq x_0$),*
(iii) $T^{f(n)}(x_0) \preceq T^{f(n)}(x_n) \preceq z$ *(respectively, $z \preceq T^{f(n)}(x_n) \preceq T^{f(n)}(x_0)$),*
(iv) $x_n \preceq x_{n+1} \preceq T^{f(n)}(x_n)$ *(respectively, $T^{f(n)}(x_n) \preceq x_{n+1} \preceq x_n$),*

for any $n \in \mathbb{N}$.

https://doi.org/10.1017/S0004972717000120 Published online by Cambridge University Press

Proof. Using the convexity of the order intervals and the monotonicity of T, we can easily deduce (i), (ii) and (iii). We prove (iv) by induction. Without loss of generality, assume that $x_0 \preceq T(x_0)$. First note that $x_0 \preceq x_1 \preceq T^{f(0)}(x_0) = T(x_0)$. The monotonicity of T implies $T(x_0) \preceq T(x_1)$, which yields $x_0 \preceq x_1 \preceq T^{f(1)}(x_1)$. Using the convexity of the order intervals, we get $x_1 \preceq x_2 \preceq T^{f(1)}(x_1)$. Fix $n \geq 2$. Assume that $x_k \preceq x_{k+1} \preceq T^{f(k)}(x_k)$, for any $k \in [0, n-1]$. We claim that

$$x_n \preceq x_{n+1} \preceq T^{f(n)}(x_n).$$

By the convexity of the order intervals, this will hold if we prove that $x_n \preceq T^{f(n)}(x_n)$. Our assumption implies

$$x_n \preceq T^{f(n-1)}(x_{n-1}) \preceq T^{f(n-1)+f(n-2)}(x_{n-2}) = T^{f(n)}(x_{n-2}),$$

where we used the monotonicity of T, $x_{n-1} \preceq T^{f(n-2)}(x_{n-2})$ and the definition of the Fibonacci sequence. Since $x_{n-2} \preceq x_{n-1} \preceq x_n$, the monotonicity of T implies that $x_n \preceq T^{f(n)}(x_n)$. The induction argument completes the proof of (iv). □

Property (iv) ensures the monotonicity of the sequence $\{x_n\}$. This is hugely important as the following results show.

Proposition 3.3. *Let $\{x_n\}$ be a bounded monotone increasing or decreasing sequence in X, and assume that X is reflexive.*

(1) *$\{x_n\}$ is weakly convergent.*

(2) *If $\lim_{n\to\infty} d(x_n, K) = 0$, where K is a compact nonempty subset of X, then $\{x_n\}$ converges strongly.*

Proof. Without loss of generality, assume that $\{x_n\}$ is monotone increasing. Since X is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{\psi(n)}\}$ of $\{x_n\}$ which converges weakly to some point $x \in X$. We claim that any other subsequence $\{x_{\phi(n)}\}$ of $\{x_n\}$ also converges weakly to x. Assume that $\{x_{\phi(n)}\}$ converges weakly to z. Since $\{x_n\}$ is monotone increasing, then we must have $x_n \preceq z$, for any $n \geq 1$, since order intervals are closed and convex. In particular, $x_{\psi(n)} \preceq z$, for any $n \geq 1$, which implies $x \preceq z$. Clearly this will force $x = z$. The proof of (1) is complete.

We now prove (2). Our assumption implies the existence of $\{y_n\}$ in K such that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since $\{x_n\}$ converges weakly to x, then $\{y_n\}$ also converges weakly to x. Note that a weakly convergent sequence which belongs to a compact subset must be strongly convergent. Assume, on the contrary, that $\{y_n\}$ does not converge strongly to x. Then there exist $\varepsilon > 0$ and a subsequence $\{y_{\phi(n)}\}$ such that $\inf_{n\geq 1} \|y_{\phi(n)} - x\| \geq \varepsilon$. Since K is compact, there exists a subsequence $\{y_{\overline{\phi}(n)}\}$ of $\{y_{\phi(n)}\}$ which converges strongly. Since $\{y_n\}$ converges weakly to x, $\{y_{\overline{\phi}(n)}\}$ converges strongly to x. This contradiction implies that $\{y_n\}$ converges strongly to x and consequently $\{x_n\}$ converges strongly to x. □

