Nagoya Math. J., **224**(1) (2016), 168–201
DOI 10.1017/nmj.2016.34

BLOWUPS AND FIBERS OF MORPHISMS

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Abstract. Our object of study is a rational map $\Psi\colon \mathbb{P}_k^{s-1} \dashrightarrow \mathbb{P}_k^{n-1}$ defined by homogeneous forms $g_1, \dots, g_n$, of the same degree d, in the homogeneous coordinate ring $R = k[x_1, \dots, x_s]$ of $\mathbb{P}_k^{s-1}$. Our goal is to relate properties of Ψ, of the homogeneous coordinate ring $A = k[g_1, \dots, g_n]$ of the variety parameterized by Ψ, and of the Rees algebra $\mathcal{R}(I)$, the bihomogeneous coordinate ring of the graph of Ψ. For a regular map Ψ, for instance, we prove that $\mathcal{R}(I)$ satisfies Serre's condition R_i, for some $i > 0$, if and only if A satisfies R_{i-1} and Ψ is birational onto its image. Thus, in particular, Ψ is birational onto its image if and only if $\mathcal{R}(I)$ satisfies R_1. Either condition has implications for the shape of the *core*, namely, $\operatorname{core}(I)$ is the multiplier ideal of I^s and $\operatorname{core}(I) = (x_1, \dots, x_s)^{sd-s+1}$. Conversely, for $s = 2$, either equality for the core implies birationality. In addition, by means of the generalized rows of the syzygy matrix of $g_1, \dots, g_n$, we give an explicit method to reduce the nonbirational case to the birational one when $s = 2$.

§1. Introduction

Let k be a field. We investigate rational maps $\Psi\colon \mathbb{P}_k^{s-1} \dashrightarrow \mathbb{P}_k^{n-1}$. Such a map Ψ is defined by homogeneous forms $g_1, \dots, g_n$, of the same degree d, in the homogeneous coordinate ring $R = k[x_1, \dots, x_s]$ of $\mathbb{P}_k^{s-1}$. One of our main goals is to determine necessary and sufficient conditions for Ψ to be birational onto its image. This problem has been studied extensively; see, for example, Hacon [22] for a technique that uses the nonvanishing theorem of Kollár [35]. The advantage of having convenient criteria for determining if a given parameterization is birational onto its image is quite obvious. Papers about parameterization of curves and surfaces are now ubiquitous (see, for

Received January 7, 2015. Accepted October 31, 2015.

2010 Mathematics subject classification. Primary 13A30, 13H15, 14A10; Secondary 13D02, 14E05, 14F18.

The first author was partially supported by the Simons Foundation. The second author was partially supported by the NSF. The last author was partially supported by the NSF and as a Simons Fellow.

example [2–4, 7–11, 15, 16, 24, 25, 37, 50]). It is much better to have "the parameterization is birational" as a conclusion, rather than as a hypothesis.

In this paper we employ the syzygies of the forms $[g_1, \ldots, g_n]$ to determine if the rational map $\Psi = [g_1 : \cdots : g_n] : \ \mathbb{P}^{s-1} \dashrightarrow \mathbb{P}^{n-1}$ is birational. For $s = n$ this approach appears already in the work of Hulek *et al.* [26], and for $n \geqslant s$ it has been further developed in [47]. In [17, 48] the method has been advanced by emphasizing the role of the Rees algebra associated to the ideal $I = (g_1, \ldots, g_n)$ of R. The Rees algebra $\mathcal{R}(I)$ gives the bihomogeneous coordinate ring of the graph of Ψ, whereas, the subalgebra $A = k[g_1, \ldots, g_n]$ of R is the homogeneous coordinate ring of the image of Ψ. In fact, A is isomorphic to the special fiber ring $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ for $\mathfrak{m}$ equal to the maximal homogeneous ideal $\mathfrak{m} = (x_1, \ldots, x_s)$ of R. The rings $\mathcal{R}(I)$ and $\mathcal{F}(I)$ are known as *blowup rings* associated to I. In this paper we relate geometric properties of Ψ, algebraic information about the homogeneous coordinate ring A of the image, and the bihomogeneous coordinate ring $\mathcal{R}(I)$ of the graph.

If Ψ is a morphism (that is, if I is primary to the maximal homogeneous ideal $\mathfrak{m}$ of R), then the degree of $\Psi : \mathbb{P}_k^{s-1} \to \operatorname{Im} \Psi$ is equal to $d^{s-1}/e(A)$, where $e(A)$ is the multiplicity of the standard-graded k-algebra A. We prove, for instance, that $\mathcal{R}(I)$ satisfies Serre's condition R_i, for some $i > 0$, if and only if A satisfies R_{i-1} and Ψ is birational onto its image (that is, $\Psi : \mathbb{P}_k^{s-1} \to \operatorname{Im} \Psi$ has degree 1). Thus, in particular, Ψ is birational onto its image if and only if $\mathcal{R}(I)$ satisfies R_1. Furthermore, $\Psi : \mathbb{P}_k^{s-1} \to \operatorname{Im} \Psi$ is a birational morphism with a smooth image if and only if the Rees ring $\mathcal{R}(I)$ has an isolated singularity. Even if the rational map Ψ is not a morphism, if the dimension of $\operatorname{Im} \Psi$ is $s - 1$ (that is, if the Krull dimension of A is s), then the degree of $\Psi : \mathbb{P}_k^{s-1} \to \operatorname{Im} \Psi$ is equal to the multiplicity of the local ring $\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)}$. Moreover, we are able to relate the degree of the map Ψ, the degree of the image, and the j-multiplicity of the ideal I. Results of this type were obtained before by Simis *et al.* [49], Validashti [52], Xie [53] and by Jeffries *et al.* [32].

