FORMAL CATEGORIES

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1. Introduction. A category $\mathfrak{C}$ is made up of its class of objects $\mathfrak{O}$, of a function M (or $M_{\mathfrak{C}}$) which assigns to each $(A, B) \in \mathfrak{O} \times \mathfrak{O}$ the class $M(A, B)$ of all morphisms from A to B, of an operation μ which is a class of maps

$$\mu_{ABC}\colon M(B, C) \times M(A, B) \to M(A, C),$$

$A, B, C \in \mathfrak{O}$, and of a family of maps

$$k_A\colon O \to M(A, A),$$

$A \in \mathfrak{O}$, where O is a set with a single element. The operation is associative, i.e. for any $A, B, C, D \in \mathfrak{O}$,

1. $\mu_{ACD}(M(C, D) \times \mu_{ABC}) = \mu_{ABD}(\mu_{BCD} \times M(A, B))$,

and each map k_A picks out from M(A, A) the identity morphism of A, so that

2. $\mu_{ABB}(k_B \times M(A, B)) = 1_{M(A,B)} = \mu_{AAB}(M(A, B) \times k_A)$.

This characterization of the notion of category leads quite naturally to the following generalization. One starts out with a class of objects $\mathfrak{O}$. Then, instead of assigning to each $(A, B) \in \mathfrak{O} \times \mathfrak{O}$ a set $M(A, B)$, one could choose $M(A, B)$ in the class of objects of some ordinary category $\mathfrak{C}$. One must then have a functor $\otimes\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ to be able to define an operation μ consisting of morphisms

$$\mu_{ABC}\colon M(B, C) \otimes M(A, B) \to M(A, C)$$

and one must choose some special object I of $\mathfrak{C}$ to play the role of the set O. Then, to be able to express generalizations of 1 and 2, one must assume that $\otimes$ is associative, i.e. that $(\mathfrak{C} \otimes \mathfrak{C}) \otimes \mathfrak{C}$ and $\mathfrak{C} \otimes (\mathfrak{C} \otimes \mathfrak{C})$ are equal or identifiable, and that I acts as an identity for $\otimes$, i.e. the functors $I \otimes \mathfrak{C}$ and $\mathfrak{C} \otimes I$ are equal to or identifiable with $1_{\mathfrak{C}}$. We thus need to assume precisely that $(\mathfrak{C}, \otimes, I)$ is a multiplicative category in the sense of Bénabou **(1)**. A system $(\mathfrak{O}, \mathfrak{M}, \mu, k)$ satisfying these conditions will be called a $(\mathfrak{C}, \otimes, I)$-formal category, or simply a $(\mathfrak{C}, \otimes, I)$-category. One may then generalize the notion of ordinary functor to a notion of $(\mathfrak{C}, \otimes, I)$-functor. The $(\mathfrak{C}, \otimes, I)$-categories with classes of objects in some fixed universe (with fixed class of objects $\mathfrak{O}$), and their $(\mathfrak{C}, \otimes, I)$-functors (leaving the elements of $\mathfrak{O}$ invariant), form a category $\mathbf{P}(\mathfrak{C}, \otimes, I)$ $(\mathbf{P}_{\mathfrak{O}}(\mathfrak{C}, \otimes, I))$.

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If the objects and morphisms of $\mathfrak{C}$ are categories and functors, multiplied in the usual way, if $\otimes$ is the ordinary direct product of categories and functors, and if I is a trivial category with a single morphism, then the $(\mathfrak{C}, \otimes, I)$-categories are what we shall call categories of the second type. This notion is studied in §2. It is characterized by the five "laws of calculation with functors and natural transformations" of Godement **(4**, Appendice**)** and is a special case of the notion of double category of Ehresmann **(3)**. It is also a natural abstract setting for the study of adjoint functors.

In §3, to the notion of multiplicative category and morphism of multiplicative categories of Bénabou **(1)**, is added a notion of natural transformation of these morphisms. If one starts out from a class of categories, the multiplicative categories $(\mathfrak{C}, \otimes, I)$, where $\mathfrak{C}$ is in this class, with their morphisms and the natural transformations of these, form a category of the second type $\mathbf{C}_m$. Necessary and sufficient conditions for a morphism of multiplicative categories to have an adjoint are then given. Given two multiplicative categories $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$, if $G\colon \mathfrak{C} \to \mathfrak{C}_1$ is a functor permuting with tensor products, then

$$(G, 1_{G\otimes}, 1_{I_1})\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

and as a corollary of the preceding results, if G has a right adjoint, $(G, 1_{G\otimes}, 1_{I_1})$ also has a right adjoint.

In §4, the functions $\mathbf{P}$ and $\mathbf{P}_{\mathfrak{D}}$ are extended to double functors on $\mathbf{C}_m$. That a morphism of multiplicative categories

$$(G, \phi, \delta)\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}, \otimes, I_1)$$

induces a functor

$$\mathbf{P}_{\mathfrak{D}}(G, \phi. \delta)\colon \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I) \to \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}_1, \otimes, I_1)$$

generalizes Theorem 4.2 of **(5)**. Results of the preceding section are then applied to show that, in particular, the forgetful functor which assigns to each additive category its underlying ordinary category has a left adjoint, and then to show that if $\mathfrak{J}$ is any category and if $(\mathfrak{C}, \otimes, I)$ is a multiplicative category, where every $\mathfrak{J}$-diagram in $\mathfrak{C}$ has an inverse limit, then every $\mathfrak{J}$-diagram in $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$ has an inverse limit, and furthermore, if

$$(G. \phi, \delta)\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

where $\mathfrak{C}_1$ also has the property that every $\mathfrak{J}$-diagram has an inverse limit and where G permutes with inverse limits of $\mathfrak{J}$-diagrams, then $\mathbf{P}_{\mathfrak{D}}(G, \phi, \delta)$ also permutes with inverse limits of $\mathfrak{J}$-diagrams. These results are modified to analogous results for the functor $\mathbf{P}$ in the fifth and last section. Direct limits of diagrams in $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$ are also studied there for particular multiplicative categories $(\mathfrak{C}, \otimes, I)$.

2. Some remarks on functors and natural transformations.

DEFINITION 1. *"Category of the second type" is the name we shall give to a system consisting of*

(1) *objects* $A, B, C, \ldots,$

(2) *for each ordered couple* (A, B), *a category* $\mathbf{M}(A, B)$, *with objects* $f, g, h, \ldots$ *and morphisms* $\phi, \psi, \chi, \ldots,$

(3) *a function which assigns to each object* A *an object* 1_A *of* $\mathbf{M}(A, A)$,

(4) *for each ordered triple* (A, B, C), *a functor*

$$\mathbf{M}(B, C) \times \mathbf{M}(A, B) \xrightarrow{*} \mathbf{M}(A, C)$$

satisfying the following conditions:

A1. *If* $\psi \in \mathbf{M}(A, B)$, $1_B * \psi = \psi = \psi * 1_A$.

A2. *If* $\phi \in \mathbf{M}(A, B)$, $\psi \in \mathbf{M}(B, C)$, *and* $\chi \in \mathbf{M}(C, D)$, *then*

$$\chi * (\psi * \phi) = (\chi * \psi) * \phi.$$

The hypothesis that $*$ is a functor of two variables may be written down as follows:

A3. $1_g * 1_f = 1_{f*g}$.

A4. $(\psi'\psi) * (\phi'\phi) = (\psi' * \phi')(\psi * \phi)$.

Also it is well known that A4 is equivalent to the following:

B1. $(\psi'\psi) * f = (\psi' * f)(\psi * f)$ and $g * (\phi'\phi) = (g * \phi')(g * \phi)$.

B2. $(g' * \phi)(\psi * f) = (\psi * f')(g * \phi)$.

Furthermore, A2 obviously implies the following:

B3. $(h * g) * \phi = h * (g * \phi)$ and $\chi * (g * f) = (\chi * g) * f$.

B4. $(h * \psi) * f = h * (\psi * f)$.

Conversely, if we assume that B1 and B2 hold, then B3 and B4 imply A2, for we have that if $\phi: f \to f'$, $\psi: g \to g'$, and $\chi: h \to h'$, then

$$\begin{aligned}
\chi * (\psi * \phi) &= (\chi * (g' * f'))(h * (\psi * \phi)) \\
&= (\chi * (g' * f'))(h * ((\psi * f')(g * \phi))) \\
&= ((\chi * g') * f')(h * (\psi * f'))(h * (g * \phi)) \\
&= ((\chi * g') * f')((h * \psi) * f')((h * g) * \phi) \\
&= (((\chi * g')(h * \psi)) * f')((h * g) * \phi) \\
&= ((\chi * \psi) * f')((h * g) * \phi) \\
&= (\chi * \psi) * \phi.
\end{aligned}$$

Thus, in the definition of the notion of category of the second type, one may replace Axiom (4) by the following, denoting the class of objects of $\mathbf{M}(A, B)$ by $\mathbf{M}_0(A, B)$:

(4′) for each ordered triple (A, B, C), there exist functions

$$\mathbf{M}(B, C) \times \mathbf{M}_0(A, B) \to \mathbf{M}(A, C),$$
$$\mathbf{M}_0(B, C) \times \mathbf{M}(A, B) \to \mathbf{M}(A, C)$$

satisfying conditions A1, A3 (modified in an obvious way), and B1 to B4.

The notion of category of the second type is a particular case of the notion of double category of Ehresmann (**3**).

We notice that an ordinary category may be considered as a category of the second type in which each $M(A, B)$ is simply a class, i.e. a category whose only morphisms are all identity morphisms. In general, in a category of the second type $\mathbf{C}$, the function $\mathbf{M}_0$ defines an ordinary category $\mathbf{C}_0$, the product gf of $g \in M_0(B, C)$ and $f \in M_0(A, B)$ being just $g * f$. With this new notation in mind, one notices the similarity between the laws B1 to B4 and the five rules of calculation with functors and natural transformations of Godement (**4**, Appendice). This gives us a means of constructing categories of the second type. One takes as objects categories $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ... For each ordered couple (A, B), one denotes by $\mathbf{F}(A, B)$ the category whose objects are the functors from A to B and whose morphisms are the natural transformations of these functors. For each category $\mathfrak{A}$, $1_{\mathfrak{A}}$ denotes the identity functor from $\mathfrak{A}$ to $\mathfrak{A}$. Thus, for each ordered triple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, one defines the functions

$$\mathbf{F}(\mathfrak{B}, \mathfrak{C}) \times \mathbf{F}_0(\mathfrak{A}, \mathfrak{B}) \to \mathbf{F}(\mathfrak{A}, \mathfrak{C}),$$
$$\mathbf{F}_0(\mathfrak{B}, \mathfrak{C}) \times \mathbf{F}(\mathfrak{A}, \mathfrak{B}) \to \mathbf{F}(\mathfrak{A}, \mathfrak{C}),$$

in the usual way, i.e. given $\beta\colon S \to S'$ in $\mathbf{F}(\mathfrak{B}, \mathfrak{C})$ and given $U \in \mathbf{F}_0(\mathfrak{A}, \mathfrak{B})$, $(\beta * U)_A = \beta_{U(A)}$ for each object A of $\mathfrak{A}$ so that $\beta * U\colon SU \to S'U$ and given $S \in \mathbf{F}_0(\mathfrak{B}, \mathfrak{C})$ and $\alpha\colon U \to U'$ in $\mathbf{F}(\mathfrak{A}, \mathfrak{B})$, $(S * \alpha)_A = S(\alpha_A)$ for each object A of $\mathfrak{A}$ so that $(S * \alpha)\colon SU \to SU'$. The five rules of calculation of Godement state that these functions satisfy B1 to B4. That they also satisfy A1 and A3 is obvious. Thus, one has defined a true category of the second type. We shall say that these categories of the second type are "concrete."

A very simple example of a category of the second type is that of a pre-ordered semi-group (it has just one object).

Given two categories of the second type $\mathbf{C}$ and $\mathbf{D}$, a functor from the first to second is just a functor from the double category $\mathbf{C}$ to the double category $\mathbf{D}$ in the sense of Ehresmann (**3**), i.e. it is a function $\mathbf{T}$ which assigns to each object A of $\mathbf{C}$ an object $\mathbf{T}(A)$ of $\mathbf{D}$, to each object f of $\mathbf{M}(A, B)$, an object $\mathbf{T}(f)$ of $\mathbf{M}(\mathbf{T}(A), \mathbf{T}(B))$ and to each morphism $\phi\colon f \to f'$ in $\mathbf{M}(A, B)$ a morphism $\mathbf{T}(\phi)\colon\ \mathbf{T}(f) \to \mathbf{T}(f')$ in $\mathbf{M}(\mathbf{T}(A), \mathbf{T}(B))$ such that $\mathbf{T}(1_A) = 1_{\mathbf{T}(A)}$ $\mathbf{T}(1_f) = 1_{\mathbf{T}(f)}$, $\mathbf{T}(\phi'\phi) = \mathbf{T}(\phi')\mathbf{T}(\phi)$, $\mathbf{T}(\psi * \phi) = \mathbf{T}(\psi) * \mathbf{T}(\phi)$. This is equivalent to saying that the restriction $\mathbf{T}(A, B)$ of $\mathbf{T}$ to $\mathbf{M}(A, B)$ is a functor, that $\mathbf{T}(1_A) = 1_{\mathbf{T}(A)}$, and that each diagram

$$\begin{array}{ccc}
\mathbf{M}(B, C) \times \mathbf{M}(A, B) & \xrightarrow{\quad * \quad} & \mathbf{M}(A, C) \\
\mathbf{T}(B, C) \times \mathbf{T}(A, B) \downarrow & & \downarrow \mathbf{T}(A, C) \\
\mathbf{M}(\mathbf{T}(B), \mathbf{T}(C)) \times \mathbf{M}(\mathbf{T}(A), \mathbf{T}(B)) & \xrightarrow{*} & \mathbf{M}(\mathbf{T}(A), \mathbf{T}(C))
\end{array}$$

is commutative. We shall simply call these functors of categories of the second type, or more generally of double categories, "double functors." We notice that the restriction of $\mathbf{T}$ to $\mathbf{C}_0$ is an ordinary functor $\mathbf{T}_0$ from $\mathbf{C}_0$ to $\mathbf{D}_0$. It is obvious what the definition of the notion of double functor of more than one variable should be.

Let $\mathbf{C}$ be a category of the second type. Given four objects A, B, C, and D of $\mathbf{C}$, an object f of $\mathbf{M}(A, B)$, and an object h of $\mathbf{M}(C, D)$, we denote by $\mathbf{M}(f, h)$ the function which assigns to each $\psi \in \mathbf{M}(B, C)$ the morphism $(h * \psi) * f = h * (\psi * f)$ (by B4) of $\mathbf{M}(A, D)$. By A3 and B1, $\mathbf{M}(f, h)$ is a functor from $\mathbf{M}(B, C)$ to $\mathbf{M}(A, D)$, i.e. an object of $\mathbf{F}(\mathbf{M}(B, C), \mathbf{M}(A, D))$. Thus given $\phi: f \to f'$ and $\psi: h \to h'$, for each object g of $\mathbf{M}(B, C)$, we define

$$M(\phi, \chi)_g = \chi * g * \phi: hgf = \mathbf{M}(f, h)(g) \to \mathbf{M}(f', h')(g) = h'gf',$$

and one can show easily that the family $\mathbf{M}(\phi, \chi)$ of all the $\mathbf{M}(\phi, \chi)_g$ is a natural transformation from $\mathbf{M}(f, h)$ to $\mathbf{M}(f', h')$. Now, we assert that the function $\mathbf{M}$ we have just defined is a double functor in two variables, contravariant in the first variable with respect to the $*$-operation, taking its values in the concrete category of the second type whose objects are the $\mathbf{M}(B, C)$. Obviously, we have that

$$\mathbf{M}(1_f, 1_h) = 1_{\mathbf{M}(f,h)} \quad \text{and} \quad \mathbf{M}(1_A, 1_C) = 1_{\mathbf{M}(A,C)}$$

and then, for each object g of $\mathbf{M}(B, C)$, if $\phi: f \to f'$ in $\mathbf{M}(A, B)$ and $\chi: h \to h'$ in $\mathbf{M}(C, D)$,

$$\begin{aligned}
\mathbf{M}(\phi'\phi, \chi'\chi)_g &= (\chi'\chi) * g * (\phi'\phi) \\
&= (\chi'\chi) * ((g * \phi')(g * \phi)) \\
&= (\chi' * g * \phi')(\chi * g * \phi) \\
&= \mathbf{M}(\phi', \chi')_g \mathbf{M}(\phi, \chi)_g = (\mathbf{M}(\phi', \chi')\mathbf{M}(\phi, \chi))_g
\end{aligned}$$

so that $\mathbf{M}(\phi'\phi, \chi'\chi) = \mathbf{M}(\phi', \chi')\mathbf{M}(\phi, \chi)$ while if $\lambda: 1 \to 1'$ in $\mathbf{M}(X, A)$ and $\mu: m \to m'$ in $\mathbf{M}(D, Y)$, then

$$\begin{aligned}
\mathbf{M}(\phi * \lambda, \mu * \chi)_g &= (\mu * \chi) * g * (\phi * \lambda) \\
&= (m' * \chi * g * \phi * 1')(\mu * h * g * f * \lambda) \\
&= \mathbf{M}(1', m')(\mathbf{M}(\phi, \chi)_g). \mathbf{M}(\lambda, \mu)_{\mathbf{M}(f,h)(g)} \\
&= (\mathbf{M}(\lambda, \mu) * \mathbf{M}(\phi, \chi))_g
\end{aligned}$$

so that

$$\mathbf{M}(\phi * \lambda, \mu * \chi) = \mathbf{M}(\lambda, \mu) * \mathbf{M}(\phi, \chi).$$

The restriction $\mathbf{M}_0$ of $\mathbf{M}$ to $\mathbf{C}_0$ is obviously an ordinary functor of two variables, contravariant in the first variable. In the case where $\mathbf{C}$ is a concrete category of the second type, we use the symbols $\mathbf{F}$ and $\mathbf{F}_0$ instead of $\mathbf{M}$ and $\mathbf{M}_0$.

Many notions and results concerning functors and natural transformations may be generalized to categories of the second type. This is the case for the notion of an adjoint functor. Let $\mathbf{C}$ be a category of the second type. For any two objects A and B of $\mathbf{C}$, by an adjoint morphism from A to B we mean a quadruple (f, g, ζ, η), where

$$f \in \mathbf{M}_0(A, B), \qquad g \in \mathbf{M}_0(B, A), \qquad \zeta: 1_B \to fg \text{ in } \mathbf{M}(B, B)$$

and $\eta: gf \to 1_A$ in $\mathbf{M}(A, A)$, such that

$$(f * \eta)(\zeta * f) = 1_f \quad \text{and} \quad (\eta * g)(g * \zeta) = 1_g;$$

see the characterization of adjoint functors given in (**5**, Theorem 4.1). We shall also say that f is a right adjoint of g and that g is a left adjoint of f.

Adjoint morphisms

$$A \xrightarrow{(f,\, g,\, \zeta,\, \eta)} B \xrightarrow{(\bar{f},\, \bar{g},\, \bar{\zeta},\, \bar{\eta})} C$$

may be multiplied as follows (**8**, pp. 106–107):

$$(\bar{f}, \bar{g}, \bar{\zeta}, \bar{\eta})(f, g, \zeta, \eta) = (\bar{f}f, g\bar{g}, (\bar{f} * \zeta * \bar{g})\bar{\zeta}, \eta(g * \bar{\eta} * f)) : A \to C.$$

This product is associative and with respect to it, every object A of $\mathbf{C}$ has an identity adjoint morphism $(1_A, 1_A, 1_{1_A}, 1_{1_A})$. Given two adjoint morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "{(f,\, g,\, \zeta,\, \eta)}"] \arrow[r, shift right, "{(f',\, g',\, \zeta',\, \eta')}"'] & B, \end{tikzcd}$$

by a morphism from the first to the second, we mean a pair (α, β), where $\alpha: f \to f'$ in $\mathbf{M}(A, B)$ and $\beta: g \to g'$ in $\mathbf{M}(B, A)$, such that $(\alpha * \beta)\zeta = \zeta'$ and $\eta'(\beta * \alpha) = \eta$. It is obvious that if

$$(f, g, \zeta, \eta) \xrightarrow{(\alpha,\, \beta)} (f', g', \zeta', \eta') \xrightarrow{(\alpha',\, \beta')} (f'', g'', \zeta'', \eta''),$$

then

$$(f, g, \zeta, \eta) \xrightarrow{(\alpha'\alpha,\, \beta'\beta)} (f'', g'', \zeta'', \eta'')$$

and we define $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta'\beta)$. This product is obviously associative, and with respect to it, each adjoint morphism (f, g, ζ, η) has an identity morphism $(1_f, 1_g)$. Thus, the adjoint morphisms from A to B with their morphisms, multiplied in this way, form a category $\mathbf{M}^{\#}(A, B)$.

