ON THE GENERALIZED HADAMARD PRODUCT
AND THE JORDAN-HADAMARD PRODUCT

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The generalized Hadamard product $S * T$ and the Jordan-Hadamard product $S \circ T$ of two operator-matrices S and T are introduced. They coincide with the usual Hadamard product of two complex matrices when the underlying Hilbert spaces are one-dimensional. Some inequalities which hold true for the usual Hadamard product of positive definite complex matrices are shown to be true for these two new products of positive invertible operator-matrices.

In [2] the concavity of certain tensor product maps are applied to obtain old and new inequalities of Hadamard product of positive definite complex matrices (*cf.* [8]). Recall that the Hadamard product of two complex matrices is the matrix of entrywise product of the two complex matrices. In this paper we introduce two binary operations of operator-matrices, namely the generalized Hadamard product and the Jordan-Hadamard product. Following Ando's approach we derive some inequalities involving these products, some of which take precisely the same form as those obtained in [2].

1. Notations and preliminary results

Given a Hilbert space H , we denote by $L(H)$ the C^*-algebra of all (bounded) operators on H . An operator $A \in L(H)$ is said to be *positive*

(denoted $A \geq 0$) if $A = B^*B$ for some $B \in L(H)$. By $A \geq B$ we mean $A - B \geq 0$. The cone of all positive operators on H is denoted by $L_+(H)$.

For positive operators A and B , the *harmonic mean* $A \; ! \; B$ and the *geometric mean* $A \; \# \; B$ are introduced by Ando [1] and Pusz and Woronowicz [6] respectively. By definition

$$A \; ! \; B = \max\left\{X \geq 0 \;\middle|\; \begin{pmatrix} 2A & 0 \\ 0 & 2B \end{pmatrix} \geq \begin{pmatrix} X & X \\ X & X \end{pmatrix}\right\}$$

and

$$A \; \# \; B = \max\left\{X \geq 0 \;\middle|\; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\} \; .$$

If A, B are invertible, then

$$A \; ! \; B = \left\{ \tfrac{1}{2}\left(A^{-1}+B^{-1}\right) \right\}^{-1} = 2\left\{B - B(A+B)^{-1}B\right\}$$

and

$$A \; \# \; B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}} \; .$$

The harmonic mean and geometric mean have the following properties (*cf.* [1]):

(1) $A \; ! \; B = B \; ! \; A$ and $A \; \# \; B = B \; \# \; A$;

(2) $\tfrac{1}{2}(A+B) \geq A \; \# \; B \geq A \; ! \; B$;

(3) $(A+B) \; ! \; (C+D) \geq (A!C) + (B!D)$, $(A+B) \; \# \; (C+D) \geq (A\#C) + (B\#D)$.

If A and B are invertible, then

(4) $(A!B)^{-1} = \tfrac{1}{2}\left(A^{-1}+B^{-1}\right)$ and $(A\#B)^{-1} = A^{-1} \; \# \; B^{-1}$.

Recall that a map ϕ from a convex subset of $L_+(H_1) \times \ldots \times L_+(H_k)$ to $L(H)$ is said to be *convex* if

$$\phi\left(\lambda A_1 + (1-\lambda)B_1, \; \ldots, \; \lambda A_k + (1-\lambda)B_k\right) \leq \lambda\phi\left(A_1, \; \ldots, \; A_k\right) + (1-\lambda)\phi\left(B_1, \; \ldots, \; B_k\right)$$

for any $0 < \lambda < 1$. ϕ is *concave* if $-\phi$ is convex.

(5) The map $A \longmapsto A^p$ on $L_+(H)$ is concave if $0 \leq p \leq 1$ and is convex if $1 \leq p \leq 2$ or $-1 \leq p \leq 0$.

A linear map ϕ from $L(H_1)$ to $L(H_2)$ is said to be *positive* if it

takes positive operators to positive operators. A positive linear map is said to be *normalized* if it preserves the identity operators.

For any positive linear map ϕ and any positive operators A and B the following inequalities hold (*cf.* [1], [2]):

(6) $$\phi(A\#B) \leq \phi(A) \# \phi(B) \quad \text{and} \quad \phi(A!B) \leq \phi(A) \text{ ! } \phi(B) \ .$$

If A, B are invertible, then

(7) $$\phi(B)\phi(A)^{-1}\phi(B) \ \leq \phi\left(BA^{-1}B\right) \ .$$

If ϕ is normalized, then

(8)
$$
\begin{cases}
\phi\left(A^p\right) \geq \phi(A)^p & (1 \leq p \leq 2) \ , \\[2mm]
\phi\left(A^p\right) \leq \phi(A)^p & (0 \leq p \leq 1) \ , \\[2mm]
\phi(A) \leq \phi\left(A^p\right)^{1/p} & (1 \leq p < \infty) \ , \\[2mm]
\phi(A) \geq \phi\left(A^p\right)^{1/p} & (\tfrac{1}{2} \leq p \leq 1) \ , \\[2mm]
\phi(A) \geq \phi\left(A^{-p}\right)^{-1/p} & (1 \leq p < \infty) \ , \\[2mm]
\phi\left(A^p\right)^{1/p} \leq \phi\left(A^q\right)^{1/q} & (1 \leq p \leq q \ \text{ or } \ \tfrac{1}{2}q \leq p \leq 1 \leq q) \ .
\end{cases}
$$

In [1] and [2] Ando studies the concavity and convexity of the tensor product maps

$$\left(A_1, \ \ldots, \ A_k\right) \longmapsto \overset{k}{\underset{i=1}{\otimes}} A_i^{p_i}$$

defined on $L_+\left(H_1\right) \times \ldots \times L_+\left(H_k\right)$ to obtain the following results.