The j-multiplicity is a generalization of the classical Hilbert–Samuel multiplicity that applies to ideals that are not necessarily zero-dimensional. The notion was introduced by Achilles and Manaresi [1] and has found applications in intersection theory and equisingularity theory. It is interesting to find formulas for the j-multiplicity of classes of ideals [31, 32, 42]. Our formula serves this purpose if the degree of the map and of its image are known. Conversely, we obtain the degree of the image of a rational

https://doi.org/10.1017/nmj.2016.34 Published online by Cambridge University Press

map if the j-multiplicity can be computed, for instance by using residual intersection techniques (see, e.g. [38]). We also express the degree of certain dual varieties in terms of the j-multiplicity of Jacobian ideals.

The starting point for our investigation is the Eisenbud–Ulrich interpretation [19] of the fibers of the morphism Ψ over the point p in $\mathbb{P}^{n-1}$ in terms of the corresponding generalized row of the homogeneous syzygy matrix for $[g_1, \dots, g_n]$. This technique is explained and extended in Section 3; it also plays a significant role in [16].

In Section 4, we prove an algebraic analogue of a consequence of Hurwitz' theorem. Let r be the degree of the morphism $\Psi : \mathbb{P}^1_k \dashrightarrow \operatorname{Im}(\Psi)$, which at the level of coordinate rings corresponds to the embedding $A \hookrightarrow k[R_d]$. Thus r is the degree of the field extension $\operatorname{Quot}(A) \subset \operatorname{Quot}(k[R_d])$. We show that there exist homogeneous forms f_1, f_2 of degree r in R such that the entries of the matrix φ are homogeneous polynomials in the variables f_1 and f_2. In particular, the ideal I is extended from an ideal in $k[f_1, f_2]$. Thus, replacing $k[x_1, x_2]$ by $k[f_1, f_2]$ we can reduce the nonbirational case to the birational one. Furthermore, we provide an explicit description of f_1 and f_2 in terms of φ. If q_1 and q_2 are general points in $\mathbb{P}^1_k$, then $f_i = \gcd(I_1(p_i\varphi))$ with $p_i = \Psi(q_i)$. This method gives an efficient algorithm for reparameterizing the rational map Ψ.

In Section 5, we relate the birationality of Ψ to the shape of $\operatorname{core}(I)$, the core of I, the intersection of all reductions of I. Since reductions, even minimal ones, are highly nonunique, one uses the core to encode information about all of them. The concept was introduced by Rees and Sally [46], and has been studied further by Huneke and Swanson, by Corso, Polini, and Ulrich, by Polini and Ulrich, and by Huneke and Trung [12–14, 27, 28, 44]. The core appears naturally in the contexts of Briançon–Skoda theorems that compare the integral closure filtration with the adic filtration of an ideal [6, 40]. Another aspect that makes the core very appealing is its connection to adjoints and multiplier ideals, and, as discovered by Hyry and Smith, to Kawamata's conjecture on the nonvanishing of sections of certain line bundles [29, 30]. We prove that if Ψ is birational onto its image then $\operatorname{core}(I)$ is the adjoint ideal of I^s and $\operatorname{core}(I) = (x_1, \dots, x_s)^{sd-s+1}$. The converse of this statement holds for $s = 2$. Indeed, if $\operatorname{Im}\Psi$ is a curve, Ψ is a morphism, and φ is a homogeneous Hilbert–Burch matrix for the row vector $[g_1, \dots, g_n]$, then in 5.14 we prove the following result.

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THEOREM. *Statements (1)–(8) are equivalent.*

(1) *The morphism* Ψ *is birational onto its image.*
(2) *The Rees ring* $\mathcal{R}(I)$ *satisfies Serre's condition* (R_1).
(3) *One has the equality of canonical modules* $\omega_{\mathcal{R}(I)} = \omega_{\mathcal{R}(\mathfrak{m}^d)}$.
(4) *One has the equality of endomorphism rings*

$$\mathrm{End}_{\mathcal{R}(I)}(\omega_{\mathcal{R}(I)}) = \mathrm{End}_{\mathcal{R}(\mathfrak{m}^d)}(\omega_{\mathcal{R}(\mathfrak{m}^d)}).$$

(5) $e(A) = d$.
(6) $\mathrm{core}(I) = \mathfrak{m}^{2d-1}$.
(7) *One has the equality of the core and an adjoint* $\mathrm{core}(I) = \mathrm{adj}(I^2)$.
(8) *The ideal* $\mathrm{core}(I)$ *is integrally closed.*

Furthermore, statements (1)–(8) are all implied by

(9) $\gcd($*column degrees of* $\varphi) = 1$.

We highlight the fact that the integral closedness of the core of I, which is a single graded component of the canonical module $\omega_{\mathcal{R}(I)}$, forces the shape of the entire canonical module. Also, we emphasize that these equivalent conditions may sometimes be read from numerical information about a homogeneous presentation matrix φ for I (that is, from information about the graded Betti numbers of the homogeneous ideal I in the ring $R = k[x_1, x_2]$). In general, the sufficient condition (9) is far from necessary; however, if I is generated by monomials, then condition (9) is equivalent to conditions (1)–(8).