Then, in the situation

$$\begin{tikzcd}[column sep=huge] A \arrow[r, shift left=2, "{(f,\, g,\, \zeta,\, \eta)}"] \arrow[r, shift right=2, "{(f',\, g',\, \zeta',\, \eta')}"'] \arrow[r, phantom, "\downarrow (\alpha,\, \beta)"] & B \arrow[r, shift left=2, "{(\bar{f},\, \bar{g},\, \bar{\zeta},\, \bar{\eta})}"] \arrow[r, shift right=2, "{(\bar{f}',\, \bar{g}',\, \bar{\zeta}',\, \bar{\eta}')}"'] \arrow[r, phantom, "\downarrow (\bar{\alpha},\, \bar{\beta})"] & C \end{tikzcd}$$

we define

$$(\bar{\alpha}, \bar{\beta}) * (\alpha, \beta) = (\bar{\alpha} * \alpha, \beta * \bar{\beta})$$

and this is a morphism from $(\bar{f}, \bar{g}, \bar{\zeta}, \bar{\eta})(f, g, \zeta, \eta)$ to $(\bar{f}', \bar{g}', \bar{\zeta}', \bar{\eta}')(f', g', \zeta', \eta')$, for

$$\begin{aligned}(\bar{\alpha} * \alpha * \beta * \bar{\beta})(\bar{f} * \zeta * \bar{g})\bar{\zeta} &= (\bar{\alpha} * ((\alpha * \beta)\zeta) * \bar{\beta})\bar{\zeta} \\ &= (\bar{\alpha} * \zeta' * \bar{\beta})\bar{\zeta} = (\bar{f}' * \zeta' * \bar{g}')(\bar{\alpha} * 1_B * \bar{\beta})\bar{\zeta} \\ &= (\bar{f}' * \zeta' * \bar{g}')(\bar{\alpha} * \bar{\beta})\bar{\zeta}) = (\bar{f}' * \zeta' * \bar{g}')\bar{\zeta}'\end{aligned}$$

and, similarly, one shows that

$$\eta'(g' * \bar{\eta}' * f')(\beta * \bar{\beta} * \bar{\alpha} * \alpha) = \eta(g * \bar{\eta} * f).$$

It is then obvious that we have just defined a new category of the second type $\mathbf{C}^{\#}$. Furthermore, it is obvious that if $\mathbf{T}: \mathbf{C} \to \mathbf{D}$ is a double functor and if, in the situation

$$A \quad \begin{array}{c} \xrightarrow{(f,\, g,\, \zeta,\, \eta)} \\ \downarrow (\alpha, \beta) \\ \xrightarrow[(f',\, g',\, \zeta',\, \eta')]{} \end{array} \quad B,$$

one sets

$$\begin{aligned} \mathbf{T}^{\#}(A) &= A, \\ \mathbf{T}^{\#}(f, g, \zeta, \eta) &= (\mathbf{T}(f), \mathbf{T}(g), \mathbf{T}(\zeta), \mathbf{T}(\eta)), \\ \mathbf{T}^{\#}(\alpha, \beta) &= (\mathbf{T}(\alpha), \mathbf{T}(\beta)), \end{aligned}$$

then $\mathbf{T}^{\#}$ is a double functor from $\mathbf{C}^{\#}$ to $\mathbf{D}^{\#}$. If $\mathbf{T}$ were contravariant with respect to the $*$-operation, then, setting, for each adjoint morphism, (f, g, ζ, η): $A \to B$,

$$\mathbf{T}^{\#}(f, g, \zeta, \eta) = (\mathbf{T}(g), \mathbf{T}(f), \mathbf{T}(\zeta), \mathbf{T}(\eta)),$$

one could verify that $\mathbf{T}^{\#}(f, g, \zeta, \eta)$ is an adjoint morphism from $\mathbf{T}(A)$ to $\mathbf{T}(B)$ and that if one defines $\mathbf{T}^{\#}(A)$ and $\mathbf{T}^{\#}(\alpha, \beta)$ exactly as above, then $\mathbf{T}^{\#}$ is a double functor from $\mathbf{C}^{\#}$ to $\mathbf{D}^{\#}$ covariant with respect to both operations.

These definitions are immediately generalizable to functors of the second type of more than one variable. In particular, given any category of the second type $\mathbf{C}$, in the situation

$$A \xleftarrow{(f,\, g,\, \zeta,\, \eta)} B, \qquad C \xrightarrow{(\bar{f},\, \bar{g},\, \bar{\zeta},\, \bar{\eta})} D,$$

$$\begin{aligned} &\mathbf{M}^{\#}((f, g, \zeta, \eta)(\bar{f}, \bar{g}, \bar{\zeta}, \bar{\eta})) \\ &\qquad = (\mathbf{M}(g, \bar{f}), \mathbf{M}(f, \bar{g}), \mathbf{M}(\zeta, \bar{\zeta}), \mathbf{M}(\eta, \bar{\eta})) : \mathbf{M}(B, C) \to \mathbf{M}(A, D). \end{aligned}$$

Given a category of the second type $\mathbf{C}$, we now use the adjoint morphisms of $\mathbf{C}$ to define an ordinary category $\mathbf{C}_{\#}$. The objects of $\mathbf{C}_{\#}$ are simply the adjoint morphisms of $\mathbf{C}$. Given two such objects

$$(f, g, \zeta, \eta): A \to B, \qquad (f', g', \zeta', \eta'): A' \to B',$$

the morphisms from the first to the second are the couples (v, w), where $v \in \mathbf{M}_0(A, A')$ and $w \in \mathbf{M}_0(B, B')$ such that

(1) $wf = f'v$, $vg = g'w$;
(2) $w * \zeta = \zeta' * w$, $v * \eta = \eta' * v$.

We notice that given a pair (v, w) with property (1), if it satisfies half of condition (2), then it satisfies the other half. For example, if $w * \zeta = \zeta' * w$, then

$$\begin{aligned}(f'v * \eta)(\zeta' * f'v) &= (wf * \eta)(\zeta' * wf) \\ &= (wf * \eta)(w * \zeta * f) \\ &= w * ((f * \eta)(\zeta * f)) \\ &= w * 1_f = 1_{wf} = 1_{f'v}.\end{aligned}$$

However, we know that $(f' * \eta')(\zeta' * f') = 1_{f'}$ so that

$$(f' * \eta' * v)(\zeta' * f'v) = 1_{f'v}$$

and, therefore, $v * \eta = \eta' * v$.

Given two such morphisms

$$(f, g, \zeta, \eta) \xrightarrow{(v, w)} (f', g', \zeta', \eta') \xrightarrow{(v', w')} (f'', g'', \zeta'', \eta''),$$

we define their product as follows:

$$(v', w')(v, w) = (v'v, w'w) : (f, g, \zeta, \eta) \to (f'', g'', \zeta'', \eta'').$$

This operation is associative and, with respect to it, every adjoint morphism (f, g, ζ, η) has an identity morphism $(1_f, 1_g)$. Thus, we have effectively defined a category $\mathbf{C}_\#$. Now if $\mathbf{T}: \mathbf{C} \to \mathbf{D}$ is a double functor, it is obvious that if for each adjoint morphism (f, g, ζ, η) of $\mathbf{C}$, one sets

$$\mathbf{T}_\#(f, g, \zeta, \eta) = (\mathbf{T}(f), \mathbf{T}(g), \mathbf{T}(\zeta), \mathbf{T}(\eta)) = \mathbf{T}^\#(f, g, \zeta, \eta)$$

and if for each morphism

$$(v, w) : (f, g, \zeta, \eta) \to (f', g', \zeta', \eta')$$

one sets

$$\mathbf{T}_\#(v, w) = (\mathbf{T}(v), \mathbf{T}(w)),$$

then $\mathbf{T}_\#$ is a functor from $\mathbf{C}_\#$ to $\mathbf{D}_\#$. Similarly, if $\mathbf{T}$ is contravariant with respect to the $*$-operation and if one sets

$$\begin{aligned}\mathbf{T}_\#(f, g, \zeta, \eta) &= (\mathbf{T}(g), \mathbf{T}(f), \mathbf{T}(\zeta), \mathbf{T}(\eta)) = \mathbf{T}^\#(f, g, \zeta, \eta), \\ \mathbf{T}_\#(v, w) &= (\mathbf{T}(v), \mathbf{T}(w)),\end{aligned}$$

then $\mathbf{T}_\#$ is a *contravariant* functor from $\mathbf{C}_\#$ to $\mathbf{D}_\#$. These definitions extend trivially to double functors of more than one variable.

We shall say that a category $\mathfrak{C}$ is a direct $\mathfrak{J}$-category if every functor $F: \mathfrak{J} \to \mathfrak{C}$ has a direct limit in $\mathfrak{C}$. This is equivalent **(6)** to saying that the canonical functor

$$E_{\mathfrak{J}}: \mathfrak{C} \to \mathbf{F}(\mathfrak{J}, \mathfrak{C})$$

has a left adjoint, i.e. that there is an adjoint morphism

$$(E_{\mathfrak{J}}, L, \lambda, 1_{1_{\mathfrak{C}}}) : \mathfrak{C} \to \mathbf{F}(\mathfrak{J}, \mathfrak{C}).$$

Given a functor $V: \mathfrak{B} \to \mathfrak{C}$, we shall say that it a direct $\mathfrak{J}$-functor if for every functor $F: \mathfrak{J} \to \mathfrak{B}$ with direct limit $\lambda_F: F \to E_{\mathfrak{J}}(B)$,

$$V * \lambda_F \colon VF \to VE_{\mathfrak{J}}(B) = E_{\mathfrak{J}}(V(B))$$

is a direct limit of VF. When $\mathfrak{B}$ and $\mathfrak{C}$ are direct $\mathfrak{J}$-categories, this just means that $(V, \mathbf{F}(\mathfrak{J}, V))$ is a morphism from $(E_{\mathfrak{J}}, L, \lambda, 1_{1_{\mathfrak{B}}})$ to $(E_{\mathfrak{J}}, L, \lambda, 1_{1_{\mathfrak{C}}})$. We notice that in this case, since the diagram

$$\begin{tikzcd} \mathfrak{B} \arrow[r, "E_{\mathfrak{J}}"'] \arrow[d, "V"'] & \mathbf{F}(\mathfrak{J}, \mathfrak{B}) \arrow[d, "\mathbf{F}(\mathfrak{J}, V)"] \\ \mathfrak{C} \arrow[r, "E_{\mathfrak{J}}"] & \mathbf{F}(\mathfrak{J}, \mathfrak{C}) \end{tikzcd}$$

is commutative, to show that V is a direct $\mathfrak{J}$-functor, it suffices to verify that $VL = L\mathbf{F}(\mathfrak{J}, V)$ and that $\mathbf{F}(\mathfrak{J}, V) * \lambda = \lambda * \mathbf{F}(\mathfrak{J}, V)$.

PROPOSITION. *Given two categories $\mathfrak{B}$ and $\mathfrak{C}$, if $\mathfrak{C}$ is a direct $\mathfrak{J}$-category, then so is* $\mathbf{F}(\mathfrak{B}, \mathfrak{C})$.

Given two functors: $V\colon \mathfrak{A} \to \mathfrak{B}$ and $\bar{V}\colon \mathfrak{C} \to \mathfrak{D}$, if $\mathfrak{C}$ and $\mathfrak{D}$ are direct $\mathfrak{J}$-categories and if V is a direct $\mathfrak{J}$-functor, then

$$\mathbf{F}(V, \bar{V}) \colon \mathbf{F}(\mathfrak{B}, \mathfrak{C}) \to \mathbf{F}(\mathfrak{A}, \mathfrak{D})$$

is a direct $\mathfrak{J}$-functor.

Proof. We know that

$$(\mathbf{F}(\mathfrak{B}, E_{\mathfrak{J}}), \mathbf{F}(\mathfrak{B}, L), \mathbf{F}(\mathfrak{B}, \lambda), \mathbf{F}(\mathfrak{B}, 1_{1_{\mathfrak{C}}}))$$

is an adjoint morphism from $\mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{C}))$ to $\mathbf{F}(\mathfrak{B}, \mathfrak{C})$. However, since

$$\mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{C})) \cong \mathbf{F}(\mathfrak{B} \times \mathfrak{J}, \mathfrak{C}) \cong \mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{B}, \mathfrak{C})),$$

the categories $\mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{C}))$ and $\mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{B}, \mathfrak{C}))$ may be identified and then $\mathbf{F}(\mathfrak{B}, E_{\mathfrak{J}})$ is identified with

$$E_{\mathfrak{J}} \colon \mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{B}, \mathfrak{C})) \to \mathbf{F}(\mathfrak{B}, \mathfrak{C}),$$

which proves the first part of the proposition. One, of course, identifies $\mathbf{F}(\mathfrak{B}, L)$ with L and $\mathbf{F}(\mathfrak{B}, \lambda)$ with λ.

The diagram

$$\begin{tikzcd} \mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{C})) \arrow[r, "\mathbf{F}(\mathfrak{B}, L)"] \arrow[d, "\mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \bar{V}))"'] & \mathbf{F}(\mathfrak{B}, \mathfrak{C}) \arrow[d, "\mathbf{F}(\mathfrak{B}, \bar{V})"] \\ \mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{D})) \arrow[r, "\mathbf{F}(\mathfrak{B}, L)"] \arrow[d, "\mathbf{F}(V, \mathbf{F}(\mathfrak{J}, \mathfrak{D}))"'] & \mathbf{F}(\mathfrak{B}, \mathfrak{D}) \arrow[d, "\mathbf{F}(V, \mathfrak{D})"] \\ \mathbf{F}(\mathfrak{A}, \mathbf{F}(\mathfrak{J}, \mathfrak{D})) \arrow[r, "\mathbf{F}(\mathfrak{A}, L)"] & \mathbf{F}(\mathfrak{A}, \mathfrak{D}) \end{tikzcd}$$

is commutative. The first square is commutative because $\bar{V}$ is a direct $\mathfrak{J}$-functor. The second square is commutative simply because $\mathbf{F}$ is a functor. Thus, the diagram

$$\begin{tikzcd} \mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{C})) \arrow[r, "\mathbf{F}(\mathfrak{B}, L)"'] \arrow[d, "{F(V, F(\mathfrak{J}, \bar{V}))}"'] & \mathbf{F}(\mathfrak{B}, \mathfrak{C}) \arrow[d, "{\mathbf{F}(V, \bar{V})}"] \\ \mathbf{F}(\mathfrak{A}, \mathbf{F}(\mathfrak{J}, \mathfrak{D})) \arrow[r, "\mathbf{F}(\mathfrak{A}, L)"] & \mathbf{F}(\mathfrak{A}, \mathfrak{D}) \end{tikzcd}$$

is commutative. But if the categories $\mathbf{F}(\mathfrak{B}, \mathbf{F}(\mathfrak{J}, \mathfrak{C}))$ and $\mathbf{F}(\mathfrak{A}, \mathbf{F}(\mathfrak{J}, \mathfrak{D}))$ are identified with the categories $\mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{B}, \mathfrak{C}))$ and $\mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{A}, \mathfrak{D}))$ as indicated above, then the functor $\mathbf{F}(V, \mathbf{F}(\mathfrak{J}, \bar{V}))$ is identified with $\mathbf{F}(\mathfrak{J}, \mathbf{F}(V, \bar{V}))$. Thus, we have proved the commutativity of the diagram

$$\begin{tikzcd} \mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{B}, \mathfrak{C})) \arrow[r, "L"'] \arrow[d, "{\mathbf{F}(\mathfrak{J}, \mathbf{F}(V, \bar{V}))}"'] & \mathbf{F}(\mathfrak{B}, \mathfrak{C}) \arrow[d, "{\mathbf{F}(V, \bar{V})}"] \\ \mathbf{F}(\mathfrak{J}, \mathbf{F}(\mathfrak{A}, \mathfrak{D})) \arrow[r, "L"] & \mathbf{F}(\mathfrak{A}, \mathfrak{D}) \end{tikzcd}$$

Furthermore, we have that

$$\begin{aligned} \mathbf{F}(V, \mathbf{F}(\mathfrak{J}, \bar{V}))*\mathbf{F}(\lambda, \lambda) &= \mathbf{F}(\lambda * V, \mathbf{F}(\mathfrak{J}, \bar{V})*\lambda) \\ &= \mathbf{F}(\lambda * V, \lambda * \mathbf{F}(\mathfrak{J}, \bar{V})) = \mathbf{F}(\lambda, \lambda)*\mathbf{F}(V, \mathbf{F}(\mathfrak{J}, \bar{V})) \end{aligned}$$

so that, carrying out the proper identifications, we have that

$$\mathbf{F}(\mathfrak{J}, \mathbf{F}(V, \bar{V}))*\lambda = \lambda*\mathbf{F}(\mathfrak{J}, \mathbf{F}(V, \bar{V})).$$

Dually, we shall say that $\mathfrak{C}$ is an inverse $\mathfrak{J}$-category if the functor

$$E_{\mathfrak{J}}: \mathfrak{C} \to \mathbf{F}(\mathfrak{J}, \mathfrak{C})$$

has a right adjoint, i.e. if there exists an adjoint morphism

$$(L, E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}}}, \lambda): \mathbf{F}(\mathfrak{J}, \mathfrak{C}) \to \mathfrak{C},$$

and that a functor $V : \mathfrak{B} \to \mathfrak{C}$ is an inverse $\mathfrak{J}$-functor if for every functor $F : \mathfrak{J} \to \mathfrak{B}$ with inverse limit

$$\lambda_F : E_{\mathfrak{J}}(B) \to F,$$

$$V*\lambda_F : VE_{\mathfrak{J}}(B) = E_{\mathfrak{J}}(V(B)) \to VF$$

is an inverse limit of VF. The dualization of the preceding proposition is left to the reader.