If $0 \leq p_i \leq 1$ $(i = 1, \ldots, k)$ and $\sum\limits_{i=1}^{k} p_i \leq 1$, then the map

(9) $\left(A_1, \ \ldots, \ A_k\right) \longmapsto \overset{k}{\underset{i=1}{\otimes}} A_i^{p_i}$ is concave on $L_+\left(H_1\right) \times \ldots \times L_+\left(H_k\right)$.

If $0 < p_i \leq 1$ $(i = 1, \ldots, k)$, then the map

(10) $\left(A_1, \ \ldots, \ A_k\right) \longmapsto \overset{k}{\underset{i=1}{\otimes}} A_i^{-p_i}$ is convex on $L_+\left(H_1\right) \times \ldots \times L_+\left(H_k\right)$.

If $1 \leq q \leq 2$, $0 \leq p_i \leq 1$ $(i = 1, \ldots, k)$ and $\sum_{i=1}^{k} p_i \leq q - 1$,

then the map

$$(11) \quad \left(A_0, \ldots, A_k\right) \longmapsto A_0^q \otimes \left(\bigotimes_{i=1}^{k} A_i^{-p_i}\right) \quad \text{is convex on}$$

$$L_+\left(H_0\right) \times \ldots \times L_+\left(H_k\right) .$$

2.   Generalized Hadamard product and Jordan-Hadamard product

For the rest of this paper we fix a positive integer n . Given a Hilbert space H the C^* -algebra of all n -square matrices whose entries are operators on H is denoted by $M_n\left(L(H)\right)$. Its elements, called n -square operator-matrices, can be considered as operators acting on the direct sum $\overset{n}{\underset{1}{\oplus}} H$ of n copies of H . If P_t is the projection on $\overset{n}{\underset{1}{\oplus}} H$ with range the t th direct summand $(t = 1, \ldots, n)$, then each $S \in M_n\left(L(H)\right)$ has a decomposition of the form $S = \sum_{s,t=1}^{n} P_s S P_t$.

DEFINITION.   Let H, H_1 and H_2 be Hilbert spaces.   For

$$S = \sum_{s,t=1}^{n} P_s S P_t \in M_n\left(L\left(H_1\right)\right) \quad \text{and} \quad T = \sum_{s,t=1}^{n} Q_s T Q_t \in M_n\left(L\left(H_2\right)\right) , \text{ their}$$

generalized Hadamard product $S * T$ is defined by

$$S * T = \sum_{s,t=1}^{n} \left[\left(P_s S P_t\right) \otimes \left(Q_s T Q_t\right)\right] \in M_n\left(L\left(H_1\right) \otimes L\left(H_2\right)\right) .$$

In terms of the matricial representations of S and T , the generalized Hadamard product $S * T$ is the n -square matrix of entrywise tensor product. If S and T are in $M_n\left(L(H)\right)$, their *Jordan-Hadamard product* $S \circ T$ is defined to be $\frac{1}{2}(S*T+T*S)$.

When the Hilbert spaces H_1 and H_2 are both 1-dimensional, it is easily seen that the generalized Hadamard product and the Jordan-Hadamard product just defined coincide with the usual Hadamard product of complex matrices (*cf.* [2], [8]). The following properties of generalized Hadamard

product and Jordan-Hadamard product can be derived directly from the definition. The underlying Hilbert spaces are assumed to be the same when addition or Jordan-Hadamard product are dealt with:

(12)
$$
\begin{cases}
(S+T) * R = S * R + T * R , \\[4pt]
R * (S+T) = R * S + R * T , \\[4pt]
(S*T) * R = S * (T*R) , \\[4pt]
(\alpha S) * T = S * (\alpha T) = \alpha(S*T) \quad \text{for} \quad \alpha \in \mathbb{C} , \\[4pt]
(S+T) \circ R = S \circ R + T \circ R , \\[4pt]
R \circ (S+T) = R \circ S + R \circ T , \\[4pt]
(\alpha S) \circ T = S \circ (\alpha T) = \alpha(S \circ T) \quad \text{for} \quad \alpha \in \mathbb{C} , \\[4pt]
\quad S \circ S = S * S .
\end{cases}
$$

Following Ando's approach [2], we shall develop some inequalities involving generalized Hadamard product and Jordan-Hadamard product. Many of the inequalities obtained in [2] for the usual Hadamard product of complex matrices hold true for the generalized Hadamard product or Jordan-Hadamard product. However, the validity of some inequalities in [2] depends heavily on the commutativity of the usual Hadamard product. Examples which reveal this fact can easily be found. For such inequalities we manage to replace them by others in weaker forms.