The fact that the core can detect geometric properties was already apparent in the work of Hyry and Smith [29, 30] and in [20], where the Cayley–Bacharach property of zero-dimensional schemes is characterized in terms of the structure of the core of the maximal ideal of their homogeneous coordinate ring. The equality $\mathrm{core}(I) = \mathrm{adj}(I^g)$ (where g is the height of the ideal I) has also been investigated by Hyry and Smith [29, 30] in their work on the conjecture of Kawamata. Adjoints of ideals in regular domains were introduced by Lipman [40], and in rings essentially of finite type over a field of characteristic zero they coincide with multiplier ideals, which play an important role in algebraic geometry due to their connection with vanishing theorems [39]. For the core to be the adjoint of an ideal, it needs to be integrally closed, which is always the case if $\mathcal{R}(I)$ satisfies Serre's condition R_1 (see [44]). Surprisingly, however, in 5.14 the integral closeness of the core is sufficient to guarantee the equality between the core and the multiplier ideal.

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§2. Notation, conventions, and preliminary results

2.1. If I and J are ideals of a ring R, then the *saturation of I with respect to J* is $I : J^\infty = \bigcup_{i=1}^{\infty}(I : J^i)$. Recall that $I : J^\infty$ is obtained from I by removing all primary components whose radical contain J. We write gcd to mean *greatest common divisor*. If I is a homogeneous ideal in $k[x, y]$, then we denote the gcd of a generating set of I by $\gcd(I)$; notice that this polynomial generates the saturation $I : (x, y)^\infty$.

2.2. If R is a ring, then we write $\operatorname{Quot}(R)$ for the *total ring of quotients* of R; that is,

$$\operatorname{Quot}(R) = U^{-1}R,$$

where U is the set of nonzerodivisors on R. If R is a domain, then the total ring of quotients of R is usually called the *quotient field* of R. An extension of domains $A \subset B$ is called *birational* if A and B have the same quotient field.

If $A \subset B$ is an extension of domains so that $\operatorname{Quot}(B)$ is algebraic over $\operatorname{Quot}(A)$, then we use the three notations

$$[B : A], \quad \operatorname{rank}_A B, \quad \text{and} \quad [\operatorname{Quot}(B) : \operatorname{Quot}(A)]$$

interchangeably. Indeed, in this situation,

$$\operatorname{Quot}(A) \otimes_A B = \operatorname{Quot}(B). \tag{2.2.1}$$

The rank of B as an A-module, denoted $\operatorname{rank}_A B$, is defined to be the dimension of the left hand side of (2.2.1) as a vector space over $\operatorname{Quot}(A)$. The dimension of the right side of (2.2.1), as a vector space over $\operatorname{Quot}(A)$, is denoted $[\operatorname{Quot}(B) : \operatorname{Quot}(A)]$.

2.3. If R is a Noetherian ring, M is a finitely generated R-module of Krull dimension s, and $\mathfrak{a}$ is an ideal of R with the Krull dimension of $R/\mathfrak{a}$ equal to zero, then *the multiplicity of the R-module M with respect to the ideal $\mathfrak{a}$* is

$$e_{\mathfrak{a}}(M) = s! \lim_{n\to\infty} \frac{\lambda_R(M/\mathfrak{a}^n M)}{n^s},$$

where $\lambda_R(\_)$ represents the length of an R-module. If R is local with maximal ideal $\mathfrak{m}$, then we often write $e(M)$ or $e_R(M)$ in place of $e_{\mathfrak{m}}(M)$. Similarly, if R is a standard-graded algebra over a field (see 2.4) with maximal homogeneous ideal $\mathfrak{m}$, and M is a graded R-module, then we

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often write $e(M)$ or $e_R(M)$ in place of $e_{\mathfrak{m}}(M)$. In either case, we call the common value $e(M) = e_R(M) = e_{\mathfrak{m}}(M)$ the *multiplicity of the R-module M*; in particular, $e(R)$ means $e_R(R)$.

2.4. Let $R = \bigoplus_{0 \leqslant i} R_i$ be a Noetherian graded algebra and M be a finitely generated graded R-module. It follows that

$$\lambda_{R_0}(M_n) = \lambda_R \left(\frac{\bigoplus_{n \leqslant i} M_i}{\bigoplus_{n < i} M_i} \right).$$

This nonnegative integer is the value of the *Hilbert function* of the R-module M at n, denoted $\mathrm{HF}_M(n)$.

Let k be a field. The graded algebra $R = \bigoplus_{0 \leqslant i} R_i$ is a *standard-graded k-algebra* if R_0 is equal to k, R_1 is a finitely generated R_0-module, and R is generated by R_1 as an algebra over R_0. If R is a standard-graded k-algebra and M is a finitely generated, graded R-module with positive Krull dimension s, then the multiplicity of M may be expressed in terms of Hilbert functions:

$$e_R(M) = (s-1)! \lim_{n \to \infty} \frac{\mathrm{HF}_M(n)}{n^{s-1}}.$$

2.5. If R is a standard-graded k-algebra with maximal homogeneous ideal $\mathfrak{m}$, then the multiplicity $e(R)$ of the standard-graded k-algebra R is equal to the multiplicity $e(R_{\mathfrak{m}})$ of the local ring $R_{\mathfrak{m}}$.

2.6. One important application of multiplicity is the following well-known theorem of Nagata [41, 40.6]; see also [51, Exercise 11.8, Example 11.1.11]. If C is a Noetherian formally equidimensional local ring, then $e(C) = 1$ if and only if C is a regular local ring. When we apply this result, C is a one-dimensional Noetherian local domain and the hypothesis that C be formally equidimensional is automatically satisfied.