This proposition shows the good behaviour monotone sequences have. In fact, it is surprising to see that despite the fact that the Lebesgue function spaces $L_p([0, 1])$, with $1 < p < +\infty$, fail the weak Opial condition [8], they enjoy a similar conclusion for monotone sequences. This fact will be discussed later in this section.

https://doi.org/10.1017/S0004972717000120 Published online by Cambridge University Press

LEMMA 3.4. *Let K be a convex and bounded nonempty subset of X. Assume that the map $T : K \to K$ is monotone asymptotically nonexpansive with the Lipschitz constants $\{k_n\}$ and that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $x_0 \in K$ be such that $x_0 \preceq T(x_0)$ (respectively, $T(x_0) \preceq x_0$) and $\{t_n\} \subset [0, 1]$ and consider the sequence $\{x_n\}$ generated by (3.1). Let z be a fixed point of T such that $x_0 \preceq z$ (respectively, $z \preceq x_0$). Then $\lim_{n\to\infty} \|x_n - z\|$ exists.*

PROOF. Without loss of generality, assume that $x_0 \preceq T(x_0)$. From the definition of $\{x_n\}$,

$$\begin{aligned}\|x_{n+1} - z\| &\le t_n\|T^{f(n)}(x_n) - z\| + (1 - t_n)\|x_n - z\| \\ &= t_n\|T^{f(n)}(x_n) - T^{f(n)}(z)\| + (1 - t_n)\|x_n - z\|,\end{aligned}$$

for any $n \ge 1$. Since T is monotone asymptotically nonexpansive,

$$\|x_{n+1} - z\| \le k_{f(n)}\, \|x_n - z\| = (k_{f(n)} - 1)\, \|x_n - z\| + \|x_n - z\|,$$

for any $n \ge 1$. In particular,

$$\|x_{n+1} - z\| - \|x_n - z\| \le (k_{f(n)} - 1)\delta(K),$$

for any $n \in \mathbb{N}$, where $\delta(K) = \sup\{\|c_1 - c_2\| : c_1, c_2 \in K\}$ is the diameter of K. Hence

$$\|x_{n+m} - z\| - \|x_n - z\| \le \delta(K) \sum_{i=0}^{m-1}(k_{f(n+i)} - 1),$$

for any $n, m \ge 1$. Letting $m \to \infty$ gives

$$\limsup_{m\to\infty} \|x_m - z\| \le \|x_n - z\| + \delta(K) \sum_{i=n}^{\infty}(k_{f(i)} - 1),$$

for any $n \ge 1$. Next we let $n \to \infty$ and get

$$\limsup_{m\to\infty} \|x_m - z\| \le \liminf_{n\to\infty} \|x_n - z\| + \delta(K) \liminf_{n\to\infty} \sum_{i=n}^{\infty}(k_{f(i)} - 1) = \liminf_{n\to\infty} \|x_n - z\|.$$

Therefore, $\limsup_{m\to\infty} \|x_m - z\| = \liminf_{n\to\infty} \|x_n - z\|$, which implies the desired conclusion. □

In the proof of the following lemma, we use the concept of ultrapower of a Banach space. For the basic definitions and properties of ultrafilters, see [1]. Let $\mathcal{U}$ be a nontrivial ultrafilter over $\mathbb{N}$. It is known that $\lim_{n,\mathcal{U}} \alpha_n$ exists for any bounded sequence of real numbers $\{\alpha_n\}$. Let X be a Banach space. The vector space

$$\ell_\infty(X) = \Big\{\{x_n\} \subset X : \|\{x_n\}\|_\infty = \sup_n \|x_n\| < \infty\Big\},$$

endowed with the norm $\|\cdot\|_\infty$, is a Banach space. The set

$$X_0 = \Big\{\{x_n\} \in \ell_\infty(X) : \lim_{n,\mathcal{U}} \|x_n\| = 0\Big\}$$

is a closed subspace of $\ell_\infty(X)$. The quotient space $(X_{\mathcal{U}}) = \ell_\infty(X)/X_0$ is known as the ultrapower of the Banach space X. In particular, for any $\tilde{x} \in (X)_{\mathcal{U}}$,

$$\|\tilde{x}\|_{\mathcal{U}} = \lim_{n,\mathcal{U}} \|x_n\|,$$

where $\{x_n\}$ is any representative of $\tilde{x}$. For more on the properties of the ultrapower of a Banach space, see [1, 3].