We end this section by noticing that given a category of the second type $\mathbf{C}$ one may define a double category $\mathbf{C}^{\square}$ containing both $\mathbf{C}^{\#}$ and $\mathbf{C}_{\#}$. It consists of all configurations

(1)
$$\begin{tikzcd}
A \arrow[r, "{(f,\, g,\, \zeta,\, \eta)}"] \arrow[d, "v"'] \arrow[dr, phantom, "\searrow\beta \qquad \alpha\swarrow"] & B \arrow[d, "w"] \\
A' \arrow[r, "{(f',\, g',\, \zeta',\, \eta')}"'] & B'
\end{tikzcd}$$

where (f, g, ζ, η) and (f', g', ζ', η') are adjoint morphisms of $\mathbf{C}$, where v and w are objects of $\mathbf{M}(A, A')$ and $\mathbf{M}(B, B')$ respectively, and where $\alpha\colon wf \to f'v$ and $\beta\colon vg \to g'w$ are such that

$$(f' * \beta)(\alpha * g)(w * \zeta) = \zeta' * w,$$
$$(\eta' * v)(g * \alpha)(\beta * f) = v * \eta,$$

and two operations which we define as follows: given two configurations

(2)
$$\begin{tikzcd}
A' \arrow[r, "{(f',\, g',\, \zeta',\, \eta')}"] \arrow[d, "v'"'] \arrow[dr, phantom, "\searrow\beta' \qquad \alpha'\swarrow"] & B' \arrow[d, "w'"] \\
A'' \arrow[r, "{(f'',\, g'',\, \zeta'',\, \eta'')}"'] & B''
\end{tikzcd}$$
(3)
$$\begin{tikzcd}
B \arrow[r, "{(\bar f,\, \bar g,\, \bar\zeta,\, \bar\eta)}"'] \arrow[d, "w"'] \arrow[dr, phantom, "\searrow\bar\beta \qquad \bar\alpha\swarrow"] & C \arrow[d, "x"] \\
B' \arrow[r, "{(\bar f',\, \bar g',\, \bar\zeta',\, \bar\eta')}"'] & C'
\end{tikzcd}$$

the "vertical" product of (1) by (2) is the configuration

$$\begin{tikzcd}
A' \arrow[r, "{(f,\, g,\, \zeta,\, \eta)}"'] \arrow[d, "v'v"'] \arrow[dr, phantom, "\searrow(\beta' * w)(v' * \beta) \qquad (\alpha' * v)(w' * \alpha)\swarrow"] & B \arrow[d, "w'w"] \\
A'' \arrow[r, "{(f'',\, g'',\, \zeta,\, \eta)}"'] & B_{,,}
\end{tikzcd}$$

while the "horizontal product" of (1) by (3) is the configuration

$$\begin{tikzcd}
A \arrow[r, "{(\bar f f,\, g\bar g,\, (\bar f * \zeta * \bar g)\bar\zeta,\, \eta(g * \bar\eta * f))}"'] \arrow[d, "v"'] \arrow[dr, phantom, "\searrow(g' * \bar\beta)(\beta * \bar g) \qquad (\bar f' * \alpha)(\bar\alpha * f)\swarrow"] & C \arrow[d, "x"] \\
A' \arrow[r, "{(\bar f' f',\, g'\bar g',\, (\bar f * \zeta' * \bar g')\bar\zeta',\, \eta'(g' * \bar\eta' * f'))}"'] & C'
\end{tikzcd}$$

If one only considers configurations (1) for which $v = 1_A$ and $w = 1_B$, then one is working in $\mathbf{C}^{\#}$. If one only considers configurations (1) for which $\alpha = 1_{wf}$ and $\beta = 1_{vg}$, and the "vertical" operation, then one is working in $\mathbf{C}_{\#}$. Any double functor $\mathbf{T}\colon \mathbf{C} \to \mathbf{D}$ induces a double functor $\mathbf{T}^{\square} : \mathbf{C}^{\square} \to \mathbf{D}^{\square}$.

3. Multiplicative categories. We begin by recalling some definitions of Bénabou **(1)** in trivially altered form. To simplify the notation, whenever it is convenient, we shall identify functors $F, F'\colon \mathfrak{C} \to \mathfrak{C}'$ that are isomorphic in the sense that there is a natural equivalence between them.

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A multiplicative category is a triple $(\mathfrak{C}, \otimes, I)$ when $\mathfrak{C}$ is a category, $\otimes$ is a functor from $\mathfrak{C} \times \mathfrak{C}$ to $\mathfrak{C}$ that is associative, i.e. the functors $(\mathfrak{C} \otimes \mathfrak{C}) \otimes \mathfrak{C}$ and $\mathfrak{C} \otimes (\mathfrak{C} \otimes \mathfrak{C})$ are equal (or identifiable), and I is an object of $\mathfrak{C}$ such that $I \otimes \mathfrak{C}$ and $\mathfrak{C} \otimes I$ are both equal to (or identifiable with) $1_{\mathfrak{C}}$.

Given two multiplicative categories $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$, a morphism from the first to the second is a triple (G, ϕ, δ), where G is a functor from $\mathfrak{C}$ to $\mathfrak{C}_1$, ϕ is a natural transformation from $\otimes\, (G \times G)$ to $G \otimes$ such that for any three objects, A, B, C of $\mathfrak{C}$,

$C1.$ $\phi_{A \otimes B, C}(\phi_{AB} \otimes G(C)) = \phi_{A, B \otimes C}(G(A) \otimes \phi_{BC})$
and where $\delta \colon I_1 \to G(I)$ is a morphism in $\mathfrak{C}_1$ such that
$C2.$ $\phi_{IA}(\delta \otimes G(A)) = 1_{G(A)} = \phi_{AI}(G(A) \otimes \delta)$.

Given two such morphisms

$$(\mathfrak{C}, \otimes, I) \xrightarrow{(G, \phi, \delta)} (\mathfrak{C}_1, \otimes, I_1) \xrightarrow{(G_1, \phi_1, \delta_1)} (\mathfrak{C}_2, \otimes, I_2),$$

their product is

$$(G_1 G, (G_1 * \phi)(\phi_1 * (G \times G)), G_1(\delta)\delta_1) \colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_2, \otimes, I_2).$$

This product is associative and, with respect to it, every multiplicative category $(\mathfrak{C}, \otimes, I)$ has an identity morphism $(1_{\mathfrak{C}}, 1_{\otimes}, 1_I)$.

To these definitions of Bénabou, we add the following. Given two morphisms of multiplicative categories

$$(G, \phi, \delta), (G', \phi', \delta') \colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

a natural transformation from the first to the second is a natural transformation α from G to G' satisfying the condition that the diagram

$$\begin{tikzcd}
\otimes (G \times G) \arrow[r, "\otimes * (\alpha \times \alpha)"'] \arrow[d, "\phi"] & \otimes (G' \times G') \arrow[d, "\phi'"] \\
G \otimes \arrow[r, "\alpha * \otimes"] & G' \otimes
\end{tikzcd}$$

is commutative and the condition that $\delta' = \alpha_I \delta$.

PROPOSITION 1. *Given two natural transformations*

$$(G, \phi, \delta) \xrightarrow{\alpha} (G', \phi', \delta') \xrightarrow{\alpha'} (G'', \phi'', \delta''),$$

their product $\alpha'\alpha$ is a natural transformation from (G, ϕ, δ) to (G'', ϕ'', δ'').

Proof.
$$\begin{aligned}
\phi''(\otimes * (\alpha'\alpha \times \alpha'\alpha)) &= \phi''(\otimes * (\alpha' \times \alpha'))(\otimes * (\alpha \times \alpha)) \\
&= (\alpha' * \otimes)\phi'(\otimes * (\alpha \times \alpha)) \\
&= (\alpha' * \otimes)(\alpha * \otimes)\phi \\
&= (\alpha'\alpha * \otimes)\phi,
\end{aligned}$$
$$(\alpha'\alpha)_I\, \delta = \alpha'_I\, \alpha_I\, \delta = \alpha'_I\, \delta' = \delta''.$$

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Thus, the morphisms from $(\mathfrak{C}, \otimes, I)$ to $(\mathfrak{C}_1, \otimes, I_1)$ and their natural transformations, multiplied in the ordinary way, form a category

$$\mathbf{F}_m((\mathfrak{C}, \otimes, I), (\mathfrak{C}_1, \otimes, \bar{I}_1)).$$

PROPOSITION 2. *In the situation*

$$(\mathfrak{C}, \otimes, I) \underset{(G', \phi', \delta')}{\overset{(G, \phi, \delta)}{\longrightarrow}} \downarrow\alpha \quad (\mathfrak{C}_1, \otimes, I_1) \underset{(G'_1, \phi'_1, \delta'_1)}{\overset{(G_1, \phi_1, \delta_1)}{\longrightarrow}} \downarrow\alpha_1 \quad (\mathfrak{C}_2, \otimes, I_2),$$

$$\alpha_1 * \alpha\colon (G_1, \phi_1, \delta_1)(G, \phi, \delta) \to (G'_1, \phi'_1, \delta'_1)(G', \phi', \delta').$$

Proof. Let

$$(G_2, \phi_2, \delta_2) = (G_1, \phi_1, \delta_1)(G, \phi, \delta),$$
$$(G'_2, \phi'_2, \delta'_2) = (G'_1, \phi'_1, \delta'_1)(G', \phi', \delta').$$

Then

$$\begin{aligned}
&\phi_2(\otimes * ((\alpha_1 * \alpha) \times (\alpha_1 * \alpha))) \\
&\quad = (G'_1 * \phi')(\alpha_1 * \otimes * (G' \times G'))(\otimes * (\alpha_1 \times \alpha_1) * (G' \times G')) \\
&\qquad\qquad (\otimes * ((G_1 \times G_1) * (\alpha \times \alpha))) \\
&\quad = (G'_1 * \phi')(\alpha_1 * \otimes * (G' \times G'))(\phi_1 * (G' \times G'))((\otimes(G_1 \times G_1)) * (\alpha \times \alpha)) \\
&\quad = (\alpha_1 * \phi')(\phi_1 * (\alpha \times \alpha)) \\
&\quad = (\alpha_1 * (G' \otimes))(G_1 * \phi')((G_1 \otimes) * (\alpha \times \alpha))(\phi_1 * (G \times G)) \\
&\quad = (\alpha_1 * (G' \otimes))(G_1 * \alpha * \otimes)(G_1 * \phi)(\phi_1 * (G \times G)) \\
&\quad = (\alpha_1 * \alpha * \otimes)\phi_2
\end{aligned}$$

and

$$\begin{aligned}
(\alpha_1 * \alpha)_I\, \delta_1 &= (\alpha_1)_{G'(I)}\, G_1(\alpha_I) G_1(\delta)\delta_1 \\
&= (\alpha_1)_{G'(I)}\, G_1(\delta')\delta_1 \\
&= G'_1(\delta')(\alpha_1)_{I_1}\delta_1 \\
&= G'_1(\delta')\delta'_1 = \delta'_2.
\end{aligned}$$

Thus, if we start out from a concrete category of the second type $\mathbf{C}$,[1] we may define a new category of the second type $\mathbf{C}_m$. The objects of $\mathbf{C}_m$ are the multiplicative categories $(\mathfrak{C}, \otimes, I)$, where $\mathfrak{C}$ is an object of $\mathbf{C}$. Given two such objects $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$ the category of morphisms from the first to the second is

$\mathbf{F}_m((\mathfrak{C}, \otimes, I), (\mathfrak{C}_1, \otimes, I_1))$. The $*$-operation of $\mathbf{C}_m$ is just the $*$-operation of $\mathbf{C}$. One may obviously define a canonical double functor $\mathbf{T}_m\colon \mathbf{C}_m \to C$ by setting

$$\mathbf{T}_m(\mathfrak{C}, \otimes, I) = \mathfrak{C}, \qquad \mathbf{T}_m(G, \phi, \delta) = G, \qquad \mathbf{T}_m(\alpha) = \alpha.$$

[1]What we are doing in the first half of this section (up to the Corollary of Theorem 4 inclusively) can be done more abstractly in an arbitrary category of the second type.

We are now able to make the following assertion: Given two morphisms

$$(\mathfrak{C}, \otimes, I) \underset{(H, \psi, \beta)}{\overset{(G, \phi, \delta)}{\rightleftarrows}} (\mathfrak{C}_1, \otimes, I_1),$$

if $\rho: 1_{\mathfrak{C}} \to HG$ and $\sigma: GH \to 1_{\mathfrak{C}_1}$ are natural transformations, then

$$((H, \psi, \beta), (G, \phi, \delta), \rho, \sigma)$$

is an adjoint morphism from $(\mathfrak{C}_1, \otimes, \mathfrak{I}_1)$ to $(\mathfrak{C}, \otimes, \mathfrak{I})$ in $\mathbf{C}_m$ if and only if (H, G, ρ, σ) is an adjoint morphism from $\mathfrak{C}_1$ to $\mathfrak{C}$ and ρ and σ are natural transformations in $\mathbf{C}_m$, from $1_{(\mathfrak{C}, \otimes, I)} = (1_{\mathfrak{C}}, 1_{\otimes}, 1_I)$ to $(H, \psi, \beta)(G, \phi, \delta)$ and from $(G, \phi, \delta)(H, \psi, \beta)$ to $1_{(\mathfrak{C}, \otimes, I_1)} = (1_{\mathfrak{C}_1}, 1_{\otimes}, 1_{I_1})$ respectively. This immediately yields the following theorem.

THEOREM 1. *Given two morphisms of multiplicative categories*

$$(\mathfrak{C}, \otimes, I) \underset{(H, \psi, \beta)}{\overset{(G, \phi, \delta)}{\rightleftarrows}} (\mathbf{C}_1, \otimes, I_1)$$

and two natural transformations

$$\rho: 1_{\mathfrak{C}} \to HG,$$
$$\sigma: GH \to 1_{\mathfrak{C}_1},$$

$((H, \psi, \beta), (G, \phi, \delta), \rho, \sigma)$ is an adjoint morphism from $(\mathfrak{C}_1, \otimes, I_1)$ to $(\mathfrak{C}, \otimes, I)$ in $\mathbf{C}_m$ if and only if (H, G, ρ, σ) is an adjoint morphism from $\mathfrak{C}_1$ to $\mathfrak{C}$ and the following diagrams are commutative:

$$\begin{tikzcd}
\otimes(HG \times HG) \arrow[r, "\psi * (G \times G)"] & H \otimes (G \times G) \\
\otimes \arrow[u, "\otimes * (\rho \times \rho)"] & HG\otimes \arrow[l, "\rho * \otimes"'] \arrow[u, "H * \phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
HG(I) & & H(I_1) \arrow[ll, "H(\delta)"'] \\
& I \arrow[ul, "\rho_I"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\otimes(GH \times GH) \arrow[r, "\phi * (H \times H)"] \arrow[d, "\otimes * (\sigma \times \sigma)"'] & G \otimes (H \times H) \arrow[d, "G * \psi"] \\
\otimes & GH \otimes \arrow[l, "\sigma * \otimes"']
\end{tikzcd}
\qquad
\begin{tikzcd}
GH(I_1) \arrow[d, "\sigma_{I_1}"'] & G(I) \arrow[l, "G(\beta)"'] \\
I_1 & I_1 \arrow[l, "I_{I_1}"'] \arrow[u, "\delta"']
\end{tikzcd}$$

Another criterion for adjoint morphisms in $\mathbf{C}_m$ will be given in Theorem 4.

If $(\mathfrak{C}, \otimes, I)$ is a multiplicative category, then $\otimes$ may be thought of as a functor from $\mathfrak{C}^0 \times \mathfrak{C}^0$ to $\mathfrak{C}^0$ and then $(\mathfrak{C}^0, \otimes, I)$ is also a multiplicative category. Then, if $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$ are multiplicative categories, a morphism from $(\mathfrak{C}^0, \otimes, I)$ to $(\mathfrak{C}_1{}^0, \otimes, I_1)$ is a triple (G, χ, γ) where

$$G: \mathfrak{C} \to \mathfrak{C}_1, \quad \chi: G\otimes \to \otimes(G \times G), \quad \text{and } \gamma: G(I) \to I_1$$

such that

$$(\chi_{AB} \otimes G(C))\chi_{A\otimes B,C} = (G(A) \otimes \chi_{BC})\chi_{A,B\otimes C},$$
$$(\gamma \otimes G(A))\chi_{IA} = 1_{G(A)} = (G(A) \otimes \gamma)\chi_{AI}.$$

THEOREM 2. *If* (G, χ, γ) *is a morphism from* $(\mathfrak{C}^0, \otimes, I)$ *to* $(\mathfrak{C}_1{}^0, \otimes, I_1)$ *and if* (H, G, ρ, σ) *is an adjoint morphism from* $\mathfrak{C}_1$ *to* $\mathfrak{C}$, *then there exists a unique natural transformation* $\psi\colon \otimes\ (H \times H) \to H \otimes$ *and a unique morphism* $\beta\colon I \to H(I_1)$ *such that the diagrams*

$$\begin{tikzcd} GH \otimes \arrow[d, "\sigma * \otimes"'] & G \otimes (H \times H) \arrow[l, "G * \psi"'] \arrow[d, "\chi * (H \times H)"] \\ \otimes & \otimes (GH \times GH) \arrow[l, "\otimes * (\sigma \times \sigma)"'] \end{tikzcd} \qquad \begin{tikzcd} GH(I_1) \arrow[dr, "\sigma_{I_1}"'] & & G(I) \arrow[ll, "G(\beta)"'] \arrow[dl, "\gamma"] \\ & I_1 & \end{tikzcd}$$

are commutative. Furthermore,

$$\Phi(G, \chi, \gamma) = (H, \psi, \beta)\colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I).$$

Proof. First we notice that

$$(\mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, H), \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, G), \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, \rho), \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, \sigma))$$

is an adjoint morphism from $\mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, \mathfrak{C}_1)$ to $\mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, \mathfrak{C})$. Then, since

$$(\otimes * (\sigma \times \sigma))(\chi * (H \times H))\colon G \otimes (H \times H) = \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, G)(\otimes\ (H \times H)) \to \otimes,$$

there exists a unique natural transformation

$$\psi\colon \otimes\ (H \times H) \to \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, H)(\otimes)$$

such that the diagram

$$\begin{tikzcd} \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, G)\mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, H)(\otimes) \arrow[dd, "F(\mathfrak{C}_1 \times \mathfrak{C}_1, \sigma)"'] & \mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, G)(\otimes (H \times H)) \arrow[l, "\mathbf{F}(\mathfrak{C}_1 \times \mathfrak{C}_1, G)(\psi)"'] \arrow[d, equal] \\ & G \otimes (H \times H) \arrow[d, "\chi * (H \times H)"] \\ \otimes & \otimes (GH \times GH) \arrow[l, "\otimes * (\sigma \times \sigma)"'] \end{tikzcd}$$

is commutative, i.e. such that the first diagram in the theorem is commutative. The existence and uniqueness of β is obvious. Let us show that

$$(H, \psi, \beta)\colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I).$$

First of all, if A, B, and C are arbitrary objects of $\mathfrak{C}_1$, we have that

$$\begin{aligned}
&\sigma_{A\otimes B\otimes C}G(\psi_{A\otimes B,C}(\psi_{AB} \otimes H(C))) \\
&\quad = \sigma_{A\otimes B\otimes C}G(\psi_{A\otimes B,C})G(\psi_{AB} \otimes H(C)) \\
&\quad = (\sigma_{A\otimes B} \otimes \sigma_C)\chi_{H(A\otimes B)H(C)}G(\psi_{AB} \otimes H(C)) \\
&\quad = (\sigma_{A\otimes B} \otimes \sigma_C)(G(\psi_{AB}) \otimes GH(C))\chi_{H(A)\otimes H(B),H(C)} \\
&\quad = (\sigma_{A\otimes B}G(\psi_{AB}) \otimes \sigma_C)\chi_{H(A)\otimes H(B),H(C)} \\
&\quad = ((\sigma_A \otimes \sigma_B)\chi_{H(A)H(B)} \otimes \sigma_C)\chi_{H(A)\otimes H(B),H(C)} \\
&\quad = ((\sigma_A \otimes \sigma_B) \otimes \sigma_C)(\chi_{H(A)H(B)} \otimes GH(C))\chi_{H(A)\otimes H(B),H(C)} \\
&\quad = (\sigma_A \otimes (\sigma_B \otimes \sigma_C))(GH(A) \otimes \chi_{H(B),H(C)})\chi_{H(A),H(B)\otimes H(C)} \\
&\quad = (\sigma_A \otimes ((\sigma_B \otimes \sigma_C)\chi_{H(B)H(C)}))\chi_{H(A),H(B)\otimes H(C)} \\
&\quad = (\sigma_A \otimes (\sigma_{B\otimes C}G(\psi_{BC})))\chi_{H(A),H(B)\otimes H(C)} \\
&\quad = (\sigma_A \otimes \sigma_{B\otimes C})(GH(A) \otimes G(\psi_{BC}))\chi_{H(A),H(B)\otimes H(C)} \\
&\quad = (\sigma_A \otimes \sigma_{B\otimes C})\chi_{H(A),H(B\otimes C)}G(H(A) \otimes \psi_{BC}) \\
&\quad = \sigma_{A\otimes B\otimes C}G(\psi_{A,B\otimes C})G(H(A) \otimes \psi_{BC}) \\
&\quad = \sigma_{A\otimes B\otimes C}G(\psi_{A,B\otimes C}(H(A) \otimes \psi_{BC}))
\end{aligned}$$

so that

$$\psi_{A\otimes B,C}(\psi_{AB} \otimes H(C)) = \psi_{A,B\otimes C}(H(A) \otimes \psi_{BC})$$

and then

$$\begin{aligned}
\sigma_A\, G(\psi_{IA}(\beta \otimes H(A))) &= \sigma_A\, G(\psi_{IA})G(\beta \otimes H(A)) \\
&= \sigma_{I\otimes A}G(\psi_{IA})G(\beta \otimes H(A)) \\
&= (\sigma_I \otimes \sigma_A)\chi_{H(A)H(I)}G(\beta \otimes H(A)) \\
&= (\sigma_I \otimes \sigma_A)(G(\beta) \otimes GH(A))\chi_{I,H(A)} \\
&= (\sigma_I\, G(\beta) \otimes \sigma_A)\chi_{I,H(A)} \\
&= (\gamma \otimes \sigma_A)\chi_{I,H(A)} \\
&= ({}_I \otimes \sigma_A)(\gamma \otimes GH(A))\chi_{I,H(A)} \\
&= \sigma_A\, 1_{GH(A)} = \sigma_A\, G(1_{H(A)})
\end{aligned}$$

so that $\psi_{IA}(\beta \otimes H(A)) = 1_{H(A)}$. In similar fashion, one shows that

$$\psi_{AI}(H(A) \otimes \beta) = 1_{H(A)}.$$

The preceding theorem may be applied in the following situation. Again, $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$ are multiplicative categories and (H, G, ρ, σ) is an adjoint morphism from $\mathfrak{C}_1$ to $\mathfrak{C}$. Then, (G, H, σ, ρ) is an adjoint morphism from $\mathfrak{C}^0$ to $\mathfrak{C}_1{}^0$ and if

$$(H, \psi, \beta)\colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I),$$

then

$$(H, \psi, \beta)\colon (\mathfrak{C}_1{}^{00}, \otimes, I_1) \to (\mathfrak{C}^{00}, \otimes, I)$$

and, by Theorem 1, there exists a unique natural transformation

$$\chi\colon G \otimes \to \otimes\ (G \times G)$$

and a unique morphism $\gamma\colon G(I) \to I_1$ such that the diagrams

$$\begin{tikzcd} HG\otimes \arrow[rr, "H * \chi"] & & H \otimes (G \times G) \arrow[d, "\psi * (G \times G)"] \\ \otimes \arrow[u, "\rho * \otimes"] \arrow[rr, "\otimes * (\rho \times \rho)"] & & \otimes (HG \times HG) \end{tikzcd} \qquad \begin{tikzcd} HG(I) \arrow[rr, "H(\gamma)"] & & H(I_1) \\ & I \arrow[ul, "\rho_I"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

are commutative. Furthermore,

$$\Psi(H, \psi, \beta) = (G, \chi, \gamma) : (\mathfrak{C}^0, \otimes, I) \to (\mathfrak{C}_1{}^0, \otimes, I_1).$$