2.7. We use the associativity formula for multiplicity; see for example, [5, Corollary 4.6.8] or [51, Theorem 11.2.4]. Let R be a Noetherian ring, $\mathfrak{a}$ be an ideal of R with the Krull dimension of $R/\mathfrak{a}$ equal to zero, and M be a finitely generated R-module. Then

$$e_{\mathfrak{a}}(M) = \sum_P \lambda_{R_P}(M_P) e_{\mathfrak{a}}(R/P),$$

where P varies over the prime ideals in the support of M with the property that the Krull dimension of R/P is equal to the Krull dimension of M.

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The proof makes use of a filtration of M whose factors are cyclic modules defined by prime ideals of R.

The following result is elementary. See [49, Proposition 6.1] for a more sophisticated version and [5, Corollary 4.6.9] for a local version.

Observation 2.8. Assume $A \subseteq B$ is a module-finite extension of standard-graded k-algebras which also are domains. Then $e_B(B) = e_A(A) \operatorname{rank}_A B$.

Proof. One has $e_B(B) = e_A(B) = e_A(A) \operatorname{rank}_A B$. The first equality holds because B has the same Hilbert function independent of whether B is viewed as an A-module or a B-module. The second equality is the associativity formula for multiplicity. □

2.9. A dominant rational map $\Psi : X \dashrightarrow Y$ of projective varieties is *birational* if the induced map of function fields $K(Y) \hookrightarrow K(X)$ is an isomorphism. In general, the *degree* of the rational map Ψ is the dimension of the field extension $[K(X) : K(\operatorname{Im} \Psi)]$.

REMARK 2.10. Let $\psi : A \to B$ be a homomorphism of standard-graded k-algebras, where k is a field and A and B are domains. Assume that $\psi(A_i) \subset B_i$ for all i, and that $\psi(A_+) \neq 0$. Then the degree of the rational map $\operatorname{Proj}(\psi) : \operatorname{Proj}(B) \to \operatorname{Proj}(A)$ is $[\operatorname{Quot}(B) : \operatorname{Quot}(\psi(A))]$.

Proof. According to 2.9, the degree of $\operatorname{Proj}(\psi)$ is defined to be $[B_{(0)} : (\psi(A))_{(0)}]$, where $B_{(0)}$ and $(\psi(A))_{(0)}$ are the subfields of $\operatorname{Quot}(B)$ and $\operatorname{Quot}(\psi(A))$, respectively, which consist of the homogeneous elements of degree zero. Let a_1 be an element of A_1 with $\psi(a_1) \neq 0$. It is easy to see that $\psi(a_1)$ is transcendental over $B_{(0)}$, $\operatorname{Quot}(B) = B_{(0)}(\psi(a_1))$, and $\operatorname{Quot}(\psi(A)) = (\psi(A))_{(0)}(\psi(a_1))$. It follows that

$$[B_{(0)} : (\psi(A))_{(0)}] = [\operatorname{Quot}(B) : \operatorname{Quot}(\psi(A))]. \qquad \square$$

2.11. If $R = \bigoplus_{0 \leqslant i} R_i$ is a graded ring and s is a positive integer, then the sth Veronese ring of R is equal to $R^{(s)} = \bigoplus_{0 \leqslant i} R_{is}$. One regrades the Veronese ring in order to have the component of $R^{(s)}$ in degree i be R_{is}. The sth Veronese of a graded module $M = \bigoplus M_i$ is formed in a similar manner: $M^{(s)} = \bigoplus M_{is}$, with M_{is} having degree i.

2.12. Recall that the *Rees algebra* $\mathcal{R}(I)$ of an ideal I in a commutative ring R is the graded subalgebra $R[It]$ of the polynomial ring $R[t]$. If R has a distinguished maximal ideal $\mathfrak{m}$ (that is, if R is graded with maximal

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homogeneous ideal $\mathfrak{m}$ or if R is local with maximal ideal $\mathfrak{m}$), then the *special fiber ring* of I is $\mathcal{R}(I) \otimes_R R/\mathfrak{m}$.

In the typical situation in the present paper, the ring R will be a standard-graded polynomial ring over a field k, and I will be a homogeneous ideal of R generated by homogeneous forms of the same degree d, and, after regrading, the k-subalgebra $k[I_d]$ of R will be the coordinate ring of a projective variety. In this case, there is a k-algebra isomorphism from the projective coordinate ring $k[I_d]$, to the special fiber ring $\mathcal{F}(I)$. Indeed,

$$\begin{array}{ccc} & \mathcal{R}(I) & \\ \nearrow & & \searrow \\ k[I_d] \cong k[I_d t] & \xrightarrow{\cong} & \mathcal{R}(I) \otimes_R R/\mathfrak{m} = \mathcal{F}(I) \end{array}$$

where $\mathfrak{m}$ is the maximal homogeneous ideal of R.

2.13. Let $J \subset I$ be ideals in a commutative Noetherian ring R. The following conditions are equivalent:

(a) there exists a nonnegative integer m with $JI^m = I^{m+1}$, and
(b) the Rees algebra $\mathcal{R}(I)$ is finitely generated as a module over $\mathcal{R}(J)$.

When these conditions occur, one says that J is a *reduction* of I or I is *integral* over J. (For details, see, for example, [51, 8.21, 1.25, 1.1.1, 1.2.1].)