Lemma 3.5. *Let K be a nonempty weakly compact convex subset of a uniformly convex X. Let $T : K \to K$ be a continuous monotone asymptotically nonexpansive mapping with the Lipschitz constants $\{k_n\}$. Assume that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $x_0 \in K$ be such that $x_0 \preceq T(x_0)$ (respectively, $T(x_0) \preceq x_0$) and $\{t_n\} \sqsubset [0, 1]$ and consider the sequence $\{x_n\}$ generated by (3.1). Then*

$$\lim_{n\to\infty} \|x_n - T^{f(n)}(x_n)\| = 0.$$

Proof. Without loss of generality, assume that $x_0 \preceq T(x_0)$. By Theorem 2.2, there is a fixed point, z, of T such that $x_0 \preceq z$. By Lemma 3.4, $r = \lim_{n\to\infty} \|x_n - z\|$ exists. If $r = 0$, the conclusion is trivial. Assume that $r > 0$. Then

$$\limsup_{n\to\infty} \|T^{f(n)}(x_n) - z\| = \limsup_{n\to\infty} \|T^{f(n)}(x_n) - T^{f(n)}(z)\| \le \limsup_{n\to\infty} k_{f(n)}\, \|x_n - z\| = r,$$

since $x_n \preceq z$, for any $n \ge 1$. On the other hand,

$$\|x_{n+1} - z\| \le t_n \|T^{f(n)}(x_n) - z\| + (1 - t_n)\|x_n - z\|,$$

for any $n \ge 1$. Let $\mathcal{U}$ be a nontrivial ultrafilter over $\mathbb{N}$. Then $\lim_{\mathcal{U}} t_n = t \in [a, b]$, where $0 < a \le b < 1$. Hence

$$r = \lim_{\mathcal{U}} \|x_{n+1} - z\| \le t\, \lim_{\mathcal{U}} \|T^{f(n)}(x_n) - z\| + (1 - t)r.$$

Since $t \neq 0$, we get $\lim_{\mathcal{U}} \|T^{f(n)}(x_n) - z\| \ge r$. Hence

$$r \le \lim_{\mathcal{U}} \|T^{f(n)}(x_n) - z\| \le \limsup_{n\to\infty} \|T^{f(n)}(x_n) - z\| \le r,$$

which implies $\lim_{\mathcal{U}} \|T^{f(n)}(x_n) - z\| = r$. Let $(X)_{\mathcal{U}}$ be the ultrapower of X and set $\tilde{x} = (\{x_n\})_{\mathcal{U}}$, $\tilde{y} = (\{T^{f(n)}(x_n)\})_{\mathcal{U}}$ and $\tilde{z} = (\{z\})_{\mathcal{U}}$. Then

$$\|\tilde{x} - \tilde{z}\|_{\mathcal{U}} = \|\tilde{y} - \tilde{z}\|_{\mathcal{U}} = \|t\tilde{x} + (1 - t)\tilde{y} - \tilde{z}\|_{\mathcal{U}} = r.$$

From the uniform convexity of X, we know that $(X)_{\mathcal{U}}$ is strictly convex. Since $t \in (0, 1)$, we get $\tilde{x} = \tilde{y}$, that is, $\lim_{\mathcal{U}} \|x_n - T^{f(n)}(x_n)\| = 0$. Since $\mathcal{U}$ was an arbitrary nontrivial ultrafilter, we conclude that $\lim_{n\to\infty} \|x_n - T^{f(n)}(x_n)\| = 0$, which completes the proof. □

The map T is said to be compact if it maps bounded sets into relatively compact ones. The following result is the monotone version of [10, Theorem 2.2].

https://doi.org/10.1017/S0004972717000120 Published online by Cambridge University Press

THEOREM 3.6. *Let K be a nonempty weakly compact convex subset of a uniformly convex X. Let $T : K \to K$ be a monotone asymptotically nonexpansive mapping with the Lipschitz constants $\{k_n\}$ and assume that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and T^m is compact for some $m \geq 1$. Let $x_0 \in K$ be such that x_0 and $T(x_0)$ are comparable. Let $\{t_n\} \sqsubset [0, 1]$. Consider the sequence $\{x_n\}$ generated by (3.1). Then $\{x_n\}$ converges strongly to a fixed point of T which is comparable to x_0.*