We begin with the following important theorem which shows that the tensor product of two n-square operator-matrices and their generalized Hadamard product are related (*cf.* [5]). For convenience we write

$$
\prod_{i=1}^{k} * S_i = S_1 * \dots * S_k \quad \text{and} \quad \prod_{1}^{k} * S = S * \dots * S \quad (k \text{ times}).
$$

THEOREM 1. *For each* $k \geq 1$ *there is a normalized positive linear map* $\phi = \phi_k$ *from* $\displaystyle \bigotimes_{i=1}^{k} M_n\big(L(H_i)\big)$ *into* $M_n\!\left(\displaystyle \bigotimes_{i=1}^{k} L\big(H_i\big)\right)$ *such that*

$$
\phi\!\left(\bigotimes_{i=1}^{k} X_i\right) = \prod_{i=1}^{k} * X_i \quad \text{for} \quad X_i \in M_n\big(L\big(H_i\big)\big) .
$$

Proof. Write $X_i = \displaystyle \sum_{u,v=1}^{n} P_u^{(i)} X_i P_v^{(i)}$ as in the definition of

generalized Hadamard product. To simplify the notations let F be the set
of all maps from the set $K = \{1, 2, \ldots, k\}$ into the set
$N = \{1, 2, \ldots, n\}$. By the operation rule of tensor product we have

$$\underset{i \in K}{\otimes} X_i = \sum_{s,t \in F} \left[\underset{i \in K}{\otimes} P^{(i)}_{s(i)} X_i P^{(i)}_{t(i)} \right]$$

$$= \sum_{s,t \in F} \left[\left(\underset{i \in K}{\otimes} P^{(i)}_{s(i)} \right) \left(\underset{i \in K}{\otimes} X_i \right) \left(\underset{i \in K}{\otimes} P^{(i)}_{t(i)} \right) \right] .$$

Let ϕ be the normalized positive linear map from $\underset{i \in K}{\otimes} M_n \big(L(H_i) \big)$ into

$M_n \Big(\underset{i \in K}{\otimes} L(H_i) \Big)$ defined by

$$\phi \left(\underset{i \in K}{\otimes} Y_i \right) = \left(\sum_{u \in N} \underset{i \in K}{\otimes} P^{(i)}_u \right) \left(\underset{i \in K}{\otimes} Y_i \right) \left(\sum_{v \in N} \underset{i \in K}{\otimes} P^{(i)}_v \right)$$

for $Y_i \in M_n \big(L(H_i) \big)$. It follows from the mutual orthogonality of the
projections that

$$\phi \left(\underset{i \in K}{\otimes} X_i \right) = \sum_{u,v \in N} \sum_{s,t \in F} \left[\left(\underset{i \in K}{\otimes} P^{(i)}_u \right) \left(\underset{i \in K}{\otimes} P^{(i)}_{s(i)} \right) \right.$$
$$\left. \times \left(\underset{i \in K}{\otimes} X_i \right) \left(\underset{i \in K}{\otimes} P^{(i)}_{t(i)} \right) \left(\underset{i \in K}{\otimes} P^{(i)}_v \right) \right]$$

$$= \sum_{u,v \in N} \left(\underset{i \in K}{\otimes} P^{(i)}_u \right) \left(\underset{i \in K}{\otimes} X_i \right) \left(\underset{i \in K}{\otimes} P^{(i)}_v \right)$$

$$= \sum_{u,v \in N} \left(\underset{i \in K}{\otimes} P^{(i)}_u X_i P^{(i)}_v \right) = \underset{i \in K}{\prod} * X_i .$$

This completes the proof.

(13) As a consequence of Theorem 1 it is clear that the generalized
      Hadamard product (or Jordan-Hadamard product) of positive
      (respectively invertible) n-square operator-matrices is
      positive (respectively invertible).

Furthermore, if S_i, $T_i \in M_n \big(L(H_i) \big)$ are Hermitian and $-T_i \leq S_i \leq T_i$
$(i = 1, \ldots, k)$ then

(14) $$-\prod_{i=1}^{k} * T_i \leq \prod_{i=1}^{k} * S_i \leq \prod_{i=1}^{k} * T_i .$$

Indeed, by induction, (14) reduces to the case $k = 2$. By (13) the positivity of $T_i \pm S_i$ $(i = 1, 2)$ implies that

$$\left(T_1 \pm S_1\right) * \left(T_2 \pm S_2\right) = \left(T_1 * T_2\right) + \left(S_1 * S_2\right) \pm \left(T_1 * S_2\right) \pm \left(S_1 * T_2\right) \geq 0$$

and

$$\left(T_1 \pm S_1\right) * \left(T_2 \mp S_2\right) = \left(T_1 * T_2\right) - \left(S_1 * S_2\right) \mp \left(T_1 * S_2\right) \pm \left(S_1 * T_2\right) \geq 0 .$$

Hence

$$-\left(T_1 * T_2 \pm S_1 * S_2\right) \leq \left(S_1 * T_2\right) \pm \left(T_1 * S_2\right) \leq \left(T_1 * T_2\right) \pm \left(S_1 * S_2\right)$$

and so

$$\left(T_1 * T_2\right) \pm \left(S_1 * S_2\right) \geq 0 .$$

We conclude that $-\left(T_1 * T_2\right) \leq S_1 * S_2 \leq T_1 * T_2$.