2.14. If R is a Noetherian domain, then the ring S is an *S_2-ification of R* if $R \subset S \subset \mathrm{Quot}(R)$, S is module-finite over R, S satisfies Serre's condition (S_2) as an R-module, and for each $s \in S$, the ideal $R :_R s$ of R has height at least 2. It follows from [23, 2.3] that the S_2-fication of R is unique and from [23, 2.7] that if R has a canonical module ω_R then $\mathrm{End}_R(\omega_R)$ is the S_2-ification of R.

2.15. The concept of "generalized row ideals" appears widely in the literature; see, for example, [18, 19, 21]. Let M be a matrix with entries in a k-algebra and p be a nonzero row vector with entries from k, where k is a field. A *generalized row* of M is the product pM. If pM is a generalized row of M, then the ideal $I_1(pM)$ is called a *generalized row ideal* of R.

2.16. Let X be a topological space. We say that a *general point of X* has a certain property if there exists a dense, open subset U of X so that every point of U has the property.

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§3. Fibers, multiplicity, and row ideals

Data 3.1. Let k be an infinite field and $\Psi : \mathbb{P}_k^{s-1} \dashrightarrow \mathbb{P}_k^{n-1}$ be a rational map defined by k-linearly independent homogeneous forms $g_1, \dots, g_n$ of degree d in $R = k[x_1, \dots, x_s]$, a standard-graded polynomial ring over k in s variables with maximal homogeneous ideal $\mathfrak{m}$, I be the homogeneous ideal $(g_1, \dots, g_n)$ in R, and φ be a homogeneous syzygy matrix of $[g_1, \dots, g_n]$. Let $S = k[T_1, \dots, T_n]$ be a standard-graded polynomial ring over k in n variables. The map Ψ corresponds to the k-algebra homomorphism $\psi : S \to R$, which sends T_i to g_i. Let A be the image $k[I_d]$ of this homomorphism and B be the Veronese ring $k[R_d]$. After regrading, we view $A \subset B$ as standard-graded k-algebras. Notice that A is the homogeneous coordinate ring of the image of Ψ. Notice also that, according to Remark 2.10, $[B : A]$ is the degree of the rational map Ψ.

Observation 3.2. Adopt the data of 3.1. If I is $\mathfrak{m}$-primary then $d^{s-1} = e(A)[B : A]$.

Proof. The hypothesis that the ideal I is $\mathfrak{m}$-primary forces the ring extension $A \subset B$ to be module-finite; since $\mathfrak{m}^{dm} \subset I$ for some m, hence $B_m = A_1 B_{m-1}$. Observation 2.8 yields $e(A)[B : A] = e(B)$. On the other hand from the Hilbert function of B one sees $e(B) = d^{s-1}$. □

As in [19] we define the fibers of the rational map Ψ to be the following schemes.

Definition 3.3. Adopt the data of 3.1. Let p be a rational closed point in $\mathbb{P}_k^{n-1}$ and let $\mathfrak{P} \in \operatorname{Proj}(S)$ be the homogeneous prime ideal corresponding to p. The *fiber of* Ψ *over* p, (denoted $\Psi^{-1}(p)$), is the scheme $\operatorname{Proj}(R/(\mathfrak{P}R :_R I^\infty))$.

Remarks 3.4. (1) Definition 3.3 gives the correct notion of fiber as a set. Indeed, since the ideal I is the extension to R of the homogeneous maximal ideal of S, the prime ideals of $\operatorname{Proj}(R/(\mathfrak{P}R :_R I^\infty))$ correspond to the primes of $\operatorname{Proj}(R)$ that contract to the prime $\mathfrak{P}$ of $\operatorname{Proj}(S)$. In particular, the rational closed points of $\operatorname{Proj}(R/(\mathfrak{P}R :_R I^\infty))$ correspond to the points in the domain of Ψ that map to the point p of $\mathbb{P}_k^{n-1}$.

(2) If p is the point $[\alpha_1 : \cdots : \alpha_n]$ of $\mathbb{P}_k^{n-1}$, then $\mathfrak{P}$ is the prime ideal

$$\mathfrak{P} = I_2 \left(\begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ T_1 & \dots & T_n \end{bmatrix} \right) \tag{3.4.1}$$

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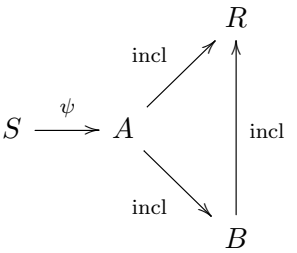
of S, and $\mathfrak{P}R$ and $\mathfrak{P}B$ are the extensions of $\mathfrak{P}$ to the rings R and B, respectively, under the ring homomorphisms:



where incl is the inclusion map. So, in particular, the ideals $\mathfrak{P}R$ and $\mathfrak{P}B$ both are generated by the 2×2 minors of

$$I_2\left(\begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ g_1 & \dots & g_n \end{bmatrix}\right). \tag{3.4.2}$$

Remark 3.5 shows that, not surprisingly, the fiber of Ψ, as defined in Definition 3.3, does not change when the rational map Ψ is composed with a t-uple embedding. Furthermore, if the point is general, then the fiber of Ψ does not change when Ψ is composed with a birational map.

REMARK 3.5. Adopt the data of 3.1 and let $\mathfrak{Q}$ be the homogeneous prime ideal in R which corresponds to the rational point q in $\mathbb{P}_k^{s-1}$.