PROOF. Without loss of generality, assume that $x_0 \preceq T(x_0)$. Lemma 3.2 implies $x_0 \preceq x_n$, for any $n \in \mathbb{N}$. By Lemma 3.5, $\lim_{n\to\infty} \|x_n - T^{f(n)}(x_n)\| = 0$. Fix $m \geq 1$ such that T^m is compact and set $K_0 = \overline{T^m(K)}$. Then K_0 is a compact nonempty set. For any $n > m$ and $x \in K$, we have $T^n(x) \in K_0$. Since $f(n) > m$ for $n > m$, we get $T^{f(n)}(x_n) \in K_0$. Hence

$$\lim_{n\to\infty} d(x_n, K_0) \leq \lim_{n\to\infty} \|x_n - T^{f(n)}(x_n)\| = 0$$

and $\lim_{n\to\infty} d(T^n(x_0), K_0) = 0$. Since X is reflexive and both sequences $\{x_n\}$ and $\{T^n(x_0)\}$ are monotone increasing, Proposition 3.3 implies that $\{x_n\}$ and $\{T^n(x_0)\}$ are strongly convergent. Hence $\{T^{f(n)}(x_n)\}$ is also strongly convergent and has the same limit as $\{x_n\}$. Let z be the strong limit of $\{T^n(x_0)\}$. We claim that z is a fixed point of T. Indeed, since $\{T^n(x_0)\}$ is monotone increasing, $T^n(x_0) \preceq z$ for any $n \in \mathbb{N}$. Hence

$$\|T^{n+1}(x_0) - T(z)\| \leq k_1 \|T^n(x_0) - z\|,$$

for any $n \geq 1$, which implies that $\{T^{n+1}(x_0)\}$ converges to $T(z)$ and z, that is, $T(z) = z$. Lemma 3.2 implies

$$T^{f(n)}(x_0) \preceq T^{f(n)}(x_n) \preceq z,$$

for any $n \in \mathbb{N}$. Since order intervals are closed, we conclude that z is also the limit of $\{T^{f(n)}(x_n)\}$ and $\{x_n\}$, that is, $\{x_n\}$ converges strongly to a fixed point of T. □

Next we discuss the weak convergence of the Fibonacci–Mann sequence. Usually this is done via the weak Opial condition [8], a property satisfied by any Hilbert space and the classical Banach spaces ℓ_p, with $1 < p < +\infty$. It is also well known that the classical Banach function spaces $L_p([0, 1])$, with $1 < p < +\infty$, fail the weak Opial condition [8]. We will show that a weaker Opial condition is necessary to obtain the weak convergence of the Fibonacci–Mann sequences. This weaker version holds in $L_p([0, 1])$ with $1 < p < +\infty$.

DEFINITION 3.7. We say that $(X, \|\cdot\|, \preceq)$ satisfies the monotone weak Opial condition if, for any monotone increasing (respectively, decreasing) sequence $\{x_n\}$ which weakly converges to x,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for any $y \neq x$ with $x_n \preceq y$ (respectively, $y \preceq x_n$) for any $n \in \mathbb{N}$.

In the next result, we give a class of Banach spaces which satisfy the monotone weak Opial condition. The norm $\|\cdot\|$ of X is said to be monotone if $u \preceq v \preceq w$ implies $\max\{\|v - u\|, \|w - v\|\} \leq \|w - u\|$ for any $u, v, w \in X$. If the norm is monotone and

https://doi.org/10.1017/S0004972717000120 Published online by Cambridge University Press

$\{x_n\}$ is monotone increasing (respectively, decreasing), then the sequence $\{\|x_n - y\|\}$ is decreasing for any y such that $x_n \leq y$ (respectively, $y \leq x_n$), for any $n \in \mathbb{N}$. In this case,

$$\liminf_{n\to\infty} \|x_n - y\| = \lim_{n\to\infty} \|x_n - y\| = \inf_{n\in\mathbb{N}} \|x_n - y\|.$$

Before we state the next result, the following proposition is needed. Let K be a subset of a Banach space $(X, \|\cdot\|)$ and $\varphi : K \to [0, +\infty)$. Recall that $\{z_n\}$ is a minimising sequence of φ if $\lim_{n\to\infty} \varphi(z_n) = \inf\{\varphi(x);\ x \in K\}$.