THEOREM 3. *Given two multiplicative categories* $(\mathfrak{C}, \otimes, I)$ *and* $(\mathfrak{C}_1, \otimes, I_1)$ *and an adjoint morphism*

$$(H, G, \rho, \sigma)\colon \mathfrak{C}_1 \to \mathfrak{C},$$

the correspondences Φ *and* Ψ *defined above are inverses of one another.*

Proof. We start out from a morphism

$$(H, \psi, \beta)\colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I)$$

and set $\Psi(H, \psi, \beta) = (G, \chi, \gamma)$ and $\Phi(G, \chi, \gamma) = (H, \psi', \beta')$. Then

$$\sigma_{I_1} G(\beta) = \sigma_{I_1} GH(\gamma) G(\rho_I) = \gamma \sigma_{G(I)}\, G(\rho_I) = \gamma$$

so that by the definition of β', $\beta = \beta'$. Also, for any objects A and B of $\mathfrak{C}_1$,

$$\begin{aligned} \sigma_{A\otimes B} G(\psi_{AB}) &= \sigma_{A\otimes B}\, G(\psi_{AB}) G(1_{H(A)} \otimes 1_{H(B)}) \\ &= \sigma_{A\otimes B}\, G(\psi_{AB}) G(H(\sigma_A) \otimes H(\sigma_B)) G(\rho_{H(A)} \otimes \rho_{H(B)}) \\ &= \sigma_{A\otimes B}\, GH(\sigma_A \otimes \sigma_B) G(\psi_{GH(A)\, GH(B)}) G(\rho_{H(A)} \otimes \rho_{H(B)}) \\ &= \sigma_{A\otimes B}\, GH(\sigma_A \otimes \sigma_B) GH(\chi_{H(A) H(B)}) G(\rho_{H(A)\otimes H(B)}) \\ &= \sigma_{A\otimes B}\, GH((\sigma_A \otimes \sigma_B)\chi_{H(A)H(B)}) G(\rho_{H(A)\otimes H(B)}) \\ &= (\sigma_A \otimes \sigma_B)\chi_{H(A)H(B)}\, \sigma_{G(H(A)\otimes H(B))}\, G(\rho_{H(A)\otimes H(B)}) \\ &= (\sigma_A \otimes \sigma_B)\chi_{H(A)H(B)} \end{aligned}$$

so that

$$(\sigma * \otimes)(G * \psi) = (\otimes * (\sigma \times \sigma))(\chi * (H \times H))$$

and therefore, by the definition of ψ', $\psi = \psi'$.

Thus, we have shown that $\Phi\Psi(H, \psi, \beta) = (H, \psi, \beta)$. The proof of the other half of the theorem is immediate, by duality.

In the situation of Theorems 1 and 2, we have thus established a duality between the morphisms

$$(H, \psi, \beta)\colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I)$$

and the morphisms

$$(G, \chi, \gamma)\colon (\mathfrak{C}^0, \otimes, I) \to (\mathfrak{C}_1{}^0, \otimes, I_1).$$

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THEOREM 4. *A morphism of multiplicative categories*

$$(G, \phi, \delta): (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1)$$

has a right adjoint if and only if there exists an adjoint morphism

$$(H, G, \rho, \sigma): \mathfrak{C}_1 \to \mathfrak{C}$$

and a morphism

$$(G, \chi, \gamma): (\mathfrak{C}^0, \otimes, I) \to (\mathfrak{C}_1{}^0, \otimes, I_1)$$

such that

$$\phi\chi = 1_{G\otimes}, \qquad \delta\gamma = 1_{G(I)},$$
$$(\otimes * (\sigma \times \sigma))(\chi\phi * (H \times H)) = \otimes * (\sigma \times \sigma), \qquad \gamma\delta = 1_{I_1}.$$

When these conditions are satisfied,

$$((H, \psi, \beta), (G, \phi, \delta), \rho, \sigma): (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I)$$

where (H, ψ, β) *is the dual of* (G, χ, γ).

Proof. Assume first of all that there exists an adjoint morphism

$$((H, \psi, \beta), (G, \phi, \delta), \rho, \sigma): (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I)$$

and let (G, χ, γ) be the dual of (H, ψ, β). By Theorem 1, we know that (H, G, ρ, σ) is an adjoint morphism from $\mathfrak{C}_1$ to $\mathfrak{C}$ and that the following conditions are satisfied:

(1) $(H * \phi)(\chi * (G \times G))(\otimes * (\rho \times \rho)) = \rho * \otimes, H(\delta)\beta = \rho_I;$

(2) $(\sigma * \otimes)(G * \psi)(\phi * (H \times H)) = \otimes * (\sigma \times \sigma), \sigma_{I_1}G(\beta)\delta = 1_{I_1}.$

Then, since (H, ψ, β) and (G, χ, γ) are dual, we have that

(3) $(H * \chi)(\rho * \otimes) = (\psi * (G \times G))(\otimes * (\rho \times \rho)), H(\gamma)\rho_I = \beta,$

(4) $(\sigma * \otimes)(G * \psi) = (\otimes * (\sigma \times \sigma))(\chi * (H \times H)), \sigma_{I_1}G(\beta) = \gamma.$

Then

$$\begin{aligned}(H * (\phi\chi))(\rho * \otimes) &= (H * \phi)(H * \chi)(\rho * \otimes)\\ &= (H * \phi)(\psi * (G \times G))(\otimes * (\rho \times \rho))\\ &= (\rho * \otimes) = H(1_{G\otimes})(\rho * \otimes)\end{aligned}$$

so that $\phi\chi = 1_{G\otimes}$ and

$$H(\delta\gamma)\rho_I = H(\delta)H(\gamma)\rho_I = H(\delta)\beta = \rho_I = H(1_{G(I)})\rho_I$$

so that $\delta\gamma = 1_{G(I)}$. Then

$$\begin{aligned}(\otimes * (\sigma \times \sigma))(\chi\phi * (H \times H)) &= (\otimes * (\sigma \times \sigma))(\chi * (H \times H))(\phi * (H \times H))\\ &= (\sigma \times \otimes)(G * \psi)(\phi * (H \times H))\\ &= \otimes * (\sigma \times \sigma),\\ \gamma\delta = \sigma_{I_1}G(\beta)\delta &= 1_{I_1}.\end{aligned}$$

Conversely, assume that (H, G, ρ, σ) is an adjoint morphism from $\mathfrak{C}_1$ to $\mathfrak{C}$ and that

$$(G, \chi, \gamma)\colon (\mathfrak{C}^0, \otimes, I) \to (\mathfrak{C}_1{}^0, \otimes, I_1)$$

satisfies the conditions of the theorem, and let (H, ψ, β) be the dual of (G, χ, γ). Then

$$\begin{aligned}
(\rho * \otimes) &= (H * 1_{G\otimes})(\rho * \otimes)\\
&= (H * \phi)(H * \chi)(\rho * \otimes)\\
&= (H * \phi)(\psi * (G \times G))(\otimes * (\rho \times \rho));\\
H(\delta)\beta &= H(\delta)H(\gamma)\rho_I = H(\delta\gamma)\rho_I = H(1_{G(I)})\rho_I = \rho_I;\\
\otimes * (\sigma \times \sigma) &= (\otimes * (\sigma \times \sigma))(\chi\phi * (H \times H))\\
&= (\otimes * (\sigma \times \sigma))(\chi * (H \times H))(\phi * (H \times H))\\
&= (\sigma * \otimes)(G * \psi)(\phi * (H \times H));\\
\sigma_{I_1} G(\beta)\delta &= \gamma\delta = 1_{I_1}.
\end{aligned}$$

Thus, conditions (1) and (2) are satisfied so that, by Theorem 1,

$$((H, \psi, \beta), (G, \phi, \delta), \rho, \sigma)\colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I).$$

Given two multiplicative categories $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$, a functor $G\colon \mathfrak{C} \to \mathfrak{C}_1$ is said to commute with tensor products if $G \otimes = \otimes (G \times G)$ (i.e. these two functors are identifiable) and $I_1 = G(I)$ (i.e. these two objects are isomorphic; Bénabou **(1)**). When this is true, then

$$(G, 1_{G\otimes}, 1_{I_1})\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

which we indicate simply by

$$G\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1)$$

and obviously

$$G\colon (\mathfrak{C}^0, \otimes, I) \to (\mathfrak{C}_1{}^0, \otimes, I_1).$$

If, furthermore, there exists an adjoint morphism

$$(H, G, \rho, \sigma)\colon \mathfrak{C}_1 \to \mathfrak{C},$$

then the conditions of Theorem 4 are trivially satisfied so that we have the following

Corollary. *Given two multiplicative categories $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$ and an adjoint morphism*

$$(H, G, \rho, \sigma)\colon \mathfrak{C}_1 \to \mathfrak{C},$$

if

$$G\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

then there exists a unique morphism

$$(H, \psi, \beta)\colon (\bar{\mathfrak{C}}\bar{\mathfrak{C}}, \otimes, \bar{I}) \to (\mathfrak{C}, \otimes, I)$$

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such that

$$((H, \psi, \beta), G, \rho, \sigma) \colon (\mathfrak{C}_1, \otimes, I_1) \to (\mathfrak{C}, \otimes, I).$$

We now consider some examples. If $\mathfrak{C}$ is a category with a functor "direct product of two factors," which we denote by $\times$, and with a zero-object O, then $(\mathfrak{C}, \times, O)$ is a multiplicative category. The category $\mathfrak{S}$ of sets and functions (in some fixed universe) has these properties, the role of zero-object being played by any set with a single element.

Given any multiplicative category $(\mathfrak{C}, \otimes, I)$, $M_{\mathfrak{C}}(I, \mathfrak{C})$ is a functor from $\mathfrak{C}$ to $\mathfrak{S}$. If, given $h \in M_{\mathfrak{C}}(I, A)$ and $k \in M_{\mathfrak{C}}(I, B)$ one defines

$$\psi_{AB}(h, k) = h \otimes k \in M_{\mathfrak{C}}(I, A, \otimes, B),$$

then the family ψ of all the ψ_{AB} is a natural transformation from $\times(M_{\mathfrak{C}}(I, \mathfrak{C}) \times M_{\mathfrak{C}}(I, \mathfrak{C}))$ to $M_{\mathfrak{C}}(I, \mathfrak{C})\otimes$. Then, if β is the function from O to $M_{\mathfrak{C}}(I, I)$ which maps the only element of O onto 1_I, it is easy to verify that $(M_{\mathfrak{C}}(I, \mathfrak{C}), \psi, \beta)$ is a morphism from $(\mathfrak{C}, \otimes, I)$ to $(\mathfrak{S}, \times, O)$. We shall say that $(M_{\mathfrak{C}}(I, \mathfrak{C}), \psi, \beta)$ is the canonical morphism from $(\mathfrak{C}, \otimes, I)$ to $(\mathfrak{S}, \times, O)$.

If $\mathfrak{Ab}$ is the category of abelian groups, considered not as an additive category but as an ordinary category, if

$$\otimes \colon \mathfrak{Ab} \times \mathfrak{Ab} \to \mathfrak{Ab}$$

is the ordinary tensor product functor, and if Z is the ring of ordinary integers, considered as an abelian group, then $(\mathfrak{Ab}, \otimes, Z)$ is a multiplicative category.

We shall denote by H the forgetful functor which assigns to each abelian group its underlying set. For any two abelian groups A and B, ν_{AB} will denote the canonical bilinear map from $H(A) \times H(B)$ to $H(A \otimes B)$. From the very definition of the functor $\otimes$, we see that the family ν of all the ν_{AB} is a natural transformation from $\times(H \times H)$ to $H\otimes$. Then, if β is the function from O to $H(Z)$ which maps the only element of O onto the generator of Z, one can verify that (H, ν, β) is essentially the canonical morphism from $(\mathfrak{Ab}, \otimes, Z)$ to $(\mathfrak{S}, \times, O)$.

Now, it is well known that there is an adjoint morphism

$$(H, G, \rho, \sigma) \colon \mathfrak{Ab} \to \mathfrak{S},$$

G being the functor which assigns to each set the free abelian group it generates.

LEMMA. *If E and F are two sets, if A is an abelian group, and if $f \colon E \times F \to H(A)$, then there is a unique bilinear function g making the diagram*

$$\begin{tikzcd} HG(E) \times HG(F) \arrow[dr, "g"] & \\ E \times F \arrow[u, "\rho_E \times \rho_F"] \arrow[r, "f"'] & (HA) \end{tikzcd}$$

commutative.

Proof. For each $x \in E$, let f_x denote the function from F to A which maps each $y \in F$ onto $f(x, y)$. There exists a unique homomorphism g_x making the diagram

$$\begin{tikzcd} HG(F) \arrow[dr, "H(g_x)"] & \\ F \arrow[u, "\rho_F"] \arrow[r, "f_x"] & H(A) \end{tikzcd}$$

commutative. Let us consider the function ϕ from E to $\mathrm{Hom}(G(F), A)$ which maps each x onto g_x. Then, there exists a unique homomorphism ψ making the diagram

$$\begin{tikzcd} HG(E) \arrow[dr, "H(\psi)"] & \\ E \arrow[u, "\rho_E"] \arrow[r, "\phi"] & H(\mathrm{Hom}(G(F), A)) \end{tikzcd}$$

commutative. We then define, for all $a \in HG(E)$ and all $b \in HG(F)$

$$g(a, b) = (\psi(a))(b) \in H(A).$$

It is easy to show that $H(g)$ is bilinear. Then, for any $x \in E$ and any $y \in F$,

$$\begin{aligned}(\rho_E \times \rho_F)(x, y) &= g(\rho_E(x), \rho_F(y)) = (\psi(\rho_E(x)))(\rho_F(y)) \\ &= (\phi(x))(\rho_F(y)) = H(g_x)\rho_F(y) = g_x(y) = f(x, y)\end{aligned}$$

so that $g(\rho_E \times \rho_F) = f$.

Now assume that g' is a bilinear function from $HG(E) \times HG(F)$ to $H(A)$ such that $g'(\rho_E \times \rho_F) = f$. For each $x \in E$, let g'_x denote the homomorphism from $HG(F)$ to $H(A)$ that maps each $b \in HG(F)$ onto $g'(\rho_E(x), b)$. Then, for each $y \in F$,

$$H(g'_x)\rho_F(y) = g'(\rho_E(x), \rho_F(y)) = f(x, y) = f_x(y) = H(g_x)\rho_F(y)$$

so that $H(g'_x)\rho_F = H(g_x)\rho_F$ and therefore $g'_x = g_x$.

Now let ψ' denote the homomorphism from $G(E)$ to $\mathrm{Hom}(G(F), A)$ which maps each $a \in G(E)$ onto the homomorphism from $G(F)$ to A which maps each $b \in G(F)$ onto $g'(a, b)$. For each $x \in E$,

$$\begin{aligned}(H(\psi')\rho_E(x))(b) &= g'(\rho_E(x), b) = g'_x(b) = g_x(b) \\ &= (\phi(x))(b) = (H(\psi)\rho_E(x))(b)\end{aligned}$$

so that $H(\psi')\rho_E = H(\psi)\rho_E$ and therefore $\psi' = \psi$. Then, for any $a \in G(E)$ and any $b \in G(F)$,

$$g'(a, b) = (\psi'(a))(b) = (\psi(a))(b) = g(a, b)$$

so that $g' = g$.

PROPOSITION 3. $G\colon (\mathfrak{S}, \times, O) \to (\mathfrak{Ab}, \otimes, Z)$.

Proof. It is obvious that $G(O) = Z$ so that we must prove that $G\times = \otimes(G \times G)$. Let E and F be any two sets and let $f\colon E \times F \to H(A)$, A being an abelian group. Then, we know that there exists a unique bilinear map $g\colon HG(E) \times HG(F) \to H(A)$, making the diagram

$$\begin{tikzcd} HG(E) \times HG(F) \arrow[dr, "g"] & \\ E \times F \arrow[u, "\rho_E \times \rho_F"] \arrow[r, "f"] & H(A) \end{tikzcd}$$

commutative. Then, we know that there exists a unique homomorphism $\phi\colon G(E) \otimes G(F) \to A$ making the diagram

$$\begin{tikzcd} H(G(E) \otimes G(F)) \arrow[dr, "H(\phi)"] & \\ HG(E) \times HG(F) \arrow[u, "\nu_{G(E)G(F)}"] \arrow[r, "g"] & H(A) \end{tikzcd}$$

commutative. Thus, ϕ is the unique homomorphism making the diagram

$$\begin{tikzcd} H(G(E) \otimes G(F)) \arrow[dr, "H(\phi)"] & \\ E \times F \arrow[u, "\nu_{G(E)G(F)}(\rho_E \times \rho_F)"] \arrow[r, "f"] & H(A) \end{tikzcd}$$

commutative so that $G(E) \otimes G(F)$ may be identified with $G(E \times F)$ and $\nu_{G(E)G(F)}(\rho_E \times \rho_F)$ may be identified with $\rho_{E\times F}$.

We are now able to conclude, by the corollary of Theorem 4, that there exists a unique natural transformation $\psi'\colon \times(H \times H) \to H\otimes$ and a unique morphism $\beta'\colon O \to H(Z)$ such that $((H, \psi', \beta'), G, \rho, \sigma)$ is an adjoint morphism from $(\mathfrak{Ab}, \otimes, Z)$ to $(\mathfrak{S}, \times, O)$.