(1) If q is any point in the domain of the rational map Ψ, then the ideals $(\mathfrak{Q} \cap A)R :_R \mathfrak{m}^\infty$ and $(\mathfrak{Q} \cap A^{(t)})R :_R \mathfrak{m}^\infty$ of R are equal for all positive integers t, where $A^{(t)}$ denotes the t-Veronese subring of A.
(2) Let C be a standard-graded k-algebra with $A \subset C \subset B$ and assume that C is birational over A. If q is a general rational point in $\mathbb{P}_k^{s-1}$, then the ideals $(\mathfrak{Q} \cap A)R :_R \mathfrak{m}^\infty$ and $(\mathfrak{Q} \cap C)R :_R \mathfrak{m}^\infty$ of R are equal.

Proof.

(1) It suffices to show that $(\mathfrak{Q} \cap A)R$ and $(\mathfrak{Q} \cap A^{(t)})R$ are equal locally at any homogeneous relevant prime ideal of R that contains either ideal. Any such prime ideal contracts to $\mathfrak{Q} \cap A$ in A, and $\mathfrak{Q} \cap A$ is a relevant prime ideal of A because q is in the domain of Ψ. Therefore, the two ideals $\mathfrak{Q} \cap A$ and $(\mathfrak{Q} \cap A^{(t)})A$ of A coincide locally at $\mathfrak{Q} \cap A$.
(2) In a similar manner it suffices to show that the two ideals $\mathfrak{Q} \cap C$ and $(\mathfrak{Q} \cap A)C$ are equal locally at the prime ideal $\mathfrak{Q} \cap A$ of A. To see this,

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notice that $A_{\mathfrak{Q}\cap A} = C_{\mathfrak{Q}\cap A}$ because the extension $A \subset C$ is birational and the point q is general. □

In the next few results we impose the hypothesis that the Krull dimension of the ring A of Data 3.1 is s. One could also say that the ideal I has maximal analytic spread. This hypothesis holds when I is $\mathfrak{m}$-primary. Indeed, in this case, B is finitely generated as an A-module, as was observed in the proof of Observation 3.2; hence, A and B have the same dimension. One advantage of the hypothesis $\dim A = s$ is that the rings $C \subset A \subset B$ all have the same Krull dimension for any Noether normalization C of A and this allows for multiplicity calculation involving these rings.

In the next proposition we show that for a general k-rational point q in $\mathbb{P}_k^{s-1}$ the multiplicity of the fiber over $\Psi(q)$ coincides with the degree of the field extension $\mathrm{Quot}(A) \subset \mathrm{Quot}(B)$. Notice that the rational map Ψ is defined at such a point q and that $\Psi(q)$ is general in the image of Ψ.

PROPOSITION 3.6. *Adopt Data 3.1 and assume that the Krull dimension of A is equal to s. Then the equation*

$$[B : A] = e(R/(\mathfrak{p}R :_R I^\infty))$$

holds, where $\mathfrak{p}$ is the homogeneous prime ideal in A of $\Psi(q)$ for a general rational point q in $\mathbb{P}_k^{s-1}$.

Proof. No harm is done if we assume that k is algebraically closed because the hypotheses and conclusions remain unchanged under this change of base.

The rational map $\mathrm{Proj}(R) \dashrightarrow \mathrm{Proj}(A)$ is defined at general points q and their images p, which correspond to prime ideals $\mathfrak{p}$, are general in $\mathrm{Proj}(A)$. Since A and B have the same Krull dimension, the field extension $\mathrm{Quot}(A) \subset \mathrm{Quot}(B)$ is algebraic and therefore finite. Since $\mathfrak{p}$ is general, the Generic Freeness Lemma implies that $B_\mathfrak{p}$ is free as an $A_\mathfrak{p}$-module. Therefore, $B_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module and

$$[\mathrm{Quot}(B) : \mathrm{Quot}(A)] = \mu_{A_\mathfrak{p}}(B_\mathfrak{p}) = \lambda_{A_\mathfrak{p}}(B \otimes_A k(\mathfrak{p})).$$

The ring $B \otimes_A k(\mathfrak{p})$ is Artinian and therefore B is equal to the direct product $\times_{\mathfrak{q}'} B_{\mathfrak{q}'}/\mathfrak{p}B_{\mathfrak{q}'}$, where $\mathfrak{q}'$ varies over all primes in $\mathrm{Proj}(B)$ with $\mathfrak{q}' \cap A = \mathfrak{p}$. Therefore,

$$\lambda_{A_\mathfrak{p}}(B \otimes_A k(\mathfrak{p})) = \sum_{\mathfrak{q}'} \lambda_{A_\mathfrak{p}}(B_{\mathfrak{q}'}/\mathfrak{p}B_{\mathfrak{q}'}).$$

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Since $B_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module, the field extension $\operatorname{Quot}(A/\mathfrak{p}) \subset \operatorname{Quot}(B/\mathfrak{q}')$ is algebraic, and therefore $A/\mathfrak{p}$ and $B/\mathfrak{q}'$ have the same Krull dimension. Thus the rings $A/\mathfrak{p}$ and $B/\mathfrak{q}'$ are standard-graded one-dimensional domains over the algebraically closed field k, hence both are polynomial rings in one variable. Since the inclusion $A/\mathfrak{p} \subset B/\mathfrak{q}'$ is homogeneous, it then follows that it is actually an equality. Therefore, $k(\mathfrak{p}) = k(\mathfrak{q}')$ and we obtain

$$\lambda_{A_{\mathfrak{p}}}(B_{\mathfrak{q}'}/\mathfrak{p}B_{\mathfrak{q}'}) = \lambda_{B_{\mathfrak{q}'}}(B_{\mathfrak{q}'}/\mathfrak{p}B_{\mathfrak{q}'}).$$