Note that (14) is also true for Jordan-Hadamard product. It follows immediately that both products preserve ordering.

THEOREM 2. *Suppose that* S_{ij} $(i = 1, \ldots, k; j = 1, \ldots, m)$ *are positive invertible operator-matrices. Then the following inequalities hold:*

$$(15) \quad \sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}\right) \leq m^{1-(1/p)} \left[\sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}^{p}\right)\right]^{1/p} \quad if \ 1 \leq p < \infty ;$$

$$(16) \quad \sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}\right) \leq m^{1-(1/p)} \left[\sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}^{p}\right)\right]^{1/p} \quad if \ \tfrac{1}{2} \leq p \leq 1 ;$$

$$(17) \quad \sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}\right) \leq m^{1+(1/p)} \left[\sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}^{-p}\right)\right]^{-1/p} \quad if \ 1 \leq p < \infty .$$

Proof. Applying (5) and (8) and Theorem 1 we have

$$\sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij} \right) = \phi\left(\sum_{j=1}^{m} \bigotimes_{i=1}^{k} S_{ij} \right)$$

$$\leq m\phi\left[\left(m^{-1} \sum_{j=1}^{m} \left(\bigotimes_{i=1}^{k} S_{ij}^{p} \right) \right)^{1/p} \right]$$

$$\leq m^{1-(1/p)} \left[\phi\left(\sum_{j=1}^{m} \left(\bigotimes_{i=1}^{k} S_{ij}^{p} \right) \right) \right]^{1/p}$$

$$= m^{1-(1/p)} \left[\sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}^{p} \right) \right]^{1/p} \qquad (1 \leq p < \infty)$$

where ϕ is as in Theorem 1. This proves (15). (16) and (17) are proved similarly.

COROLLARY. *The following inequalities hold for positive invertible operator-matrices* S_i, T_i $(i = 1, \ldots, k)$:

(18) $\qquad \sum_{i=1}^{k} (S_i \circ T_i) \leq k^{1-(1/p)} \left[\sum_{i=1}^{k} \left(S_i^{p} \circ T_i^{p} \right) \right]^{1/p} \quad if \quad 1 \leq p < \infty$;

(19) $\qquad \sum_{i=1}^{k} (S_i \circ T_i) \geq k^{1-(1/p)} \left[\sum_{i=1}^{k} \left(S_i^{p} \circ T_i^{p} \right) \right]^{1/p} \quad if \quad \tfrac{1}{2} \leq p \leq 1$;

(20) $\qquad \sum_{i=1}^{k} (S_i \circ T_i) \geq k^{1+(1/p)} \left[\sum_{i=1}^{k} \left(S_i^{-p} \circ T_i^{-p} \right) \right]^{-1/p} \quad if \quad 1 \leq p < \infty$;

(21) $\qquad \prod_{i=1}^{k} * S_i \leq \left(\prod_{i=1}^{k} * S_i^{p} \right)^{1/p} \quad if \quad 1 \leq p < \infty$;

(22) $\qquad \prod_{i=1}^{k} * S_i \geq \left(\prod_{i=1}^{k} * S_i^{p} \right)^{1/p} \quad if \quad \tfrac{1}{2} \leq p \leq 1$;

(23) $\qquad \prod_{i=1}^{k} * S_i \geq \left(\prod_{i=1}^{k} * S_i^{-p} \right)^{-1/p} \quad if \quad 1 \leq p < \infty$.

In particular we have the following inequalities:

(24) $\qquad\qquad S \circ T \leq \left(S^{p} \circ T^{p} \right)^{1/p} \quad if \quad 1 \leq p < \infty$,

(25) $S \circ T \geq \left(S^p \circ T^p\right)^{1/p}$ if $\frac{1}{2} \leq p \leq 1$,

(26) $S \circ T \geq \left(S^{-p} \circ T^{-p}\right)^{-1/p}$ if $1 \leq p < \infty$,

for positive and invertible operator-matrices S *and* T .

The inequality $\displaystyle\prod_{i=1}^{k} * S_i \leq \prod_{i=1}^{k} * \left(k^{-1} \sum_{j=1}^{k} S_j^{p_i}\right)^{1/p_i}$ $\left(p_i \geq 1$ and

$\displaystyle\sum_{i=1}^{k} 1/p_i = 1 \right)$ which holds for positive definite complex matrices S_i

$(i = 1, \ldots, k)$ and the usual Hadamard product $(cf.\ [2])$ fails in our
case. For example, consider the following 2-square matrices whose entries
are 2-square complex matrices:

$$S = \begin{pmatrix} 1 & 0 & \vdots & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 2 & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix} \quad and \quad T = \begin{pmatrix} 3 & 0 & \vdots & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{pmatrix} .$$

It is easy to verify that

$$S * T \nleq \left\{\tfrac{1}{2}\left(S^2 + T^2\right)\right\}^{\frac{1}{2}} * \left\{\tfrac{1}{2}\left(S^2 + T^2\right)\right\}^{\frac{1}{2}} .$$

However, we shall prove that for positive invertible operator-matrices S
and T ,

$$S \circ T \leq \left\{\tfrac{1}{2}\left(S^2 + T^2\right)\right\}^{\frac{1}{2}} \circ \left\{\tfrac{1}{2}\left(S^2 + T^2\right)\right\}^{\frac{1}{2}} .$$

Indeed, some inequalities in more general forms will be proved and
inequalities of similar type will also be considered in Theorem 3.