It is easy to see that $\beta' = \beta$ and we shall show that ψ' coincides with ν. If A and B are any two abelian groups,

$$\begin{aligned} H(\sigma_A \otimes_B G(\nu_{AB}))\rho_{H(A)\times H(B)} &= H(\sigma_A \otimes_B) HG(\nu_{AB})\rho_{H(A)\times H(B)} \\ &= H(\sigma_A \otimes_B)\rho_{H(A\otimes B)}\nu_{AB} = \nu_{AB} = \nu_{AB}(H(\sigma_A) \times H(\sigma_B))(\rho_{H(A)} \times \rho_{H(B)}) \\ &= H(\sigma_A \otimes \sigma_B)\nu_{GH(A)GH(B)}(\rho_{H(A)} \times \rho_{H(B)}) \\ &= H(\sigma_A \otimes \sigma_B)\rho_{H(A)\times H(B)} \end{aligned}$$

so that $\sigma_A \otimes_B G(\nu_{AB}) = \sigma_A \otimes \sigma_B$ and therefore $\psi' = \nu$.

Following are a few more examples of a similar type.

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1. If $(H, G, \rho, \sigma): \mathfrak{Ab} \to \mathfrak{S}$ is the same adjoint morphism as above,

$$H: (\mathfrak{Ab}^0, \times, O) \to (\mathfrak{S}^0, \times, O)$$

and $(G, H, \sigma, \rho): \mathfrak{S}^0 \to \mathfrak{Ab}^0$.

2. Let $\mathfrak{G}$ be the category of groups and homomorphisms, let $*$ denote the functor "free product of two groups," let 1 denote the group with one element, let $+$ denote the functor "direct sum of two sets," and, of course, let $\emptyset$ denote the void set. Then, $(\mathfrak{G}, *, 1)$ and $(\mathfrak{S}, +, \emptyset)$ are multiplicative categories, if H is the forgetful functor from $\mathfrak{G}$ to $\mathfrak{S}$, H has a left adjoint G, i.e. there is an adjoint morphism

$$(H, G, \rho, \sigma): \mathfrak{G} \to \mathfrak{S},$$

G being the functor which assigns to each set the free group it generates and

$$G: (\mathfrak{S}, +, \emptyset) \to (\mathfrak{G}, *, 1).$$

3. If $(H, G, \rho, \sigma): \mathfrak{G} \to \mathfrak{S}$ is as in the preceding example, then

$$(G, H, \sigma, \rho): \mathfrak{S}^0 \to \mathfrak{G}^0$$

and $H: (\mathfrak{G}^0, \times, 1) \to (\mathfrak{S}^0, \times, O)$.

Our next example is a relative one. We start out from a given multiplicative category $(\mathfrak{C}, \otimes, I)$. If $\mathfrak{J}$ is any category, then one may identify the categories $\mathbf{F}(\mathfrak{J}, \mathfrak{C}) \times \mathbf{F}(\mathfrak{J}, \mathfrak{C})$ and $\mathbf{F}(\mathfrak{J}, \mathfrak{C} \times \mathfrak{C})$, $(\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I))$ is a multiplicative category, and obviously

$$E_{\mathfrak{J}}: \mathfrak{C} \to \mathbf{F}(\mathfrak{J}, \mathfrak{C})$$

commutes with tensor products. Now, let us assume that $\mathfrak{C}$ is an inverse $\mathfrak{J}$-category, i.e. that there exists an adjoint morphism

$$(L, E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}}}, \lambda): \mathbf{F}(\mathfrak{J}, \mathfrak{C}) \to \mathfrak{C}.$$

By the Corollary of Theorem 4, we know that there exists a unique natural transformation $\psi: \otimes(L \times L) \to L\mathbf{F}(\mathfrak{J}, \otimes)$ and a unique morphism

$$\beta: I \to LE_{\mathfrak{J}}(I) = I$$

such that

$$(L, \psi, \beta): (\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I)) \to (\mathfrak{C}, \otimes, I)$$

and

$$((L, \psi, \beta), E_I, 1_{1_{\mathfrak{C}}}, \lambda): (\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I)) \to (\mathfrak{C}, \otimes, I).$$

Obviously, $\beta = 1_I$.

For our final example, we start out from an arbitrary category of the second type $\mathbf{C}$. It is obvious that for each object A of $\mathbf{C}$, $(\mathbf{M}(A, A), *, 1_A)$ is a multiplicative category, which we also denote by $\tilde{\mathbf{T}}(A)$.

PROPOSITION 4. *If* $(f, g, \zeta, \eta): A \to B$ *is an adjoint morphism of* $\mathbf{C}$, *if for each pair of objects* h, k *of* $\mathbf{M}(A, A)$ *one sets*

$$\phi_{hk} = fh * \eta * kg: fhgfkg \to fhkg,$$

$\phi = \{\phi_{hk}\}$ *is a natural transformation from* $*(\mathbf{M}(g,f) \times \mathbf{M}(g,f))$ *to* $\mathbf{M}(g,f)*$ *and*

$$\tilde{\mathbf{T}}(f, g, \zeta, \eta) = (\mathbf{M}(g,f), \phi, \zeta): (\mathbf{M}(A, A), *, 1_A) \to (\mathbf{M}(B, B), *, 1_B).$$

Proof. To prove that ϕ is a natural transformation, one must show that given $\gamma: h \to h'$ and $\delta: k \to k'$, the diagram

$$\begin{tikzcd} fhgfkg \arrow[r, "fh * \eta * kg"'] \arrow[d, "f * \gamma * gf * \delta * g"'] & fhkg \arrow[d, "f * \gamma * \delta * g"] \\ fh'gfk'g \arrow[r, "fh' * \eta * k'g"] & fh'k'g \end{tikzcd}$$

is commutative. But this follows from

$$\begin{aligned} (h' * \eta * k')(\gamma * gf * \delta) &= \gamma * \eta * \delta \\ &= (h' * 1_A * \delta)(\gamma * 1_A * k)(h * \eta * k) \\ &= (h' * \delta)(\gamma * k)(h * \eta * k) \\ &= (\gamma * \delta)(h * \eta * k). \end{aligned}$$

One must then show that if h, k, l are objects of $\mathbf{M}(A, A)$,

$$\phi_{hk,e}(\phi_{hk} * flg) = \phi_{h,kl}(fhg * \phi_{kl}),$$

i.e. one must show that

$$(fhk * \eta * lg)(fh * \eta * kgflg) = (fh * \eta * klg)(fhgfk * \eta * lg),$$

which follows from

$$\begin{aligned} (k * \eta)(\eta * kgf) &= (k * \eta)((\eta * k) * gh) \\ &= (\eta * k) * \eta \\ &= \eta * (k * \eta) \\ &= (\eta * k)(gh * (k * \eta)) \\ &= (\eta * k)(ghk * \eta). \end{aligned}$$

Finally, one must show that if h is an object of $\mathbf{M}(A, A)$,

$$\phi_{1_A h}(\zeta * fhg) = 1_{fhg} = \phi_{h1_A}(fhg * \zeta),$$

i.e. that

$$(f * \eta * hg)(\zeta * fhg) = 1_{fhg} = (fh * \eta * g)(fhg * \zeta).$$

But this is obvious by the very definition of the notion of adjoint morphism.

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This proposition was established in the case of a concrete category of the second type of Bénabou **(1)** and the generalization given here is trivial. However, we now add the following:

PROPOSITION 5. *If*

$$(\alpha, \beta): (f, g, \zeta, \eta) \to (f', g', \zeta', \eta')$$

in $\mathbf{C}^{\#}$ *and if one sets* $\tilde{\mathbf{T}}(\alpha, \beta) = \mathbf{M}(\beta, \alpha)$, *then*

$$\tilde{\mathbf{T}}(\alpha, \beta): \tilde{\mathbf{T}}(f, g, \zeta, \eta) \to \tilde{\mathbf{T}}(f', g', \zeta', \eta').$$

Proof. We must show first of all that the diagram

$$\begin{tikzcd}
\otimes(M(g,f) \times M(g,f)) \arrow[r, "\otimes * (M(\beta,\alpha) \times M(\beta,\alpha))"'] \arrow[d, "\phi"'] & \otimes(M(g',f') \times M(g',f')) \arrow[d, "\phi'"] \\
M(g,f) \arrow[r, "M(\beta,\alpha)\otimes"] & M(g',f')
\end{tikzcd}$$

is commutative, i.e. we must show that if h, k are any two objects of $\mathbf{M}(A, A)$, the diagram

$$\begin{tikzcd}
fhgfkg \arrow[r, "\alpha * h * \beta * \alpha * k * \beta"] & f'hg'f'kg' \\
fhkg \arrow[u, "fh * \eta * kg"] \arrow[r, "\alpha * hk * \beta"] & f'hkg' \arrow[u, "f'h * \eta' * kg'"']
\end{tikzcd}$$

is commutative:

$$\begin{aligned}
&(f'h * \eta' * kg')(\alpha * h * \beta * \alpha * k * \beta) \\
&\quad = (\alpha * h) * (\eta'(\beta * \alpha)) * (k * \beta) \\
&\quad = (\alpha * h) * \eta * (k * \beta) \\
&\quad = (\alpha * hk * \beta)(fh * \eta * kg).
\end{aligned}$$

Then, one must verify that $\zeta' = \mathbf{M}(\beta, \alpha)_{1_{\mathfrak{C}}}\zeta = (\beta * \alpha)\zeta$. But this is true by the very definition of a morphism in $\mathbf{C}^{\#}$.

THEOREM 5. $\tilde{\mathbf{T}}$ *is a double functor defined on* $\mathbf{C}^{\#}$.

Proof. Let $A \xrightarrow{(f, g, \zeta, \eta)} B \xrightarrow{(f_1, g_1, \zeta_1, \eta_1)} C$

be two adjoint morphisms of C and let

$$(f_2, g_2, \zeta_2, \eta_2) = (f_1, g_1, \zeta_1, \eta_1)(f, g, \zeta, \eta) = (f_1 f, g\, g_1, (f_1 * \zeta * g_1)\zeta_1, \eta(g * \eta_1 * f))$$

Then, if h and k are objects of $\mathbf{M}(A, A)$,

$$
\begin{aligned}
((\mathbf{M}(g,f) * \phi)(\phi_1 * (\mathbf{M}(g,f) \times \mathbf{M}(g,f))))_{hk} \\
&= (f_1 fh * \eta * kgg_1)(f_1 fhg * \eta_1 * fkgg_1) \\
&= f_1 fh * (\eta(g * \eta_1 * f)) * kgg_1 \\
&= f_2 h * \eta_2 * kg_2 = (\phi_2)_{hk}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tilde{\mathbf{T}}(f_1, g_1, \zeta_1, \eta_1)\tilde{\mathbf{T}}(f, g, \zeta, \eta) &= (M(g_1, f_1), \phi_1, \zeta_1)(M(g,f), \phi, \zeta) \\
&= (M(g_1,f_1)M(g,f), (M(g_1,f_1) * \phi)(\phi_1 * (M(g,f) \times M(g,f)), M(g_1, \zeta_1)(\zeta)\zeta_1) \\
&= (M(g_2, f_2), \phi_2, \zeta_2) = \tilde{\mathbf{T}}(f_1, g_1, \zeta_1, \eta_1)(f, g, \zeta, \eta)).
\end{aligned}
$$

In the situation

$$
\begin{tikzcd}[column sep=huge]
A \arrow[r, bend left, "{(f, g, \zeta, \eta)}"] \arrow[r, bend right, "{(f'g', \zeta', \eta')}"'] \arrow[r, phantom, "\downarrow(\alpha, \beta)"] & B \arrow[r, bend left, "{(f_1, g_1, \zeta_1, \eta_1)}"] \arrow[r, bend right, "{(f_1', g'_1, \zeta'_1, \eta'_1)}"'] \arrow[r, phantom, "\downarrow(\alpha_1, \beta_1)"] & C,
\end{tikzcd}
$$

$$
\begin{aligned}
\tilde{\mathbf{T}}((\alpha_1, \beta_1) * (\alpha, \beta)) &= \tilde{\mathbf{T}}(\alpha_1 * \alpha, \beta * \beta_1) \\
&= \mathbf{M}(\beta * \beta_1, \alpha_1 * \alpha) \\
&= \mathbf{M}(\beta_1, \alpha_1) * \mathbf{M}(\beta, \alpha) = \tilde{\mathbf{T}}(\alpha_1, \beta_1) * \tilde{\mathbf{T}}(\alpha, \beta).
\end{aligned}
$$

Then, if

$$
(f, g, \zeta, \eta) \xrightarrow{(\alpha, \beta)} (f', g', \zeta', \eta') \xrightarrow{(\alpha', \beta')} (f'', g'', \zeta'', \eta''),
$$

$$
\begin{aligned}
\tilde{\mathbf{T}}(\alpha', \beta')(\alpha, \beta)) = \tilde{\mathbf{T}}(\alpha'\alpha, \beta'\beta) = \mathbf{M}(\beta'\beta, \alpha'\alpha) &= \mathbf{M}(\beta', \alpha')\mathbf{M}(\beta, \alpha) \\
&= \tilde{\mathbf{T}}(\alpha', \beta')\tilde{\mathbf{T}}(\alpha, \beta).
\end{aligned}
$$

Now, let

$$(f, g, \zeta, \eta): A \to B$$

be an adjoint morphism of $\mathbf{C}$. We know that

$$(\mathbf{M}(g,f), \mathbf{M}(f, g), \mathbf{M}(\zeta, \zeta), \mathbf{M}(\eta, \eta))$$

is an adjoint morphism from $(\mathbf{M}(A, A), *, 1_A)$ to $(\mathbf{M}(B, B), *, 1_B)$. Thus, we may speak of the dual of

$$(M(g, f), \phi, \zeta): (M(A, A), *, 1_A) \to (M(B, B), *, 1_B).$$

Now, one may define a new category of the second type $\mathbf{C}_1$ by simply replacing each $\mathbf{M}(A, B)$ in $\mathbf{C}$ by $\mathbf{M}(A, B)^0$, keeping the same $*$-operation, and in $\mathbf{C}_1, (g, f, \eta, \zeta)$ is an adjoint morphism from B to A. One may then define, for any two objects h', k' of $\mathbf{M}(B, B)^0$,

$$\psi_{h'k'} = gh' * \zeta * k'f$$

and one has that

$$(\mathbf{M}(f, g), \psi, \eta): (\mathbf{M}(B, B)^0, *, 1_B) \to (\mathbf{M}(A, A)^0, *, 1_A).$$

One may then easily verify that this morphism is the dual of $(\mathbf{M}(g,f), \phi, \zeta)$.

4. Formal categories. Given a multiplicative category $(\mathfrak{C}, \otimes, I)$, by a $(\mathfrak{C}, \otimes, I)$-formal category, or simply a $(\mathfrak{C}, \otimes, I)$-category, we mean a quadruple $(\mathfrak{O}, M, \mu, k)$, where $\mathfrak{O}$ is a class, where M is a function from $\mathfrak{O} \times \mathfrak{O}$ to the class of objects of $\mathfrak{C}$, where μ is a family of morphisms in $\mathfrak{C}$,

$$\mu_{ABC}: M(B, C) \otimes M(A, B) \to M(A, C),$$

$A, B, C \in \mathfrak{O}$, and k is a family of morphisms in $\mathfrak{C}$,

$$k_A: I \to M(A, A),$$

$A \in O$, satisfying the following two conditions.

FC1. μ is associative, i.e. if $A, B, C, D \in \mathfrak{O}$,

$$\mu_{ACD}(M(C, D) \otimes \mu_{ABC}) = \mu_{ABD}(\mu_{BCD} \otimes M(A, B)).$$

FC2. Each k_A is a unity, i.e. if $A, B \in \mathfrak{O}$,

$$\mu_{ABB}(k_B \otimes M(A, B)) = 1_{M(A,B)} = \mu_{AAB}(M(A, B) \otimes k_A).$$

The elements of $\mathfrak{O}$ will be called the objects of $(\mathfrak{O}, M, \mu, k)$.

Given two $(\mathfrak{C}, \otimes, I)$-categories $(\mathfrak{O}, M, \mu, k)$ and $(\mathfrak{O}', M', \mu', k')$, by a $(\mathfrak{C}, \otimes, I)$-functor from the first to the second we mean a function T, assigning to each $A \in \mathfrak{O}$ an element $T(A)$ of $\mathfrak{O}'$ and assigning to each $(A, B) \in \mathfrak{O} \times \mathfrak{O}$ a morphism in $\mathfrak{C}$

$$T(A, B): M(A, B) \to M'(T(A), T(B))$$

satisfying the following two conditions.

FF1. If $A, B, C \in O$,

$$T(A, C)\mu_{ABC} = \mu'_{T(A)\,T(B)\,T(C)}(T(B, C) \otimes T(A, B)).$$

FF2. For any $A \in \mathfrak{O}$, $T(A, A)k_A = k'_{T(A)}$.

Given two $(\mathfrak{C}, \otimes, I)$-functors

$$(\mathfrak{O}, M, \mu, k) \xrightarrow{T} (\mathfrak{O}', M', \mu', k') \xrightarrow{T'} (\mathfrak{O}'', M'', \mu'', k''),$$

we define their product $T'T$ as follows: $T'T(A) = T'(T(A))$ and

$$T'T(A, B) = T'(T(A), T(B))T(A, B): M(A, B) \to M''(T'T(A), T'T(B)).$$

Let us show that $T'T$ is effectively a $(\mathfrak{C}, \otimes, I)$-functor from $(\mathfrak{O}, M, \mu, k)$ to $(\mathfrak{O}'', M'', \mu'', k'')$. If $A, B, C \in \mathfrak{O}$, then

$$\begin{aligned} T'T(A, C)\mu_{ABC} &= T'(T(A), T(B))T(A, B)\mu_{ABC} \\ &= T'(T(A), T(B))\mu'_{T(A)\,T(B)\,T(C)}(T(B, C) \otimes T(A, B)) \end{aligned}$$
$$= \mu''_{T'T(A)\,T'T(B)\,T'T(C)}(T'(T(B), T(C) \otimes T'(T(A), T(B)))(T(B, C) \otimes T(A, B))$$
$$= \mu''_{T'T(A)\,T'T(B)\,T'T(C)}(T'(T(B), T(C))T(B, C) \otimes T'(T(A), T(B))T(A, B))$$
$$= \mu''_{T'T(A)\,T'T(B)\,T'T(C)}(T'T(B, C) \otimes T'T(A, B))$$

so that $T'T$ satisfies FF1. Then

$$T'T(A, A)k_A = T'(T(A), T(A))T(A, A)k_A = T'(T(A), T(A))k'_{T(A)} = k''_{T'T(A)}$$

so that $T'T$ satisfies FF2. This operation is associative and the function which assigns to each $A \in \mathfrak{O}$ itself and to each $(A, B) \in \mathfrak{O} \times \mathfrak{O}$ the identity morphism of $M(A, B)$ is a $(\mathfrak{C}, \otimes, I)$-functor from $(O, M\mu, k)$ to itself which acts as an identity.

At this point, we are tempted to speak of the category $\mathbf{P}(\mathfrak{C}, \otimes, I)$ of all $(\mathfrak{C}, \otimes, I)$-categories and all $(\mathfrak{C}, \otimes, I)$-functors, and this one can do, if one assumes, and we assume this from now on without further mention, that the classes of objects of $(C, \otimes, I)$-categories are all taken from some fixed universe.

THEOREM 1. *If*

$$(G, \phi, \delta) \colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1)$$

is a morphism of multiplicative categories, if $(\mathfrak{O}, M, \mu, k)$ is a $(\mathfrak{C}, \otimes, I)$-category and if one sets, for $A, B, C \in \mathfrak{O}$,

$$\bar{M}(A, B) = GM(A, B), \qquad \bar{\mu}_{ABC} = G(\mu_{ABC})\phi_{M(B,C),M(A,B)}, \qquad \bar{k}_A = G(k_A)\delta,$$

then $(\mathfrak{O}, \bar{M}, \bar{\mu}, \bar{k})$ is a $(\mathfrak{C}_1, \otimes, I_1)$-category which we denote by

$$\mathbf{P}(G, \phi, \delta)(\mathfrak{O}, M, \mu, k),$$

while if

$$T \colon (\mathfrak{O}, M, \mu, k) \to (\mathfrak{O}', M', \mu', k')$$

in $\mathbf{P}(\mathfrak{C}, \otimes, I)$ and if one defines $\bar{T}(A) = T(A)$ and $\bar{T}(A, B) = GT(A, B)$, then $\bar{T}$ is a $(\mathfrak{C}_1, \otimes. I)$-functor from $(\mathfrak{O}, \bar{M}, \bar{\mu}, \bar{k})$ to $(\mathfrak{O}', \bar{M}', \bar{\mu}', \bar{k}')$ which we denote by $\mathbf{P}(G, \phi, \delta)(T)$. The correspondence $\mathbf{P}(G, \phi, \delta)$ is a functor from $\mathbf{P}(\mathfrak{C}, \otimes, I)$ to $\mathbf{P}(\mathfrak{C}_1, \otimes, I_1)$.