Since $\operatorname{Proj}(B/\mathfrak{p}B) \simeq \operatorname{Proj}(R/\mathfrak{p}R)$, there exists a unique $\mathfrak{Q}' \in \operatorname{Proj}(R)$ with $\mathfrak{Q}' \cap A = \mathfrak{p}$ corresponding to each $\mathfrak{q}'$ and furthermore

$$\lambda_{B_{\mathfrak{q}'}}(B_{\mathfrak{q}'}/\mathfrak{p}B_{\mathfrak{q}'}) = \lambda_{R_{\mathfrak{Q}'}}(R_{\mathfrak{Q}'}/\mathfrak{p}R_{\mathfrak{Q}'}).$$

In summary we obtain

$$[\operatorname{Quot}(B) : \operatorname{Quot}(A)] = \sum_{\mathfrak{q}'} \lambda_{A_{\mathfrak{p}}}(B_{\mathfrak{q}'}/\mathfrak{p}B_{\mathfrak{q}'}) = \sum_{\mathfrak{Q}'} \lambda_{R_{\mathfrak{Q}'}}(R_{\mathfrak{Q}'}/\mathfrak{p}R_{\mathfrak{Q}'}).$$

Let $\mathfrak{m}_A$ denote the homogeneous maximal ideal of A. Notice that $\mathfrak{m}_A R = I$ and recall that $\dim A/\mathfrak{p}$ is one. It follows that

$$\begin{aligned}\{\mathfrak{Q}' \mid \mathfrak{Q}' \in \operatorname{Proj}(R), \mathfrak{Q}' \cap A = \mathfrak{p}\} &= \{\mathfrak{Q}' \mid \mathfrak{Q}' \in \operatorname{Proj}(R), \mathfrak{p} \subset \mathfrak{Q}', \mathfrak{m}_A \not\subset \mathfrak{Q}'\} \\ &= \{\mathfrak{Q}' \mid \mathfrak{Q}' \in \operatorname{Proj}(R), \mathfrak{p}R \subset \mathfrak{Q}', I \not\subset \mathfrak{Q}'\} \\ &= \operatorname{Proj}(R) \cap V(\mathfrak{p}R :_R I^\infty).\end{aligned}$$

The rings $R/\mathfrak{Q}'$ are polynomial rings in one variable for every prime ideal $\mathfrak{Q}'$ in the above set. In particular, these prime ideals correspond to the minimal primes of maximal dimension of the ring $R/(\mathfrak{p}R :_R I^\infty)$. Now the associativity formula for multiplicity implies that

$$\sum_{\mathfrak{Q}'} \lambda_{R_{\mathfrak{Q}'}}(R_{\mathfrak{Q}'}/\mathfrak{p}R_{\mathfrak{Q}'}) = e(R/(\mathfrak{p}R :_R I^\infty)),$$

which completes the proof. □

In the next corollary we make use of a Noether normalization C of A to compute the multiplicity of A and express it in terms of the field degree $[A : B]$ and of the multiplicity of a ring defined by a colon ideal in R.

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COROLLARY 3.7. *Let k be an infinite field, $R = k[x_1, \ldots, x_s]$ be a standard-graded polynomial ring over k in s variables, I be a homogeneous ideal in R generated by forms of degree d, A be the subring $k[I_d]$, and B be the Veronese subring $k[R_d]$. Assume that the Krull dimension of A is equal to s and let $f_1, \ldots, f_{s-1}$ be general k-linear combinations of homogeneous minimal generators of I. Then*

$$e(A) = \frac{1}{[B : A]} \cdot e\left(\frac{R}{(f_1, \ldots, f_{s-1}) :_R I^\infty}\right).$$

Proof. Let $f_1, \ldots, f_s$ be general k-linear combinations of homogeneous minimal generators of I. Let $C = k[f_1, \ldots, f_s]$. Since A has dimension s, the homogeneous ring extension $C \subset A$ is module-finite and C is a polynomial ring. Hence $[A : C] = e(A)$ by Observation 2.8, for instance.

We have the inclusion of domains

$$C \subset A \subset B.$$

We compute the field degree $[B : C]$ in two different ways. First,

$$[B : C] = [B : A] \cdot e(A).$$

Next, we wish to apply Proposition 3.6 to express $[B : C]$. By the general choice of $f_1, \ldots, f_s$, we may assume that $[0 : \ldots : 0 : 1]$ is a general point of $\operatorname{Proj}(C) \simeq \mathbb{P}_k^{s-1}$. The homogeneous prime ideal of C corresponding to this point is $\mathfrak{p} = (f_1, \ldots, f_{s-1})C$. Therefore, Proposition 3.6 shows that

$$[B : C] = e\left(\frac{R}{(f_1, \ldots, f_{s-1}) :_R (f_s)^\infty}\right).$$

On the other hand,

$$(f_1, \ldots, f_{s-1}) :_R (f_s)^\infty = (f_1, \ldots, f_{s-1}) :_R I^\infty$$

because f_s is a general k-linear combination of the homogeneous minimal generators of I. The proof is complete. □

We promised in the introduction to relate the degree of the rational map $\Psi : \mathbb{P}_k^{s-1} \dashrightarrow \operatorname{im} \Psi$ of Data 3.1, the degree of the image, and the j-multiplicity of the ideal I. The relation is

$$j(I) = d[B : A]e(A).$$

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A very quick proof of this formula is available in the present situation. Another proof of this formula, in a more general setting, is given in Theorem 5.3.