   THEOREM 3. *Suppose that* S_{ij} $(i = 1, \ldots, k; j = 1, \ldots, m)$ *are*
positive invertible operator-matrices. Then the following inequalities
hold:

(27) $\displaystyle\sum_{j=1}^{m} \left(\prod_{i=1}^{k} * S_{ij}\right) \leq \prod_{i=1}^{k} * \left(\sum_{j=1}^{m} S_{ij}^{p_i}\right)^{1/p_i}$ if $p_i \geq 1$ and $\displaystyle\sum_{i=1}^{k} 1/p_i = 1$.

In particular,

$$(28) \quad \sum_{j=1}^{m} \left[\prod_{i=1}^{k} * s_{ij} \right] \leq \prod_{i=1}^{k} * \left(\sum_{j=1}^{m} s_{ij}^{k} \right)^{1/k} \quad ,$$

$$(29) \quad \sum_{j=1}^{m} \left[\prod_{i=1}^{k} * s_{ij} \right] \geq m \prod_{i=1}^{k} * \left(m^{-1} \sum_{j=1}^{m} s_{ij}^{-p} \right)^{-1/p} \quad if \quad 1 \leq p < \infty \ ,$$

$$(30) \quad \sum_{j=1}^{m} \left[\prod_{i=1}^{k} * s_{ij} \right] \geq \left(\sum_{j=1}^{m} s_{1j}^{q} \right)^{1/q} * \left[\prod_{i=2}^{k} * \left(\sum_{j=1}^{m} s_{ij}^{-p_i} \right)^{-1/p_i} \right] \ ,$$

$$if \quad q, \ p_i > 0 \quad and \quad 1/q - 1 = \sum_{i=2}^{k} 1/p_i = 1 \ .$$

Proof. We first note that the map

$$\left(X_1, \ \ldots, \ X_k \right) \mapsto \prod_{i=1}^{k} * X_i^{p_i} \quad \left(p_i \geq 1 \quad and \quad \sum_{i=1}^{k} 1/p_i = 1 \right)$$

defined for positive invertible operator-matrices X_i $(i = 1, \ldots, k)$ is concave because it is simply $\psi\phi_k$ where ψ is the concave map in (9). Thus

$$m^{-1} \sum_{j=1}^{m} \left[\prod_{i=1}^{k} * s_{ij} \right] = m^{-1} \sum_{j=1}^{m} \left[\prod_{i=1}^{k} * \left(s_{ij}^{p_i} \right)^{1/p_i} \right]$$

$$\leq \prod_{i=1}^{k} * \left(m^{-1} \sum_{j=1}^{m} s_{ij}^{p_i} \right)^{1/p_i}$$

$$= m^{-1} \prod_{i=1}^{k} * \left(\sum_{j=1}^{m} s_{ij}^{p_i} \right)^{1/p_i}$$

$$\text{if} \quad p_i \geq 1 \quad \text{and} \quad \sum_{i=1}^{k} 1/p_i = 1 \ .$$

This proves (27). The proofs of (29) and (30) are similar using the convexity of the maps

$$\left(X_1, \ \ldots, \ X_k \right) \mapsto \prod_{i=1}^{k} * X_i^{-1/p}$$

where $1 \leq p < \infty$, and

$$(X_1, \ldots, X_k) \mapsto X_1^{1/q} * \left[\prod_{i=2}^{k} * X_i^{-1/p_i}\right]$$

where $q, p_i > 0$ and $1/q - 1 = \sum_{i=2}^{k} 1/p_i \leq 1$.

We remark that the inequalities (31) and (32) of the following corollary are in the form of Hölder's inequality. When $p = q = 2$, (31) reduces to the form of Cauchy-Schwarz-Buniakowski inequality (*cf*. [4]).

COROLLARY. *The following inequalities hold for positive invertible operator-matrices* S_i *and* T_i *($i = 1, \ldots, k$) :*

$$\sum_{i=1}^{k} (S_i * T_i) \leq \left[\sum_{i=1}^{k} S_i^p\right]^{1/p} * \left[\sum_{i=1}^{k} T_i^q\right]^{1/q} ;$$

$$(31) \quad \sum_{i=1}^{k} (S_i \circ T_i) \leq \left[\sum_{i=1}^{k} S_i^p\right]^{1/p} \circ \left[\sum_{i=1}^{k} T_i^q\right]^{1/q}$$
$$\qquad \qquad if \ p, q \geq 1 \ and \ 1/p + 1/q = 1 ;$$