Proof. Let us show first of all that $(\mathfrak{O}, \bar{M}, \bar{\mu}, \bar{k})$ is a $(\mathfrak{C}_1, \otimes, I_1)$-category. We notice that

$$\begin{aligned}
&\bar{M}(B, C) \otimes \bar{M}(A, B) \\
&= GM(B, C) \otimes GM(A, B) \xrightarrow[\phi_{M(B,C),M(A,B)}]{} G(M(B, C) \otimes M(A, B)) \xrightarrow{G(\mu_{ABC})} \\
&\qquad\qquad GM(A, C) = \bar{M}(A, C)
\end{aligned}$$

and that

$$I_1 \xrightarrow{\delta} G(I) \xrightarrow{G(k_A)} GM(A, A) = \bar{M}(A, A).$$

Then, if $A, B, C, D \in \mathfrak{O}$,

$$\begin{aligned}
&\bar{\mu}_{ABD}(\bar{\mu}_{BCD} \otimes \bar{M}(A, B)) \\
&= G(\mu_{ABD})\phi_{M(B,D)M(A,D)}(G(\mu_{BCD}) \otimes GM(A, B))(\phi_{M(C,D)M(B,C)} \otimes GM(A, B)) \\
&= G(\mu_{ABD})G(\mu_{BCD} \otimes M(A, B))\phi_{M(C,D)\otimes M(B,C),M(A,B)}(\phi_{M(C,D)M(B,C)} \\
&\qquad \otimes GM(A, B))
\end{aligned}$$

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$$= G(\mu_{ACD})G(M(C, D) \otimes \mu_{ABC})\phi_{M(C,D),M(B,C)\otimes M(A,B)}(GM(C, D) \otimes \phi_{M(B,C)M(A,B)})$$
$$= G(\mu_{ACD})\phi_{M(C,D)M(A,C)}(GM(C, D) \otimes G(\mu_{ABC}))(GM(C, D) \otimes \phi_{M(B,C)M(A,B)})$$
$$= \bar{\mu}_{ACD}(\bar{M}(C, D) \otimes \bar{\mu}_{ABC})$$

so that $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$ satisfies FC1, and also,

$$\begin{aligned}\bar{\mu}_{ABB}(\bar{k}_B \otimes \bar{M}(A, B)) &= G(\mu_{ABB})\phi_{M(B,B)M(A,B)}(G(k_B)\delta \otimes GM(A, B)) \\ &= G(\mu_{ABB})\phi_{M(B,B)M(A,B)}(G(k_B) \otimes GM(A, B))(\delta \otimes GM(A, B)) \\ &= G(\mu_{ABB})G(k_B \otimes M(A, B))\phi_{I,M(A,B)}(\delta \otimes GM(A, B)) \\ &= G(1_{M(A,B)})1_{GM(A,B)} = 1_{\bar{M}(A,B)}\end{aligned}$$

so that $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$ satisfies half of FC2. That it satisfies the other half of FC2 may be shown similarly.

Now, let us show that

$$\bar{T}: (\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k}) \to (\mathfrak{D}', \bar{M}', \bar{\mu}', \bar{k}').$$

If $A, B, C, \in \mathfrak{D}$,

$$\begin{aligned}\bar{T}(A, C)\bar{\mu}_{ABC} &= GT(A, C)G(\mu_{ABC})\phi_{M(B,C)M(A,B)} \\ &= G(\mu'_{ABC})G(T(B, C) \otimes T(A, B))\phi_{M(B,C)M(A,B)} \\ &= G(\mu'_{ABC})\phi_{M'(B,C)M'(A,B)}(GT(B, C) \otimes GT(A, B)) \\ &= \bar{\mu}'_{ABC}(T'(B, C) \otimes T'(A, B))\end{aligned}$$

so that T satisfies FF1 and also

$$\bar{T}(A, A)\bar{k}_A = GT(A, A)G(k_A)\delta = G(T(A, A)k_A)\delta = G(k'_A)\delta = \bar{k}'_A$$

so that T satisfies FF2.

Finally, that $\mathbf{P}(G, \phi, \delta)$ is a functor is trivial. We notice that if G permutes with tensor products and if $(\mathfrak{D}, M, \mu, k)$ is a $(\mathfrak{C}, \otimes, I)$-category, then

$$\mathbf{P}(G)(\mathfrak{D}, M, \mu, k) = (\mathfrak{D}, GM, G(\mu), G(k)).$$

If $(\mathfrak{C}, \otimes, I)$ is a multiplicative category, there is a canonical morphism (see §3)

$$(M_{\mathfrak{C}}(I, \mathfrak{C}), \psi, \beta): (\mathfrak{C}, \otimes, I) \to (\mathfrak{S}, \times, O)$$

and $\mathbf{P}(M_{\mathfrak{C}}(I, \mathfrak{C}), \psi, \beta)$ assigns to each $(\mathfrak{C}, \otimes, I)$-category $(\mathfrak{D}, M, \mu, k)$ its "underlying ordinary category" $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$, where

$$\bar{M}(A, B) = M_{\mathfrak{C}}(I, M(A, B)),$$
$$\bar{\mu}_{ABC}(g, f) = \mu_{ABC}(g \otimes f): I \to M(A, C),$$

and $\bar{k}_A$ assigns k_A to the only element of O.

THEOREM 2. *Given two morphisms of multiplicative categories*

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$$(G, \phi, \delta),\ (G^1, \phi^1, \delta^1)\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1)$$

and given

$$\alpha\colon (G, \phi, \delta) \to (G^1, \phi^1, \delta^1)$$

if for each $(\mathfrak{C}, \otimes, I)$*-category* $(\mathfrak{D}, M, \mu, k)$, *one sets, for each* $A, B \in \mathfrak{D}$, $\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}(A) = A$ *and*

$$\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}(A, B) = \alpha_{M(A,B)}\colon GM(A, B) \to G^1M(A, B),$$

then $\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}$ *is a functor from* $\mathbf{P}(G, \phi, \delta)(\mathfrak{D}, M, \mu, k)$ *to*

$$\mathbf{P}(G^1, \phi^1, \delta^1)(\mathfrak{D}, M, \mu, k)$$

and the family $\mathbf{P}(\alpha)$ *of all the* $\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}$ *is a natural transformation from* $\mathbf{P}(G, \phi, \delta)$ *to* $\mathbf{P}(G^1, \phi^1, \delta^1)$.

Proof. First we set

$$\begin{aligned}\mathbf{P}(G, \phi, \delta)(\mathfrak{D}, M, \mu, k) &= (\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k}),\\ \mathbf{P}(G^1, \phi^1, \delta^1)(\mathfrak{D}, M, \mu, k) &= (\mathfrak{D}, \bar{M}^1, \bar{\mu}^1, \bar{k}^1).\end{aligned}$$

Then,

$$\begin{aligned}\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}(A, C)\bar{\mu}_{ABC} &= \alpha_{M(A,C)}G(\mu_{ABC})\phi_{M(B,C)M(A,B)}\\ &= G^1(\mu_{ABC})\alpha_{M(B,C)\otimes M(A,B)}\phi_{M(B,C)M(A,B)}\\ &= G^1(\mu_{ABC})\phi^1{}_{M(B,C)M(A,B)}(\alpha_{M(B,C)} \otimes \alpha_{M(A,B)})\\ &= \bar{\mu}^1{}_{ABC}(\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}(B, C) \otimes \mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}(A, B))\end{aligned}$$

so that $\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}$ satisfies *FF*1, and

$$\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}(A, A)\bar{k}_A = \alpha_{M(A,A)}G(k_A)\delta = G^1(k_A)\alpha_I\,\delta = G^1(k_A)\delta^1 = \bar{k}_A{}^1$$

so that $\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}$ satisfies *FF*2.

Then, to prove that $\mathbf{P}(\alpha)$ is a natural transformation, one must show that given any $(\mathfrak{C}, \otimes, I)$-functor

$$T\colon (\mathfrak{D}, M, \mu, k) \to (\mathfrak{D}', M', \mu', k'),$$

the diagram

$$\begin{tikzcd}
\mathbf{P}(G, \phi, \delta)(\mathfrak{D}, M, \mu, k) \arrow[r, "\mathbf{P}(\alpha)_{(\mathfrak{D},M,\mu,k)}"] \arrow[d, "\mathbf{P}(G, \phi, \delta)(T)"'] & \mathbf{P}(G^1, \phi^1, \delta^1)(\mathfrak{D}, M, \mu, k) \arrow[d, "\mathbf{P}(G^1, \phi^1, \delta^1)(T)"] \\
\mathbf{P}(G, \phi, \delta)(\mathfrak{D}', M', \mu', k') \arrow[r, "\mathbf{P}(\alpha)_{(\mathfrak{D}',M',\mu',k')}"'] & \mathbf{P}(G^1, \phi^1, \delta^1)(\mathfrak{D}', M', \mu', k')
\end{tikzcd}$$

is commutative. But this is equivalent to showing that for any $A, B \in \mathfrak{D}$, the diagram

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$$\begin{tikzcd} GM(A,B) \arrow[r, "\alpha_{M(A,B)}"] \arrow[d, "GT(A,B)"'] & G^1M(A,B) \arrow[d, "G^1T(A,B)"] \\ GM'(T(A),T(B)) \arrow[r, "\alpha_{M'(T(A),T(B))}"] & G^1M'(T(A),T(B)) \end{tikzcd}$$

is commutative, and this is true simply because α is a natural transformation from G to G^1.

THEOREM 3. *For any concrete category of the second type* $\mathbf{C}$, $\mathbf{P}$ *is a double functor on* $\mathbf{C}_m$.

Proof. First of all, let us consider two morphisms of multiplicative categories

$$(\mathfrak{C}, \otimes, I) \xrightarrow{(G,\phi,\delta)} (\mathfrak{C}_1, \otimes, I_1) \xrightarrow{(\bar{G},\bar{\phi},\bar{\delta})} (\mathfrak{C}_2, \otimes, I_2)$$

and let (O, M, μ, k) be a $(\mathfrak{C}, \otimes, I)$-category. Set

$$\mathbf{P}(\bar{G}, \bar{\phi}, \bar{\delta})\mathbf{P}(G, \phi, \delta)(\mathfrak{O}, M, \mu, k) = (\mathfrak{O}, M'', \mu'', k'').$$

Then, for any $A, B, C \in \mathfrak{O}$,

$$\begin{aligned} \mu''_{ABC} &= \bar{G}(G(\mu_{ABC})\phi_{M(B,C)M(A,B)})\bar{\phi}_{\overline{M}(B,C)\overline{M}(A,B)} \\ &= \bar{G}G(\mu_{ABC})\bar{G}(\phi_{M(B,C)M(A,B)})\bar{\phi}_{GM(B,C)GM(A,B)} \\ &= \bar{G}G(\mu_{ABC})((\bar{G} * \phi)(\bar{\phi} * (G \times G)))_{M(B,C)M(A,B)} \end{aligned}$$

and

$$k''_A = \bar{G}(G(k_A)\delta)\bar{\delta} = \bar{G}G(k_A)\bar{G}(\delta)\bar{\delta}$$

so that $(\mathfrak{O}, M'', \mu'', k'') = \mathbf{P}((\bar{G}, \bar{\phi}, \bar{\delta})(G, \phi, \delta))(\mathfrak{O}, M, \mu, k)$. If

$$T\colon (\mathfrak{O}, M, \mu, k) \to (\mathfrak{O}', M', \mu', k'),$$

then it is trivial to show that

$$\mathbf{P}(\bar{G}, \bar{\phi}, \bar{\delta})\mathbf{P}(G, \phi, \delta)(T) = \mathbf{P}((\bar{G}, \bar{\phi}, \bar{\delta})(G, \phi, \delta))(T).$$

Thus, we have shown that $\mathbf{P}(\bar{G}, \bar{\phi}, \bar{\delta})\mathbf{P}(G, \phi, \delta) = \mathbf{P}((\bar{G}, \bar{\phi}, \bar{\delta})(G, \phi, \delta))$. Then, it is also trivial to establish that if

$$(G, \phi, \delta) \xrightarrow{\alpha} (G', \phi', \delta') \xrightarrow{\alpha'} (G'', \phi'', \delta''),$$

then $\mathbf{P}(\alpha'\alpha) = \mathbf{P}(\alpha')\mathbf{P}(\alpha)$; while if

$$(\bar{G}, \bar{\phi}, \bar{\delta}) \xrightarrow{\bar{\alpha}} (\bar{G}', \bar{\phi}', \bar{\delta}'),$$

then $\mathbf{P}(\bar{\alpha} * \alpha) = \mathbf{P}(\bar{\alpha}) * \mathbf{P}(\alpha)$.

Before we consider some examples, let us reconsider the situation of the Corollary, Theorem 4, §3. One is given two multiplicative categories $(\mathfrak{C}, \otimes, I)$ and $(\mathfrak{C}_1, \otimes, I_1)$ and an adjoint morphism

$$(H, G, \rho, \sigma)\colon C_1 \to C,$$

where G commutes with tensor products. There exists a unique natural transformation $\psi\colon \otimes (H \times H) \to H \otimes$ and a unique morphism $\beta\colon I \to H(I_1)$ making the diagrams

$$\begin{tikzcd}
GH\otimes \arrow[d, "\sigma * \otimes"'] & \\
\otimes & \otimes(GH \times GH) = G\otimes(H\times H) \arrow[l, "\otimes * (\sigma \times \sigma)"'] \arrow[ul, "G * \psi"']
\end{tikzcd}$$

$$\begin{tikzcd}
GH(I_1) \arrow[d, "\sigma_{I_1}"'] & \\
\bar{I} & I_1 = G(I) \arrow[l, "1_{I_1}"'] \arrow[ul, "G(\beta)"']
\end{tikzcd}$$

commutative and (H, ψ, β) is a morphism, while $((H, \psi, \beta), G, \rho, \sigma)$ is an adjoint morphism from $(\mathfrak{C}_1, \otimes, I_1)$ to $(\mathfrak{C}, \otimes, I)$. Then

$$\mathbf{P}^{\#}((H, \psi, \beta), G, \rho, \sigma) = (\mathbf{P}(H, \psi, \beta), \mathbf{P}(G), \mathbf{P}(\rho), \mathbf{P}(\sigma))$$

is an adjoint morphism from $\mathbf{P}(\mathfrak{C}_1, \otimes, I_1)$ to $\mathbf{P}(\mathfrak{C}, \otimes, I)$. Of course, $\mathbf{P}(G)$ assigns to each $(\mathfrak{C}, \otimes, I)$-category $(\mathfrak{D}, M, \mu, k)$ the $(\mathfrak{C}_1, \otimes, I_1)$-category $(\mathfrak{D}, GM, G(\mu), G(k))$. Let $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$ be a $(\mathfrak{C}_1, \otimes, I_1)$-category and let

$$(\mathfrak{D}, M, \mu, k) = \mathbf{P}(H, \psi, \beta)(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k}).$$

Then, $M = H\bar{M}$. Given $A, B, C \in \mathfrak{D}$,

$$\mu_{ABC} = H(\bar{\mu}_{ABC})\psi_{\bar{M}(B,C)\bar{M}(A,B)}$$

is the only morphism in $\mathfrak{C}$ making the diagram

$$\begin{tikzcd}
G(H\bar{M}(B, C) \otimes H\bar{M}(A, B)) \arrow[r, "G(\mu_{ABC})"'] \arrow[d, equal] & GH\bar{M}(A, C) \arrow[dd, "\sigma_{\bar{M}(A,C)}"] \\
GH\bar{M}(B, C) \otimes GH\bar{M}(A, B) \arrow[d, "\sigma_{\bar{M}(B,C)} \otimes \sigma_{\bar{M}(A,B)}"'] & \\
\bar{M}(B, C) \otimes \bar{M}(A, B) \arrow[r, "\bar{\mu}_{ABC}"] & \bar{M}(A, C)
\end{tikzcd}$$

commutative, while $k_A = H(\bar{k}_A)\beta$ is the only morphism making the diagram

$$\begin{tikzcd}
 & GH\bar{M}(A,A) \arrow[d, "\sigma_{\bar{M}(A,A)}"] \\
G(I) = \bar{I} \arrow[ur, "G(k_A)"] \arrow[r, "\bar{k}_A"'] & \bar{M}(A,A)
\end{tikzcd}$$

commutative. Then, $\mathbf{P}(\rho)_{(O,M,\mu,k)}$ is a $(\mathfrak{C}, \otimes, I)$-functor with the following universal property: given a $(\mathfrak{C}, \otimes, I)$-functor

$$T\colon (\mathfrak{D}, M, \mu, k) \to \mathbf{P}(H, \psi, \beta)(\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k}),$$

there exists a unique $(\mathfrak{C}_1, \otimes, I_1)$-functor $\bar{T}$ making the diagram

$$\begin{tikzcd}
\mathbf{P}(H, \psi, \beta)\mathbf{P}(G)(\mathfrak{D}, M, \mu, k) \arrow[dr, "\mathbf{P}(H, \psi, \beta)(\bar{T})"] & \\
(\mathfrak{D}, M, \mu, k) \arrow[u, "\mathbf{P}(\rho)_{(\mathfrak{D}, M, \mu, k)}"] \arrow[r, "T"'] & \mathbf{P}(H, \psi, \beta)(\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k})
\end{tikzcd}$$

commutative. If $A, B \in \mathfrak{D}$, $\bar{T}(A, B)$ is the only morphism in $\mathfrak{C}_1$ making the diagram

$$\begin{tikzcd}
HGM(A, B) \arrow[dr, "H(\bar{T}(A, B))"] & \\
M(A, B) \arrow[u, "\rho_{M(A,B)}"] \arrow[r, "T(A, B)"'] & H\bar{M}(T(A), T(B))
\end{tikzcd}$$

commutative.

Similarly, $\mathbf{P}(\sigma)_{(\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k})}$ is a $(\mathfrak{C}_1, \otimes, I_1)$-functor with the following property: given a $(\mathfrak{C}_1, \otimes, I_1)$-functor

$$\bar{T}\colon P(G)(\mathfrak{D}, M, \mu, k) \to (\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k}),$$

there exists a unique $(\mathfrak{C}, \otimes, I)$-functor T making the diagram

$$\begin{tikzcd}
\mathbf{P}(G)\mathbf{P}(H, \psi, \beta)(\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k}) \arrow[d, "\mathbf{P}(\sigma)_{(\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k})}"'] & \\
(\bar{\mathfrak{D}}, \bar{M}, \bar{\mu}, \bar{k}) & \mathbf{P}(G)(\mathfrak{D}, M, \mu, k) \arrow[l, "\bar{T}"'] \arrow[ul, "\mathbf{P}(G)(T)"']
\end{tikzcd}$$

commutative. If $A, B \in \mathfrak{D}$, $T(A, B)$ is the only morphism in $\mathfrak{C}$ making the diagram

$$\begin{tikzcd}
GH\bar{M}(T(A), T(B)) \arrow[d, "\sigma_{\bar{M}(T(A), T(B))}"'] & \\
\bar{M}(T(A), T(B)) & GM(A, B) \arrow[l, "\bar{T}(A, B)"'] \arrow[ul, "G(T(A, B))"']
\end{tikzcd}$$

commutative.

We now consider some examples. First of all, it is obvious that the $(\mathfrak{S}, \times, O)$-categories are just the ordinary categories $\mathfrak{C}$, where for any two objects A and B of $\mathfrak{C}$, $M(A, B)$ is a set, and that the $(\mathfrak{S}, \times, O)$-functors are just the ordinary functors of these categories.