Proposition 3.8. *Let $\{x_n\}$ be a monotone increasing (respectively, decreasing) bounded sequence of a uniformly convex Banach space $(X, \|\cdot\|)$. Assume that the norm $\|\cdot\|$ is monotone. Consider the function $\varphi : K \to [0, +\infty)$ defined by*

$$\varphi(x) = \lim_{n\to\infty} \|x_n - x\|,$$

where $K = \{x : x_n \leq x$ (respectively, $x \leq x_n$) for any $n \in \mathbb{N}\}$. Let z be the weak limit of $\{x_n\}$. Then $\varphi(z) = \inf\{\varphi(x) : x \in K\}$ and any minimising sequence $\{z_n\}$ of φ in K converges strongly to z. In particular, φ has one minimum point.

Proof. Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and δ_X its modulus of uniform convexity [3]. Without loss of generality, assume that $\{x_n\}$ is monotone increasing. Since X is reflexive, $\{x_n\}$ is weakly convergent to z. Since order intervals are closed and convex, we conclude that $z \in K$. For any $x \in K$, we have $x_n \leq z \leq x$, for any $n \in \mathbb{N}$. Using the monotonicity of the norm, $\|z - x_n\| \leq \|x - x_n\|$, for any $n \in \mathbb{N}$, which implies $\varphi(z) \leq \varphi(x)$. Therefore, z is a minimum point of φ in K. Let $\{z_n\}$ be a minimising sequence of φ in K. Since $\|z_n - z\| \leq \varphi(z_n) + \varphi(z)$, for any $n \in \mathbb{N}$, we conclude that $\{z_n\}$ is bounded. Moreover, if $\varphi(z) = 0$, then $\{z_n\}$ strongly converges to z. Assume that $\{z_n\}$ is not strongly convergent to z. Then there exist $\varepsilon > 0$ and a subsequence $\{z_{\psi(m)}\}$ such that $\|z_{\psi(m)} - z\| \geq \varepsilon$. Set

$$\eta = \delta_X\Big(\frac{\varepsilon}{\max\limits_{n,m}(\|z_m - x_n\|, \|z - x_n\|) + 1}\Big).$$

Note that η is well defined since $\{z_n\}$ is bounded. The uniform convexity of X implies

$$\Big\|x_n - \frac{z_{\psi(m)} + z}{2}\Big\| \leq \max(\|x_n - z_{\psi(m)}\|, \|x_n - z\|)\,(1 - \eta),$$

for any $n, m \in \mathbb{N}$. Letting $n \to \infty$ gives

$$\varphi\Big(\frac{z_{\psi(m)} + z}{2}\Big) \leq \max(\varphi(z_{\psi(m)}), \varphi(z))(1 - \eta) = \varphi(z_{\psi(m)})(1 - \eta),$$

which implies $\varphi(z) \leq \varphi(z_{\psi(m)})(1 - \eta)$, for any $m \in \mathbb{N}$. If we let $m \to \infty$, we get $\varphi(z) \leq \varphi(z)(1 - \eta)$. Since $\{z_n\}$ is not strongly convergent to z, we must have $\varphi(z) > 0$. This is a contradiction with $\eta > 0$. □

A direct consequence of Proposition 3.8 is the following result which describes a class of Banach spaces which satisfy the monotone weak Opial condition.

https://doi.org/10.1017/S0004972717000120 Published online by Cambridge University Press

Theorem 3.9. *Let $(X, \|\cdot\|, \preceq)$ be a uniformly convex partially ordered Banach space for which order intervals are convex and closed. Assume that the norm $\|\cdot\|$ is monotone. Then X satisfies the monotone weak Opial condition.*

Since the classical Lebesgue function spaces $L_p([0,1])$, with $1 < p < +\infty$, are uniformly convex and their norm is monotone we have the following corollary.

Corollary 3.10. *The Banach function spaces $L_p([0,1])$, with $1 < p < +\infty$, satisfy the monotone weak Opial condition.*

In the next result, we discuss the weak convergence of the Fibonacci–Mann iteration sequence. This result is the monotone version of [10, Theorem 2.1].