$$\sum_{i=1}^{k} (S_i * T_i) \geq \left[\sum_{i=1}^{k} S_i^q\right]^{1/q} * \left[\sum_{i=1}^{k} T_i^{-p}\right]^{-1/p} ;$$

$$(32) \quad \sum_{i=1}^{k} (S_i \circ T_i) \geq \left[\sum_{i=1}^{k} S_i^q\right]^{1/q} \circ \left[\sum_{i=1}^{k} T_i^{-p}\right]^{-1/p}$$
$$\qquad \qquad if \ p, q > 0 \ and \ 1/q - 1 = 1/p \leq 1 ;$$

$$(33) \quad \sum_{i=1}^{k} (S_i \circ T_i) \geq k\left\{k^{-1} \sum_{i=1}^{k} S_i^{-p}\right\}^{-1/p} \circ \left\{k^{-1} \sum_{i=1}^{k} T_i^{-p}\right\}^{-1/p} \quad if \ 1 \leq p < \infty ;$$

$$(34) \quad \sum_{i=1}^{k} (S_i \circ T_i) \leq \left\{\tfrac{1}{2} \sum_{i=1}^{k} \left[S_i^p + T_i^p\right]\right\}^{1/p} \circ \left\{\tfrac{1}{2} \sum_{i=1}^{k} \left[S_i^q + T_i^q\right]\right\}^{1/q}$$
$$\qquad \qquad if \ p, q \geq 1 \ and \ 1/p + 1/q = 1 ;$$

$$(35) \quad \sum_{i=1}^{k} \left(S_i \circ T_i\right) \geq \left\{ \tfrac{1}{2} \sum_{i=1}^{k} \left[S_i^q + T_i^q\right] \right\}^{1/q} \circ \left\{ \tfrac{1}{2} \sum_{i=1}^{k} \left[S_i^{-p} + T_i^{-p}\right] \right\}^{-1/p}$$

$$\text{if } p, q > 0 \text{ and } 1/q - 1 = 1/p \leq 1 \text{ ;}$$

$$(36) \quad \sum_{i=1}^{k} \left(S_i \circ T_i\right) \geq k \left\{ (2k)^{-1} \sum_{i=1}^{k} \left[S_i^{-p} + T_i^{-p}\right] \right\}^{-1/p}$$

$$\circ \left\{ (2k)^{-1} \sum_{i=1}^{k} \left[S_i^{-p} + T_i^{-p}\right] \right\}^{-1/p} \quad \text{if } 1 \leq p < \infty \text{ .}$$

In particular for positive invertible operator-matrices S and T we have the following inequalities:

$$S \circ T \leq \left\{ \tfrac{1}{2} \left(S^p + T^p\right) \right\}^{1/p} \circ \left\{ \tfrac{1}{2} \left(S^q + T^q\right) \right\}^{1/q}$$

$$\text{if } p, q \geq 1 \text{ and } 1/p + 1/q = 1 \text{ ;}$$

$$(37) \quad S \circ T \leq \left\{ \tfrac{1}{2} \left(S^2 + T^2\right) \right\}^{\tfrac{1}{2}} \circ \left\{ \tfrac{1}{2} \left(S^2 + T^2\right) \right\}^{\tfrac{1}{2}} \text{ ;}$$

$$(38) \quad S \circ T \geq \left\{ \tfrac{1}{2} \left(S^{-p} + T^{-p}\right) \right\}^{-1/p} \circ \left\{ \tfrac{1}{2} \left(S^{-p} + T^{-p}\right) \right\}^{-1/p} \quad \text{if } 1 \leq p < \infty \text{ ;}$$

$$S \circ T \geq \left\{ \tfrac{1}{2} \left(S^q + T^q\right) \right\}^{-1/q} \circ \left\{ \tfrac{1}{2} \left(S^{-p} + T^{-p}\right) \right\}^{-1/p}$$

$$\text{if } q, p > 0 \text{ and } 1/q - 1 = 1/p \leq 1 \text{ ;}$$

$$(39) \quad S \circ T \geq \left\{ \tfrac{1}{2} \left(S^2 + T^2\right) \right\}^{\tfrac{1}{2}} \circ \left\{ \tfrac{1}{2} \left(S^{-1} + T^{-1}\right) \right\}^{-1} \text{ .}$$

The same example prior to Theorem 3 can also be used to prove that the inequalities

$$S \ast T \geq (S\#T) \ast (S\#T) \text{ ,}$$

$$S \ast T \geq \left\{ \tfrac{1}{2} \left(S^{-1} + T^{-1}\right) \right\}^{-1} \ast \left\{ \tfrac{1}{2} (S+T) \right\} \text{ ,}$$

$$S \ast S^{-1} \geq I \text{ ,}$$

$$S \ast S \geq 2(S \ast I)(S \ast S + I \ast I)^{-1}(S \ast I) \text{ ,}$$

which hold for the usual Hadamard product [2], are not valid for the generalized one. Modifying the proofs of these inequalities in [2] we obtain the next three theorems and thus obtain the above inequalities with Jordan-Hadamard product in place of the usual Hadamard product.