In the more general case of an arbitrary category $\mathfrak{C}$ with direct product and O-object, the $(\mathfrak{C}, \times, O)$-categories with a single object are the semi-group-like objects of **(2)**. If the objects and morphisms of $\mathfrak{C}$ are categories and functors, multiplied in the usual fashion, if $\times$ is the ordinary direct product of categories and functors, and, of course, if O is the trivial category consisting of a single morphism, then the $(\mathfrak{C}, \times, O)$-categories are categories of the second type and the $(\mathfrak{C}, \times, O)$-functors are double functors.

The $(\mathfrak{Ab}, \otimes, Z)$-categories are just the additive categories and the $(\mathfrak{Ab}, \otimes, Z)$-functors are just the additive functors. The $(\mathfrak{Ab}, \otimes, Z)$-categories with a single object are essentially just rings. Now we know that there is an adjoint morphism

$$((H, \nu, \beta), G, \rho, \sigma) \colon (\mathfrak{Ab}, \otimes, Z) \to (\mathfrak{S}, \times, O),$$

which induces an adjoint morphism

$$\mathbf{P}^{\#}((H, \nu, \beta), G, \rho, \sigma) = (P(H, \nu, \beta), P(G), P(\rho), P(\sigma))$$

from $\mathbf{P}(\mathfrak{Ab}, \otimes, Z)$ to $\mathbf{P}(\mathfrak{S}, \otimes, O)$. It is easy to see that in this case $\mathbf{P}(H, \nu, \beta)$ assigns to each additive category its "underlying ordinary category" and to each additive functor, itself considered as an ordinary functor. We notice that $\mathbf{P}(G)$ assigns to each $(\mathfrak{S}, \times, O)$-category with a single object, i.e. to each semi-group, the semi-group-ring over Z that it generates.

For each additive category $(\mathfrak{D}, M, \mu, k)$, we shall call $\mathbf{P}(G)(\mathfrak{D}, M, \mu, k)$ the free additive category generated by $(\mathfrak{D}, M, \mu, k)$ and $\mathbf{P}(\rho)_{(\mathfrak{D}, M, \mu, k)}$ the canonical functor from $(\mathfrak{D}, M, \mu, k)$ to $\mathbf{P}(H, \nu, \beta)\mathbf{P}(G)(\mathfrak{D}, M, \mu, k)$.

At this point, we notice that if $\mathfrak{D}$ is a fixed class of objects and if, for each multiplicative category $(\mathfrak{C}, \otimes, I)$, $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$ denotes the subcategory of $\mathbf{P}(\mathfrak{C}, \otimes, I)$ whose objects are the $(\mathfrak{C}, \otimes, I)$-categories with $\mathfrak{D}$ as class of objects and whose morphisms are the $(\mathfrak{C}, \otimes, I)$-functors leaving the elements of $\mathfrak{D}$ invariant, then $\mathbf{P}_{\mathfrak{D}}$ may be extended in the obvious way to a double subfunctor of $\mathbf{P}$, which we also denote by $\mathbf{P}_{\mathfrak{D}}$, and that what we have said so far about $\mathbf{P}$ also holds for $\mathbf{P}_{\mathfrak{D}}$.

Now, let $\mathfrak{J}$ be any category and let $(\mathfrak{C}, \otimes, I)$ be a multiplicative category, where $\mathfrak{C}$ is an inverse $\mathfrak{J}$-category. We know that there is an adjoint morphism

$$((L, \psi, 1_I), E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}}}, \lambda)\colon (\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I)) \to (\mathfrak{C}, \otimes, I)$$

so that

$$\mathbf{P}_{\mathfrak{D}}{}^{\#}((L, \psi, 1_I), E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}}}, \lambda) = (\mathbf{P}_{\mathfrak{D}}(L, \psi, 1_I), \mathbf{P}_{\mathfrak{D}}(E_{\mathfrak{J}}), \mathbf{P}_{\mathfrak{D}}(1_{1_{\mathfrak{C}}}), \mathbf{P}_{\mathfrak{D}}(\lambda))$$

is an adjoint morphism from $\mathbf{P}_{\mathfrak{D}}(\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I))$ to $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$. But the category $\mathbf{P}_{\mathfrak{D}}(\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I))$ may be identified with the category $\mathbf{F}(\mathfrak{J}, \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I))$ and then the functor $\mathbf{P}_{\mathfrak{D}}(E_{\mathfrak{J}})$ is identified with

$$E_I : \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I) \to \mathbf{F}(\mathfrak{J}, \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)).$$

Thus, we have the following theorem.

THEOREM 4. *Given a multiplicative category* $(\mathfrak{C}, \otimes, I)$, *if* $\mathfrak{C}$ *is an inverse* $\mathfrak{J}$*-category, then so is* $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$.

Let us see what inverse limits look like in $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$. Let

$$F\colon \mathfrak{J} \to \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$$

be a functor. For each object i of $\mathfrak{J}$, let $F(i) = (\mathfrak{D}, M_i, \mu^{(i)}, k^{(i)})$. Then, for $A, B \in \mathfrak{D}$, the function $\bar{M}(A, B)$ which assigns to each object $i \in \mathfrak{J}$ the object $M_i(A, B)$ and to each morphism $\iota : i \to j$ in $\mathfrak{J}$ the morphism $F(\iota)(A, B)$ is an $\mathfrak{J}$-diagram of $\mathfrak{C}$. Furthermore, the family $\bar{\mu}$ of all the $\mu^{(i)}_{ABC}$, $A, B, C \in \mathfrak{D}$, is a natural transformation from $\bar{M}(B, C) \otimes \bar{M}(A, B)$ to $\bar{M}(A, C)$ and the family $\bar{k}$ of all the $k^{(i)}_A$ is a natural transformation from $E_{\mathfrak{J}}(I)$ to $M(A, A)$. One sees that $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$ is the $(\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I))$-category with which F is identified. Applying $\mathbf{P}_{\mathfrak{D}}(L, \psi, I)$ to $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$, one obtains a $(\mathfrak{C}, \otimes, I)$-category $(\mathfrak{D}, M, \mu, k)$, where for $A, B, C \in \mathfrak{D}$,

$$M(A, B) = L\bar{M}(A, B).$$

Now μ_{ABC} is the unique morphism in $\mathfrak{C}$ making the diagram

$$\begin{tikzcd}
E_{\mathfrak{J}}(L\bar{M}(B, C) \otimes L\bar{M}(A, B)) \arrow[rr, "E_{\mathfrak{J}}(\mu_{ABC})"'] \arrow[d, equal] & & E_{\mathfrak{J}}L\bar{M}(A, C) \arrow[dd, "\lambda_{\bar{M}(A, C)}"] \\
E_{\mathfrak{J}}L\bar{M}(B, C) \otimes E_{\mathfrak{J}}L\bar{M}(A, B) \arrow[d, "\lambda_{\bar{M}(B, C)} \otimes \lambda_{\bar{M}(A, B)}"] & & \\
\bar{M}(B, C) \otimes \bar{M}(A, B) \arrow[rr, "\bar{\mu}_{ABC}"] & & \bar{M}(A, C)
\end{tikzcd}$$

commutative and k_A is the unique morphism in $\mathfrak{C}$ making the diagram

$$\begin{tikzcd}
 & E_{\mathfrak{J}}LM(A, A) \arrow[d, "\lambda_{M(A, A)}"] \\
E_{\mathfrak{J}}(I) \arrow[ur, "E_{\mathfrak{J}}(k_A)"] \arrow[r, "\bar{k}_A"] & M(A, A)
\end{tikzcd}$$

commutative. Furthermore, if we set

$$\Lambda_{(A,B)} = \lambda_{\bar{M}(A,B)} : E_{\mathfrak{J}}M(A, B) \to \bar{M}(A, B),$$

then the family Λ of all the $\Lambda(A, B)$ is a natural transformation from $E_{\mathfrak{J}}(\mathfrak{D}, M, \mu, k)$ to $(\mathfrak{D}, \bar{M}, \bar{\mu}, \bar{k})$, which is identified with F, with the following universal property: given a natural transformation

$$\Pi : E_{\mathfrak{J}}(\mathfrak{D}, M', \mu', k') \to F,$$

there exists a unique $(\mathfrak{C}, \otimes, I)$-functor T making the diagram

$$\begin{tikzcd}
E_{\mathfrak{J}}(\mathfrak{D}, M, \mu, k) \arrow[d, "\Lambda"'] & \\
F \arrow[r, "\Pi"] & E_{\mathfrak{J}}(\mathfrak{D}, M', \mu', k') \arrow[ul, "E_{\mathfrak{J}}(T)"']
\end{tikzcd}$$

commutative. For $A, B \in \mathfrak{D}$, $T(A, B)$ is the unique morphism of $\mathfrak{C}$ making the diagram

$$\begin{tikzcd}
E_{\mathfrak{J}}M(A, B) \arrow[d, "\Lambda(A{,}\,B) \,=\, \lambda_{M(A,B)}"'] & \\
M(A, B) & E_{\mathfrak{J}}M'(A, B) \arrow[l, "\Pi(A{,}\,B)"'] \arrow[ul, "E_{\mathfrak{J}}(T(A{,}\,B))"']
\end{tikzcd}$$

commutative.

As a corollary of Theorem 4, we have that for any class $\mathfrak{D}$, the categories $\mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$ and $\mathbf{P}_{\mathfrak{D}}(\mathfrak{Ab}, \otimes, Z)$ are inverse $\mathfrak{J}$-categories, at least for any proper category $\mathfrak{J}$.

THEOREM 5. *Given a morphism of multiplicative categories*

$$(G, \phi, \delta) : (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

if $\mathfrak{C}$ and $\mathfrak{C}_1$ are inverse $\mathfrak{J}$-categories and if G is an inverse $\mathfrak{J}$-functor, then $\mathbf{P}_{\mathfrak{D}}(G, \phi, \delta)$ is an inverse $\mathfrak{J}$-functor.

Proof. By hypothesis, $(\mathbf{F}(\mathfrak{J}, G), G)$ is a morphism from

$$(L, E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}}}, \lambda) : \mathbf{F}(\mathfrak{J}, \mathfrak{C}) \to \mathfrak{C}$$

to

$$(L, E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}_1}}, \lambda) : \mathbf{F}(\mathfrak{J}, \mathfrak{C}_1) \to \mathfrak{C}_1.$$

It is then easy to verify that $((\mathbf{F}(\mathfrak{J}, G), \mathbf{F}(\mathfrak{J}, \phi), E_{\mathfrak{J}}(\delta)), (G, \phi, \delta))$ is a morphism from

$$((L, \psi, 1_I), E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}}}, \lambda) : (\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes,), E_{\mathfrak{J}}(I)) \to (\mathfrak{C}, \otimes, I)$$

to

$$((L, \bar{\psi}, 1_{I_1}), E_{\mathfrak{J}}, 1_{1_{\mathfrak{C}_1}}, \lambda) : (\mathbf{F}(\mathfrak{J}, \mathfrak{C}_1), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I_1)) \to (\mathfrak{C}_1, \otimes, I_1).$$

It then suffices to apply $(\mathbf{P}_{\mathfrak{D}})_{\#}$ and remark that when the categories

$$\mathbf{P}_{\mathfrak{D}}(\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I)) \quad \text{and} \quad \mathbf{P}_{\mathfrak{D}}(\mathbf{F}(\mathfrak{J}, \mathfrak{C}_1), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I_1))$$

are identified with the categories $\mathbf{F}(\mathfrak{J}, \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I))$ and $\mathbf{F}(\mathfrak{J}, \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}_1, \otimes, I_1))$ respectively, then the functors $\mathbf{P}_{\mathfrak{D}}(\mathbf{F}(\mathfrak{J}, G), \mathbf{F}(\mathfrak{J}, \phi), E_{\mathfrak{J}}(\delta))$ and

$$\mathbf{F}(\mathfrak{J}, \mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I))$$

are identified.

Finally, let us consider a concrete category of the second type $\mathbf{C}$. In $\mathbf{C}$, we consider an adjoint morphism

$$\mathfrak{C} \xrightarrow{(T,\, U,\, \zeta,\, \eta)} \mathfrak{C}_1$$

which induces a morphism of multiplicative categories $\tilde{\mathbf{T}}(T, U, \zeta, \eta) = (\mathbf{M}(U, T), \phi, \zeta)\colon (\mathbf{M}(\mathfrak{C}, \mathfrak{C}), *, 1_{\mathfrak{C}}) \to (\mathbf{M}(\mathfrak{C}_1, \mathfrak{C}_1), *, 1_{\mathfrak{C}_1})$. Now $\mathbf{M}(U, T)$ has a left adjoint, i.e.

$$(\mathbf{M}(U, T), \mathbf{M}(T, U), \mathbf{M}(\zeta, \zeta), \mathbf{M}(\eta, \eta))$$

is an adjoint morphism from $\mathbf{M}(\mathfrak{C}, \mathfrak{C})$ to $\mathbf{M}(\mathfrak{C}_1, \mathfrak{C}_1)$. But, in general, this adjoint morphism does not satisfy the other conditions of Theorem 1, §3, so that it does not induce an adjoint morphism from $\mathbf{P}_{\mathfrak{D}}(\mathbf{M}(\mathfrak{C}, \mathfrak{C}), *, 1_{\mathfrak{C}})$ to $\mathbf{P}_{\mathfrak{D}}(\mathbf{M}(\mathfrak{C}_1, \mathfrak{C}_1), *, 1_{\mathfrak{C}_1})$. However, if $\mathfrak{C}$ and $\mathfrak{C}_1$ are inverse $\mathfrak{J}$-categories, by the Proposition of §2, in dual form, $\mathbf{M}(\mathfrak{C}, \mathfrak{C})$ and $\mathbf{M}(\mathfrak{C}_1, \mathfrak{C}_1)$ are inverse $\mathfrak{J}$-categories and then, since $\mathbf{M}(U, T)$ is a right adjoint, it is an inverse $\mathfrak{J}$-functor (**2**, II, Proposition 2.9) so that by Theorem 5,

$$\mathbf{P}_{\mathfrak{D}}(M(U, T), \phi, \zeta)\colon \mathbf{P}_{\mathfrak{D}}(M(\mathfrak{C}, \mathfrak{C}), *, 1_{\mathfrak{C}}) \to \mathbf{P}_{\mathfrak{D}}(M(\mathfrak{C}_1, \mathfrak{C}_1), *, 1_{\mathfrak{C}_1})$$

is an inverse $\mathfrak{J}$-functor.

We notice that when $\mathfrak{D}$ contains a single element $\mathbf{P}_{\mathfrak{D}}(M(\mathfrak{C}, \mathfrak{C}), *, 1_{\mathfrak{C}})$ is essentially the category of fundamental constructions of Godement (**4**) (or of dual standard constructions of Huber (**5**)) of $\mathfrak{C}$ and their morphisms, a fundamental construction of $\mathfrak{C}$ being a triple (S, π, κ), where S is a functor from $\mathfrak{C}$ to $\mathfrak{C}$ and $\pi\colon S^2 \to S$ and $\kappa\colon 1_{\mathfrak{C}} \to S$ are natural transformations such that

$$\pi(\pi * S) = \pi(S * \pi), \qquad \pi(\kappa * S) = 1_S = \pi(S * \kappa)$$

and a morphism from (S, π, κ) to (S', π', κ') is a natural transformation $\tau\colon S \to S'$ such that

$$\tau\pi = \pi'(\tau * \tau) \quad \text{and} \quad \tau\kappa = \kappa'.$$

We notice that in this context, part of Theorem I is a generalization of **(5,** Theorem 4.2**)**.

5. Appendix on inverse and direct limits. Let $\mathfrak{J}$ be a category and let $(\mathfrak{C}, \otimes, I)$ be a multiplicative category. We shall show that if one replaces $\mathbf{P}_{\mathfrak{D}}$ by $\mathbf{P}$ in Theorems 4 and 5 of §4, then these theorems remain valid, provided $\mathfrak{J}$ is not too large. Of course, one cannot use the same method of proof; the categories $\mathbf{P}(\mathbf{F}(\mathfrak{J}, \mathfrak{C}), \mathbf{F}(\mathfrak{J}, \otimes), E_{\mathfrak{J}}(I))$ and $\mathbf{F}(\mathfrak{J}, \mathbf{P}(\mathfrak{C}, \otimes, I))$ cannot be identified.

We begin with a remark. If $(\mathfrak{D}, M, \mu, k)$ is a $(\mathfrak{C}, \otimes, I)$-category and if $f: \mathfrak{D}' \to \mathfrak{D}$ is a function, then one can define a new $(\mathfrak{C}, \otimes, I)$-category $(\mathfrak{D}', M', \mu', k')$ by setting

$$M'(A', B') = M(f(A'), f(B')), \qquad \mu'_{A'B'C'} = \mu_{f(A')f(B')f(C')}, \qquad k'_{A'} = k_{f(A')}$$

and, obviously, there is a canonical functor

$$T: (\mathfrak{D}', M', \mu', k') \to (\mathfrak{D}, M, \mu, k)$$

defined by

$$T(A') = f(A'), \qquad T(A', B') = 1_{M(f(A'), f(B'))}.$$

We shall say that $(\mathfrak{D}', M', \mu', k')$ is obtained from $(\mathfrak{D}, M, \mu, k)$ by replacing $\mathfrak{D}$ by $\mathfrak{D}'$ through f.

Now let $F: \mathfrak{J} \to \mathbf{P}(\mathfrak{C}, \otimes, I)$ be a functor. We have assumed that the classes of objects of $(\mathfrak{C}, \otimes, I)$-categories are all taken from some fixed universe, and we now assume that $\mathfrak{J}$ is not too large, i.e. that the class of all its objects and morphisms is in this same universe. For each object i of $\mathfrak{J}$, let

$$F(\mathrm{i}) = (\mathfrak{D}_i, M_i, \mu_i, k_i).$$

The function which to each object i of $\mathfrak{J}$ assigns the class $\mathfrak{D}_i$ and which to each morphism $\iota: i \to j$ assigns the function $t_\iota : \mathfrak{D}_i \to \mathfrak{D}_j$ induced by $F(\iota)$ is a functor which has an inverse limit $(\mathfrak{D}', \{q_i: \mathfrak{D}' \to \mathfrak{D}_i\})$. Now, F induces a functor

$$F': \mathfrak{J} \to \mathbf{P}_{\mathfrak{D}'}(\mathfrak{C}, \otimes, I),$$

where for each object i of $\mathfrak{J}$, $F'(i) = (\mathfrak{D}', M'_i, \mu'_i, k'_i)$ is obtained from $(\mathfrak{D}_i, M_i, \mu_i, k_i)$ by replacing $\mathfrak{D}_i$ by $\mathfrak{D}'$ through q_i and where for each morphism $\iota: i \to j$ of $\mathfrak{J}$, $F'(\iota)(A) = A$ and

$$F'(\iota)(A', B') = F(\iota)(q_i(A'), q_i(B')): M_i(q_i(A'), q_i(B')) \to \\ M_j(t_\iota q_i(A'), t_\iota q_i(B')) = M_j(q_j(A'), q_j(B')).$$

Furthermore, for each object i of $\mathfrak{J}$, there is a canonical $(\mathfrak{C}, \otimes, I)$-functor

$$Q_i: (\mathfrak{D}', M'_i, \mu'_i, k'_i) \to (\mathfrak{D}_i, M_i, \mu_i, k_i)$$

and these form a natural transformation $Q\colon F' \to F$. Given a natural transformation

$$P\colon E_{\mathfrak{J}}(O'', M'', \mu'', k'') \to F$$

for each object i of $\mathfrak{J}$, P_i induces a function $p_i\colon \mathfrak{D}'' \to \mathfrak{D}_i$ and there exists a unique function $h\colon \mathfrak{D}'' \to \mathfrak{D}'$ such that $p_i = q_i h$ for each i. Then, for each i, we set $H_i(A'') = A''$ and

$$\begin{aligned} H_i(A'', B'') = P_i(A'', B'')\colon M''(A'', B'') &\to M_i(p_i(A''), p_i(B'')) \\ &= M_i(q_i h(A''), q_i h(B'')) = M'_i(h(A''), h(B'')) \end{aligned}$$

and the class H of all the H_i is the only natural transformation making the diagram

$$\begin{tikzcd} F' \arrow[d, "Q"'] & \\ F & E_{\mathfrak{J}}(\mathfrak{D}'', M'', \mu'', k'') \arrow[l, "P"'] \arrow[ul, "H"'] \end{tikzcd}$$

commutative.