Theorem 3.11. *Let K be a nonempty weakly compact convex subset of a uniformly convex Banach space $(X, \|\cdot\|)$. Assume that the norm $\|\cdot\|$ is monotone. Let $T : K \to K$ be a monotone asymptotically nonexpansive mapping with the Lipschitz constants $\{k_n\}$ and assume that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $x_0 \in K$ be such that x_0 and $T(x_0)$ are comparable. Let $\{t_n\} \sqsubset [0,1]$. Consider the sequence $\{x_n\}$ generated by (3.1). Then $\{x_n\}$ converges weakly to a fixed point of T which is comparable to x_0.*

Proof. Without loss of generality, assume that $x_0 \preceq T(x_0)$. In this case, we know that $\{T^n(x_0)\}$ is monotone increasing. Since K is weakly compact, $\{T^n(x_0)\}$ is weakly convergent to some point z. Theorem 3.9 implies that X satisfies the monotone weak Opial condition. Hence z is the minimum point of $\varphi : K_\infty \to [0, +\infty)$ defined by

$$\varphi(y) = \liminf_{n\to\infty} \|T^n(x_0) - y\| = \lim_{n\to\infty} \|T^n(x_0) - y\|,$$

where $K_\infty = \{y \in K : T^n(x_0) \preceq y \text{ for any } n \in \mathbb{N}\}$. It is easy to check that

$$\varphi(z) \leq \varphi(T^m(z)) \leq k_m\, \varphi(z),$$

for any $m \geq 1$. Since $\lim_{m\to\infty} k_m = 1$, $\{T^m(z)\}$ is a minimising sequence of φ. Proposition 3.8 implies that $\{T^m(z)\}$ converges strongly to z. Since $T^n(x_0) \preceq z$, we get $T^{n+1}(x_0) \preceq T(z)$, for any $n \geq 1$. Since $\{T^m(z)\}$ converges to z and the order intervals are closed, we get $z \preceq T(z)$. Using the monotonicity of T, we conclude that the sequence $\{T^m(z)\}$ is monotone increasing and converges to z. So we must have $T^m(z) \preceq z$, for any $m \geq 1$. In particular, $T(z) \preceq z$ which implies $T(z) = z$. By Lemma 3.2,

$$T^{f(n)}(x_0) \preceq T^{f(n)}(x_n) \preceq z,$$

for any $n \in \mathbb{N}$. Since the order intervals are closed and convex and the sequence $\{T^n(x_0)\}$ is monotone increasing and converges weakly to z, we conclude that $\{T^{f(n)}(x_n)\}$ also converges weakly to z. By Lemma 3.5,

$$\lim_{n\to\infty} \|x_n - T^{f(n)}(x_n)\| = 0,$$

which implies that $\{x_n\}$ weakly converges to z, a fixed point of T. □

Note that the conclusion of Theorem 3.11 was obtained without assuming that T is continuous, an assumption necessary in Theorem 2.2.

https://doi.org/10.1017/S0004972717000120 Published online by Cambridge University Press

References

[1] A. Aksoy and M. A. Khamsi, *Nonstandard Methods in Fixed Point Theory* (Springer, New York, 1990).
[2] M. R. Alfuraidan and M. A. Khamsi, 'A fixed point theorem for monotone asymptotically nonexpansive mappings', *Proc. Amer. Math. Soc.*, to appear.
[3] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry* (North-Holland, Amsterdam, 1985).
[4] S. Carl and S. Heikkilä, *Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory* (Springer, New York, 2011).
[5] K. Goebel and W. A. Kirk, 'A fixed point theorem for asymptotically nonexpansive mappings', *Proc. Amer. Math. Soc.* **35** (1972), 171–174.
[6] J. Jachymski, 'The contraction principle for mappings on a metric space with a graph', *Proc. Amer. Math. Soc.* **136** (2008), 1359–1373.
[7] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory* (John Wiley, New York, 2001).
[8] Z. Opial, 'Weak convergence of the sequence of successive approximations for nonexpansive mappings', *Bull. Amer. Math. Soc.* **73** (1967), 591–597.
[9] A. C. M. Ran and M. C. B. Reurings, 'A fixed point theorem in partially ordered sets and some applications to matrix equations', *Proc. Amer. Math. Soc.* **132**(5) (2004), 1435–1443.
[10] J. Schu, 'Iterative construction of fixed points of asymptotically nonexpansive mappings', *J. Math. Anal. Appl.* **158** (1991), 407–413.

M. R. ALFURAIDAN, Department of Mathematics and Statistics,
King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
e-mail: monther@kfupm.edu.sa

M. A. KHAMSI, Department of Mathematical Sciences,
The University of Texas at El Paso, El Paso, TX 79968, USA
e-mail: mohamed@utep.edu