THEOREM 4. *Suppose that* S_i, $T_i \in M_n\big(L(H_i)\big)$ $(i = 1, \ldots, k)$ *are positive invertible operator-matrices. If* S_i *commutes with* T_i $(i = 1, \ldots, k)$ *and if* $0 \leq p \leq 1$, *then*

$$(40) \quad \prod_{i=1}^{k} * \left(S_i^{1-p}T_i^{p}\right) \leq \left(\prod_{i=1}^{k} * S_i\right)^{\frac{1}{2}}$$

$$\times \left[\left(\prod_{i=1}^{k} * S_i\right)^{-\frac{1}{2}}\left(\prod_{i=1}^{k} * T_i\right)\left(\prod_{i=1}^{k} * S_i\right)^{-\frac{1}{2}}\right]^{p}\left(\prod_{i=1}^{k} * S_i\right)^{\frac{1}{2}}.$$

Proof. Let $Y = \bigotimes_{i=1}^{k} S_i$ and let ψ be the normalized positive linear map defined by

$$\psi(X) = \phi(Y)^{-\frac{1}{2}}\phi\left(Y^{\frac{1}{2}}XY^{\frac{1}{2}}\right)\phi(Y)^{-\frac{1}{2}}$$

for operator-matrix X, where ϕ is as in Theorem 1. By (8) we have $\psi(X^p) \leq \psi(X)^p$ for $X \geq 0$. With $X = \bigotimes_{i=1}^{k} S_i^{-1}T_i$ we see that

$$\left(\prod_{i=1}^{k} * S_i\right)^{-\frac{1}{2}}\left[\prod_{i=1}^{k} * \left(S_i^{1-p}T_i^{p}\right)\right]\left(\prod_{i=1}^{k} * S_i\right)^{-\frac{1}{2}}$$

$$\leq \left[\left(\prod_{i=1}^{k} * S_i\right)^{-\frac{1}{2}}\left(\prod_{i=1}^{k} * T_i\right)\left(\prod_{i=1}^{k} * S_i\right)^{-\frac{1}{2}}\right]^{p}$$

and the theorem follows.

In case $p = \frac{1}{2}$ the inequality (40) in Theorem 4 can be expressed in terms of geometric mean

$$\prod_{i=1}^{k} * \left(S_i \# T_i\right) \leq \left(\prod_{i=1}^{k} * S_i\right) \# \left(\prod_{i=1}^{k} * T_i\right).$$

This inequality is shown to be valid in the next theorem without the assumption on commutativity.

THEOREM 5. *Let* S_i *and* T_i $(i = 1, \ldots, k)$ *be positive invertible operator-matrices. Then*

$$\prod_{i=1}^{k} * \left(S_i \# T_i\right) \le \left(\prod_{i=1}^{k} * S_i\right) \# \left(\prod_{i=1}^{k} * T_i\right)$$

and

(41) $$\left(S_1 \# T_1\right) \circ \left(S_2 \# T_2\right) \le \left(S_1 \circ S_2\right) \# \left(T_1 \circ T_2\right) .$$

Proof. By Theorem 1 and (6) we have

$$\left(\prod_{i=1}^{k} * S_i\right) \# \left(\prod_{i=1}^{k} * T_i\right) = \left[\phi\left(\bigotimes_{i=1}^{k} S_i\right)\right] \# \left[\phi\left(\bigotimes_{i=1}^{k} T_i\right)\right]$$

$$\ge \phi\left[\left(\bigotimes_{i=1}^{k} S_i\right) \# \left(\bigotimes_{i=1}^{k} T_i\right)\right]$$

$$= \phi\left[\bigotimes_{i=1}^{k} \left(S_i \# T_i\right)\right]$$

$$= \prod_{i=1}^{k} * \left(S_i \# T_i\right) ,$$

where ϕ is as in Theorem 1. The second assertion follows immediately according to (3).

COROLLARY. *The following inequalities hold for positive invertible* S, T *and the identity* I *in* $M_n\left(L(H)\right)$:

$$(S \# T) * I \le (S * I) \# (T * I) ,$$

(42) $$(S \# T) \circ I \le (S \circ I) \# (T \circ I) ;$$

(43) $$S * S \le \left(S^p * S^q\right) \# \left(S^q * S^p\right) \quad if \quad p, q \ge 0 \quad and \quad p + q = 2 ;$$

(44) $$S \circ S \le S^p \circ S^q \quad if \quad p, q \ge 0 \quad and \quad p + q = 2 ;$$

(45) $$(S * T) \# (T * S) \ge (S \# T) * (S \# T) ;$$

(46) $$S \circ T \ge (S \# T) \circ (S \# T) ;$$

(47) $$S \circ S^{-1} \ge I \circ I .$$

COROLLARY. *If* S *and* T *are positive invertible operator-matrices and if* $0 < \lambda < 1$, *then*

(48) $$S \circ T \ge \left\{\lambda S^{-1} + (1-\lambda) T^{-1}\right\}^{-1} \circ \left\{(1-\lambda) S + \lambda T\right\} .$$

In particular

(49)
$$S \circ T \geq \left\{ \tfrac{1}{2} \left(S^{-1} + T^{-1} \right) \right\}^{-1} \circ \left\{ \tfrac{1}{2} (S+T) \right\}$$
$$= \left(S^{-1} + T^{-1} \right)^{-1} \circ (S+T) \ .$$