By Theorem 4, §4, F' has an inverse limit

$$\Lambda\colon E_{\mathfrak{J}}(\mathfrak{D}', M', \mu', k') \to F'$$

in $\mathbf{P}_{\mathfrak{D}'}(\mathfrak{C}, \otimes, I)$. Then, F' induces a functor

$$F''\colon \mathfrak{J} \to \mathbf{P}_{h(\mathfrak{D}'')}(\mathfrak{C}, \otimes, I),$$

where for each object i of $\mathfrak{J}$, $F''(i)$ is the $(\mathfrak{C}, \otimes, I)$-category obtained from $(\mathfrak{D}', M'_i, \mu'_i, k'_i)$ by replacing $\mathfrak{D}'$ by $h(\mathfrak{D}'')$ through the natural inclusion h' of $h(\mathfrak{D}'')$ in $\mathfrak{D}'$, and the direct limit of F'' in $\mathbf{P}_{h(\mathfrak{D}'')}(\mathfrak{C}, \otimes, I)$ is just the $(\mathfrak{C}, \otimes, I)$-category obtained from $(\mathfrak{D}', M', \mu', k')$ by replacing $\mathfrak{D}'$ by $h(\mathfrak{D}'')$ through h. It is then easy to see that there is a unique $(\mathfrak{C}, \otimes, I)$-functor K making the diagram

$$\begin{tikzcd} E_{\mathfrak{J}}(\mathfrak{D}', M', \mu', k') \arrow[d, "\Lambda"'] & \\ F' & E_{\mathfrak{J}}(\mathfrak{D}'', M'', \mu'', k'') \arrow[l, "H"'] \arrow[ul, "E_{\mathfrak{J}}(K)"'] \end{tikzcd}$$

commutative. We have thus proved the following theorem:

THEOREM 1. *If $(\mathfrak{C}, \otimes, I)$ is a multiplicative category and if $\mathfrak{C}$ is an inverse $\mathfrak{J}$-category, then $\mathbf{P}(\mathfrak{C}, \otimes, I)$ is also an inverse $\mathfrak{J}$-category provided the class of all objects and morphisms of $\mathfrak{J}$ is in the universe from which the classes of objects of $(\mathfrak{C}, \otimes, I)$-categories are taken.*

The proof of the next theorem is left as an exercise to the reader.

THEOREM 2. *Given a morphism of multiplicative categories*

$$(G, \phi, \delta)\colon (\mathfrak{C}, \otimes, I) \to (\mathfrak{C}_1, \otimes, I_1),$$

if $\mathfrak{C}$ *and* $\mathfrak{C}_1$ *are inverse* $\mathfrak{J}$*-categories and if* G *is an inverse* $\mathfrak{J}$*-functor, then* $\mathbf{P}(G, \phi, \delta)$ *is an inverse* $\mathfrak{J}$*-functor, provided* $\mathfrak{J}$ *satisfies the condition of the preceding theorem.*

It is now natural to ask under what conditions a category $\mathbf{P}_{\mathfrak{D}}(\mathfrak{C}, \otimes, I)$ is a direct $\mathfrak{J}$-category. Although we cannot answer this question in general, we shall be able to answer it for the categories $\mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$ and $\mathbf{P}_{\mathfrak{D}}(\mathfrak{Ab}, \otimes, Z)$ when $\mathfrak{J}$ is a proper category (the class of its morphisms is a set).

First of all, in $\mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$, direct sums may be constructed pretty much the same way as one constructs the direct sum (or free product) of a family of semi-groups. We shall call the objects and morphisms of $\mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$ simply $\mathfrak{D}$-categories and $\mathfrak{D}$-functors. Let $\{\mathfrak{C}_i\}_{i\in I}$ be a family of $\mathfrak{D}$-categories, where I is a set. One may construct a new $\mathfrak{D}$-category $\mathfrak{C}$ as follows. If $A, B \in \mathfrak{D}$, the morphisms from A to B in $\mathfrak{C}$ are the finite sequences

$$f : A = A_0 \xrightarrow{f_{i1}} A_1 \xrightarrow{f_{i2}} \ldots\ldots \xrightarrow{f_{im}} A_m = B \tag{1}$$

where each f_{ih} is a non-identity morphism in $\mathfrak{C}_{ih}$ and no two successive morphisms in this sequence belong to the same category $\mathfrak{C}_i$. Given another such sequence

$$g\colon B = B_0 \xrightarrow{g_{j1}} B_1 \xrightarrow{g_{j2}} \ldots\ldots \xrightarrow{g_{jn}} B_n = C, \tag{2}$$

the product gf is the sequence from A to C obtained by writing sequence (2) after sequence (1) and carrying out all possible multiplications and cancellations so that the proper conditions are satisfied. It is not any more difficult to prove that this operation is associative than to prove the operation associative in the free product of a family of semi-groups. Of course, for each $A \in \mathfrak{D}$, one must admit the existence of a void sequence from A to A to have a unity morphism for A in $\mathfrak{C}$. For each $i \in I$, there is an obvious canonical imbedding $\mathfrak{D}$-functor $T_i\colon \mathfrak{C}_i \to \mathfrak{C}$ and these functors define $\mathfrak{C}$ as a direct sum of $\{\mathfrak{C}_i\}_{i\in I}$ in $\mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$.

Given an $\mathfrak{D}$-category $\mathfrak{C}$, by a regular equivalence relation of $\mathfrak{C}$ we mean a family $R = \{R(A, B)\}_{A,B\in\mathfrak{D}}$, where each $R(A, B)$ is an equivalence relation of the set $M(A, B)$, such that if $f, f' \in M(A, B)$ and $g, g' \in M(B, C)$, then

$$f \equiv f'(R(A, B)) \text{ and } g \equiv g'(R(B, C)) \Rightarrow gf \equiv g'f'(R(A, C)).$$

The following properties of regular equivalence relations of $\mathfrak{C}$ are obvious generalizations of well-known properties of regular equivalence relations on semi-groups.

1. The regular equivalence relations of $\mathfrak{C}$ may be partially ordered in the obvious way: any family of regular equivalence relations of $\mathfrak{C}$ then has an intersection, and there is a coarsest regular equivalence relation of $\mathfrak{C}$.

2. If T is an $\mathfrak{O}$-functor defined on $\mathfrak{C}$ and if for each $A, B \in \mathfrak{O}$, one defines an equivalence relation $R_T(A, B)$ on $M(A, B)$ by

$$f \equiv f'(R_T(A, B)) \Leftrightarrow T(f) = T(f'),$$

then $R_T = \{R_T(A, B)\}_{A,B\in\mathfrak{O}}$ is a regular equivalence relation of $\mathfrak{C}$.

3. Let R be a regular equivalence relation of $\mathfrak{C}$ and for each morphism $f: A \rightarrow B$ in $\mathfrak{C}$, let $\bar{f}$ denote the equivalence class containing f determined by $R(A, B)$ in $M(A, B)$. One can define a quotient $\mathfrak{O}$-category $\mathfrak{C}/R$ as follows: for $A, B \in \mathfrak{O}$, the class of morphisms from A to B in $\mathfrak{C}/R$ is $M(A, B)/R(A, B)$ and given $f \in M(A, B)$, $g \in M(B, C)$, $\bar{g}\bar{f} = \overline{gf}$. There is an obvious canonical $\mathfrak{O}$-functor T_R from $\mathfrak{C}$ to $\mathfrak{C}/R$ defined by $T_R(f) = \bar{f}$ and one has that $R_{T_R} = R$. Furthermore, this canonical $\mathfrak{O}$-functor has the following universal property: if $T: \mathfrak{C} \rightarrow \mathfrak{D}$ is an $\mathfrak{O}$-functor and if R_T is coarser than R, then there exists a unique $\mathfrak{O}$-functor $S: \mathfrak{C}/R \rightarrow \mathfrak{D}$ such that $T = ST_R$.

One is now able to prove that any two $\mathfrak{O}$-functors $T, T': \mathfrak{B} \rightarrow \mathfrak{C}$ have a cokernel. There is at least one regular equivalence relation R on $\mathfrak{C}$ such that $T_R T = T_R T'$, namely the coarsest regular equivalence relation of $\mathfrak{C}$. There is then a finest regular equivalence relation R of $\mathfrak{C}$ with this property, namely the intersection of all the regular equivalence relations of $\mathfrak{C}$ with this property. Then, $T_R: \mathfrak{C} \rightarrow \mathfrak{C}/R$ is a cokernel of the pair (T, T'). Then, by a result in **(7)**, any diagram in $\mathbf{P}_{\mathfrak{O}}(\mathfrak{S}, \times, O)$ with a proper category of indices has a direct limit.

Remark 1. If one defines an $\mathfrak{O}$-precategory as what is left of an $\mathfrak{O}$-category when one drops the operation, then it is possible to define a notion of free $\mathfrak{O}$-category generated by a given $\mathfrak{O}$-precategory.

Remark 2. One could show that in the category $\mathbf{P}'_{\mathfrak{O}}(\mathfrak{S}, \times, O)$ of all $\mathfrak{O}$-groupoids, i.e. $\mathfrak{O}$-categories in which all morphisms are invertible, and all their $\mathfrak{O}$-functors, every diagram with a proper category of indices has a direct limit, and that every $\mathfrak{O}$-precategory generates a free $\mathfrak{O}$-groupoid. The arguments involved are now obvious generalizations of known arguments for groups.

Remark 3. If $\mathbf{C}$ is a category of the second type, then one may define a regular equivalence relation R on $\mathbf{C}_0$ as follows: if $f, f' \in \mathbf{M}_0(A, B)$, then $f R f'$ if there exists a finite sequence of morphisms

$$f \xrightarrow{\alpha_1} f_1, \quad f_2 \xrightarrow{\alpha_2} f_1, \quad f_2 \xrightarrow{\alpha_3} f_3, \quad f_4 \xrightarrow{\alpha_4} f_3, \quad \ldots$$

ending with f'. The canonical functor $T_R: \mathbf{C}_0 \rightarrow \mathbf{C}_0/R$ may be extended in a trivial fashion to a double functor $T: \mathbf{C} \rightarrow \mathbf{C}_0/R$, i.e. one defines for each $\alpha: f \rightarrow f'$ in $\mathbf{M}(A, B)$, $T(\alpha) = 1_{T_R}(f)$. Then T has the following universal property: if U is a double functor from $\mathbf{C}$ to the ordinary category $\mathfrak{D}$, then there exists a unique functor $V: \mathbf{C}_0/R \to \mathfrak{D}$ such that $U = VT$.

Before we turn our attention to the category $\mathbf{P}_{\mathfrak{D}}(\mathfrak{Ab}, \otimes, Z)$, let us consider the following situation. We are given an adjoint morphism

$$(H, G, \rho, \sigma): \mathfrak{C}_1 \to \mathfrak{C}$$

with the following properties:

C1. Given two morphisms $f, f': \bar{A} \to \bar{B}$ in $\mathfrak{C}_1$, if $H(f) = H(f')$, then $f = f'$.

C2. Given a family of morphisms $\{f_i: H(\bar{A}_i) \to H(\bar{A})\}_{i \in I}$ in $\mathfrak{C}$, there exists $\bar{g}: \bar{A} \to \bar{B}$ in $\mathfrak{C}_1$ with the following properties:

(i) For each $i \in I$, there exists $\bar{f}_i: \bar{A}_i \to \bar{B}$ such that $H(\bar{f}_i) = H(\bar{g})f_i$.

(ii) If $\bar{h}: \bar{A} \to \bar{C}$ is such that for each $i \in I$, there exists $\bar{f}'_i: \bar{A}_i \to \bar{C}$ such that $H(\bar{f}'_i) = H(\bar{h})f_i$, then there exists a unique $\bar{k}: \bar{B} \to \bar{C}$ such that $\bar{h} = \bar{k}\bar{g}$.

In this situation, we may assert the following:

If $\mathfrak{C}$ is a direct $\mathfrak{J}$-category, then so is $\mathfrak{C}_1$.

For let $F: \mathfrak{J} \to \mathfrak{C}_1$ be a functor. Then HF has a direct limit

$$\lambda: HF \to E_{\mathfrak{J}}(A).$$

Let I be the class of objects of $\mathfrak{J}$ and let $\bar{g}: G(A) \to \bar{B}$ be the morphism in $\mathfrak{C}_1$ corresponding to the family $\{\rho_A \lambda_i\}_{i \in I}$ by C2. For each $i \in I$, there exists $\bar{f}_i: F(i) \to \bar{B}$ such that $H(\bar{f}_i) = H(\bar{g})\rho_A \lambda_i$. Then $\{\bar{f}_i\}_{i \in I}$ is a natural transformation from F to $E_{\mathfrak{J}}(\bar{B})$ and we show that it is a direct limit of F. Let $\alpha: F \to E_{\mathfrak{J}}(C)$ be a natural transformation. Since λ is a direct limit of HF, there exists a unique $\beta: A \to H(\bar{C})$ such that $H(\alpha) = E_{\mathfrak{J}}(B)\lambda$. Then, there exists a unique $\bar{h}: G(A) \to \bar{C}$ such that $\beta = H(\bar{h})\rho$ so that for each $i \in I$, $H(\alpha_i) = \beta\lambda_i = H(\bar{h})\rho\lambda_i$ and therefore, by the choice of $\bar{g}$, there exists a unique $\bar{k}: \bar{B} \to \bar{C}$ such that $\bar{h} = \bar{k}\bar{g}$. Thus, $\bar{k}$ is the only morphism from $\bar{B}$ to $\bar{C}$ such that

$$H(\alpha_i) = H(\bar{k})H(\bar{g})\rho\lambda_i = H(\bar{k})H(\bar{f}_i) = H(\bar{k}\bar{f}_i),$$

i.e. by C1, such that $\alpha_i = \bar{k}\bar{f}_i$, for each $i \in I$.

As an application of the preceding criterion, we consider the case where $(H, G, \rho, \sigma): \mathfrak{Ab} \to \mathfrak{S}$, H being the forgetful functor and therefore obviously satisfying condition C1. But it is easy to show that it also satisfies condition C2, for given a family of functions $\{f_i: H(\bar{A}_i) \to H(\bar{A})\}_{i \in I}$, the natural homomorphism $\bar{g}: \bar{A} \to \bar{A}/N$, where N is the submodule of A generated by all elements

$$f_i(a + a') - f_i(a) - f_i(a'), \qquad a, a' \in \bar{A}_i, i \in I,$$

has the desired properties. Thus, we may conclude that since $\mathfrak{S}$ is a direct $\mathfrak{J}$-category for each proper category $\mathfrak{J}$, $\mathfrak{Ab}$ is also a direct $\mathfrak{J}$-category for each proper category $\mathfrak{J}$. But then, (H, G, ρ, σ) induces the adjoint morphism

$$(\mathbf{P}_{\mathfrak{D}}(H, \nu, \beta), \mathbf{P}_{\mathfrak{D}}(G), \mathbf{P}_{\mathfrak{D}}(\rho), \mathbf{P}_{\mathfrak{D}}(\sigma)): \mathbf{P}_{\mathfrak{D}}(\mathfrak{Ab}, \otimes, Z) \to \mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$$

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and $\mathbf{P}_O(H, \nu, \beta)$ is also a forgetful functor (it assigns to each additive category its "underlying ordinary category"). We show that $\mathbf{P}_{\mathfrak{D}}(H, \nu, \beta)$ satisfies condition C2. We shall call the objects and morphisms of $\mathbf{P}_{\mathfrak{D}}(\mathfrak{Ab}, \otimes, Z)$ simply additive $\mathfrak{D}$-categories and additive $\mathfrak{D}$-functors.

By a regular equivalence relation of an additive $\mathfrak{D}$-category $\mathfrak{C}$, we mean a regular equivalence relation R of the ordinary $\mathfrak{D}$-category $\mathfrak{C}$ which is compatible with addition, i.e.

$$f \equiv f'(R(A, B)) \text{ and } g \equiv g'(R(A, B)) \Rightarrow f + g \equiv f' + g'(R(A, B))$$

for any $A, B \in \mathfrak{D}$. A regular equivalence relation R of $\mathfrak{C}$ is then determined completely by the "ideal" $\mathfrak{M} = \{\mathfrak{M}(A, B)\}_{A,B\in\mathfrak{D}}$, where each

$$\mathfrak{M}(A, B) = \{f \in M(A, B) f \equiv 0(R(A, B))\}.$$

An ideal $\mathfrak{M}$ of $\mathfrak{C}$ is just a family $\{\mathfrak{M}(A, B)\}_{A,B\in\mathfrak{D}}$, where each $\mathfrak{M}(A, B)$ is a submodule of $M(A, B)$ and where

$$f \in M(A, B) \text{ and } g \in \mathfrak{M}(B, C) \Rightarrow gf \in \mathfrak{M}(A, C),$$
$$g \in \mathfrak{M}(B, C) \text{ and } h \in M(C, D) \Rightarrow hg \in \mathfrak{M}(B, D).$$

One could obviously enumerate properties of these ideals analogous to those we have given for regular equivalence relations in $\mathfrak{D}$-categories, which generalize well-known properties of ordinary ideals in rings.

Given a family $E = \{E(A, B)\}_{A,B\in\mathfrak{D}}$, the intersection of all the ideals $\mathfrak{M}$ of $\mathfrak{C}$ such that $E(A, B) \subseteq \mathfrak{M}(A, B)$ for all $A, B \in \mathfrak{D}$ will be called the ideal of $\mathfrak{C}$ generated by E.

Now, given a family of ordinary $\mathfrak{D}$-functors

$$T_i\colon \{\mathfrak{C}_i \to \mathfrak{C}\}_{i\in I}$$

there exists an additive $\mathfrak{D}$-functor $T\colon \mathfrak{C} \to \mathfrak{D}$ with the same properties as $\bar{g}$ in condition C2. For each $A, B \in \mathfrak{D}$, let $E(A, B)$ be the subset of $M_{\mathfrak{C}}(A, B)$ consisting of all morphisms

$$T_i(f + f') - T_i(f) - T_i(f'), \qquad f, f' \in M_{\mathfrak{C}_i}(A, B), i \in I,$$

and let $\mathfrak{M}$ be the ideal of $\mathfrak{C}$ generated by $E = \{E(A, B)\}_{A,B\in\mathfrak{D}'}$. Then, one may take T to be the "natural" functor from $\mathfrak{C}$ onto $\mathfrak{C}/\mathfrak{M}$.

Then, since we know that $\mathbf{P}_{\mathfrak{D}}(\mathfrak{S}, \times, O)$ is a direct $\mathfrak{J}$-category for each proper category $\mathfrak{J}$, our criterion allows us to assert that $\mathbf{P}_{\mathfrak{D}}(\mathfrak{Ab}, \otimes, Z)$ is a direct $\mathfrak{J}$-category for each proper category $\mathfrak{J}$.

References

1. J. Bénabou, *Catégories avec multiplication*, C. R. Acad. Sci. Paris, *256* (1963), 1887–1890.
2. B. Eckmann and P. J. Hilton, *Group-like structures in general categories*, I, Math. Ann., *145* (1962), 227–255; II, Math. Ann., *151* (1963), 150–186.
3. C. Ehresmann, *Catégories doubles et catégories structurées*, C. R. Acad. Sci. Paris, *256* (1963), 1198–1201.

https://doi.org/10.4153/CJM-1965-076-0 Published online by Cambridge University Press

4. R. Godement, *Théorie des faisceaux*, Act. Sci. et ind., *1252* (Paris, 1958).
5. P. J. Huber, *Homotopy theory in general categories*, Math. Ann., *144* (1961), 361–385.
6. D. M. Kan, *Adjoint functors*, Trans. Am. Math. Soc., *87* (1958), 294–329.
7. J.-M. Maranda, *Some remarks on limits in categories*, Can. Math. Bull., *5* (1962), 133–146.
8. ——— *Injective structures*, Trans. Am. Math. Soc., *110* (1964), 98–135.

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