Proof. Since $\left(XY^{-1}X \right) \# Y = X$ for all positive invertible operators X and Y, it follows from (46) that

$$\left\{ \lambda S^{-1} + (1-\lambda) T^{-1} \right\}^{-1} \circ \left\{ (1-\lambda) S + \lambda T \right\}$$
$$= \lambda^{-1} \left\{ S - (1-\lambda) S \left[(1-\lambda) S + \lambda T \right]^{-1} S \right\} \circ \left\{ (1-\lambda) S + \lambda T \right\}$$
$$= \lambda^{-1} (1-\lambda) S \circ S + S \circ T - \lambda^{-1} (1-\lambda) S \left[(1-\lambda) S + \lambda T \right]^{-1} S \circ \left\{ (1-\lambda) S + \lambda T \right\}$$
$$\leq \lambda^{-1} (1-\lambda) S \circ S + S \circ T - \lambda^{-1} (1-\lambda) S \circ S$$
$$= S \circ T \ .$$

This completes the proof.

We remark that inequality (49) implies inequalities (38) and (39) because

$$\tfrac{1}{2} (S+T) \geq \left\{ \tfrac{1}{2} \left(S^{-1} + T^{-1} \right) \right\}^{-1} \geq \left\{ \tfrac{1}{2} \left(S^{-p} + T^{-p} \right) \right\}^{-1/p} \quad \text{if} \quad 1 \leq p < \infty$$

and

$$\tfrac{1}{2} (S+T) \geq \left\{ \tfrac{1}{2} \left(S^q + T^q \right) \right\}^{1/q} \quad \text{if} \quad \tfrac{1}{2} \leq q \leq 1$$

in view of (5).

THEOREM 6. *Let S and T be positive invertible operator-matrices. Then*

$$S * T \geq (S*I + I*T) \left[S^{-1} * T + S * T^{-1} + 2I*I \right]^{-1} (S*I + I*T)$$

and

(50)
$$S \circ S \geq 2(S \circ I) \left(S \circ S^{-1} + I \circ I \right)^{-1} (S \circ I) \ .$$

Proof. Let $X = S \otimes T$, $Y = S \otimes I + I \otimes T$ and let $Z = YX^{-1}Y$. It follows from (7) that

$$\phi(X) \geq \phi(Y) \phi(Z)^{-1} \phi(Y)$$

where ϕ is as in Theorem 1. Hence

$$S * T \geq (S*I + I*T) \left[S*T^{-1} + S^{-1}*T + 2I*I \right]^{-1} (S*I + I*T)$$

and

$$S \circ S \geq 2(S \circ I)\left(S \circ S^{-1} + I \circ I\right)^{-1}(S \circ I) \ .$$

This completes the proof.

In the proof of Theorem 6 if we let $X = S^r \otimes S^r$ and
$Y = S^p \otimes S^q + S^q \otimes S^p$ $\left(\text{or} \ Y = S^p \otimes I + I \otimes S^q\right)$, where p, q, r are arbitrary, then similar proofs shows that

$$S^r \circ S^r \geq 2\left(S^p \circ S^q\right)\left[S^{2p-r} \circ S^{2q-r} + S^{p+q-r} \circ S^{p+q-r}\right]^{-1}\left(S^p \circ S^q\right)$$

and

(51) $$S^r \circ S^r \geq 2\left(S^p \circ I\right)\left[S^{2p-r} \circ S^{-r} + S^{p-r} \circ S^{p-r}\right]^{-1}\left(S^p \circ I\right) \ .$$

We conclude this paper by looking at the relationship between the generalized Hadamard product and the usual Hadamard product. Let S and T be two nm-square complex matrices. Divide the entries of both matrices into n^2 blocks, each block of which denotes an operator on an m-dimensional Hilbert space. Then we may consider the generalized Hadamard product of the n-square operator-matrices S and T as well as the Hadamard product of the nm-square complex matrices S and T . Let ψ denote the normalized positive linear map which takes the tensor product of two m-square complex matrices to their Hadamard product (cf. [5]). Then ψ is completely positive; that is, for each $p \geq 1$, the linear map

(52)
$$\begin{pmatrix} C_{11} \otimes D_{11} & \cdots & C_{1p} \otimes D_{1p} \\ \vdots & & \vdots \\ C_{p1} \otimes D_{p1} & \cdots & D_{pp} \otimes D_{pp} \end{pmatrix} \longmapsto \begin{pmatrix} \psi\left(C_{11} \otimes D_{11}\right) & \cdots & \psi\left(C_{1p} \otimes D_{1p}\right) \\ \vdots & & \vdots \\ \psi\left(C_{p1} \otimes D_{p1}\right) & \cdots & \psi\left(C_{pp} \otimes D_{pp}\right) \end{pmatrix}$$

where the C's and the D's are m-square matrices, is positive (cf. [3], [5], [7]). It is not hard to see that the positive linear map (52) with $p = n$ takes the generalized Hadamard product of S and T to the Hadamard product of S and T .

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