COROLLARY 3.8. *Adopt the data of 3.1. Then*

$$j(I) = d[B : A]e(A).$$

Proof. According to [53, 2.5] (see also [1, 3.8] and [42, 3.6])

$$j(I) = \lambda_R\left(\frac{R}{((f_1, \dots, f_{s-1}) :_R I^\infty, f_s)}\right),$$

in the language of Corollary 3.7. On the other hand, f_s is regular on $R/(f_1, \dots, f_{s-1}) :_R I^\infty$, and f_s is a homogeneous element of R of degree d; so,

$$\lambda_R\left(\frac{R}{((f_1, \dots, f_{s-1}) :_R I^\infty, f_s)}\right) = d \cdot e\left(\frac{R}{(f_1, \dots, f_{s-1}) :_R I^\infty}\right),$$

and the assertion follows from Corollary 3.7. □

We now relate the fibers of the morphism Ψ of Data 3.1 to the generalized row ideals of the presentation matrix φ of I. This material is taken from [19]. For a point $p \in \mathbb{P}_k^{n-1}$ the ideal $I_1(p\varphi)$ is well defined: one may use any representative of p when computing the matrix product $p\varphi$.

Observation 3.9. Adopt Data 3.1. If $\mathfrak{P}$ is the homogeneous prime ideal in S which corresponds to the rational point p in $\mathbb{P}_k^{n-1}$, then the ideals $\mathfrak{P}R : I^\infty$ and $I_1(p\varphi) : I^\infty$ of R are equal.

Proof. Let $p = [\alpha_1 : \cdots : \alpha_n]$ with α_i in k. Recall the generating sets given in (3.4.1) and (3.4.2) for the ideals $\mathfrak{P}$ and $\mathfrak{P}R$ of S and R, respectively. Let χ be an invertible matrix with entries in k and $(0, \dots, 0, 1) = (\alpha_1, \dots, \alpha_n)\chi$. Define $[g'_1, \dots, g'_n] = \boldsymbol{g}'$ by

$$\boldsymbol{g}' = [g_1, \dots, g_n]\chi. \tag{3.9.1}$$

The entries of $\boldsymbol{g}'$ generate I and $\varphi' = \chi^{-1}\varphi$ is a homogeneous syzygy matrix for $\boldsymbol{g}'$. One consequence of this last statement is the fact that the bottom row of φ' generates the ideal $(g'_1, \dots, g'_{n-1}) : g'_n$. On the other hand,

$$\begin{bmatrix} \alpha_1 & \dots & \alpha_n \\ g_1 & \dots & g_n \end{bmatrix} \chi = \begin{bmatrix} 0 & \dots & 0 & 1 \\ g'_1 & \dots & g'_{n-1} & g'_n \end{bmatrix};$$

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and the ideal of 2×2 minors of a matrix is unchanged by row and column operations; therefore, $\mathfrak{P}R$ is generated by $(g'_1, \dots, g'_{n-1})$. We now see that

$$I_1(p\varphi) = I_1([0, \dots, 0, 1]\varphi') = (g'_1, \dots, g'_{n-1}) : g'_n = \mathfrak{P}R : I; \tag{3.9.2}$$

hence, $I_1(p\varphi) : I^\infty = (\mathfrak{P}R : I) : I^\infty = \mathfrak{P}R : I^\infty$. □

Recall that the degree of a subscheme of projective space is the multiplicity of its homogeneous coordinate ring.

COROLLARY 3.10. *Adopt the data of 3.1. Let p be a rational point in $\mathbb{P}_k^{n-1}$.*

(1) *If k is algebraically closed, then p is in the image of Ψ if and only if $I_1(p\varphi) : I^\infty \neq R$.*
(2) *If p is in the image of Ψ, then the degree of the fiber of Ψ over p is equal to the multiplicity of $R/(I_1(p\varphi) : I^\infty)$.*
(3) *If the Krull dimension of A is equal to s and $p = \Psi(q)$ for q a general point in $\mathbb{P}_k^{s-1}$, then $e(R/(I_1(p\varphi) : I^\infty)) = [\mathrm{Quot}(B) : \mathrm{Quot}(A)]$.*

Proof. For part (1) we use Observation 3.9, Remark 3.4(1) and the fact that the ideal $I_1(p\varphi) : I^\infty$ is not the unit ideal if and only it has dimension at least one. Item (2) follows from Observation 3.9. To prove item (3) we apply Observation 3.9 and Proposition 3.6. □

More information about the rational map Ψ of Data 3.1 may be read from φ when I is an $\mathfrak{m}$-primary ideal. In this case, Ψ is actually a morphism and the image of Ψ is a closed subscheme of $\mathbb{P}_k^{n-1}$. Even more information can be read when $s = 2$. In this case the saturation of the row ideal $I_1(p\varphi)$ is the ideal generated by the gcd of the entries of the product matrix $p\varphi$ and the image of Ψ is a curve $\mathcal{C}$.

COROLLARY 3.11. *Adopt the data of 3.1. In addition assume that I is $\mathfrak{m}$-primary and $s = 2$. Let $\mathcal{C}$ be the image of Ψ. The following statements hold.*

(1) *The point p is on the curve $\mathcal{C}$ if and only if $\gcd(I_1(p\varphi))$ is not in k.*
(2) *If p is on the curve $\mathcal{C}$, then the multiplicity of the fiber of Ψ over p is equal to*
$$\deg(\gcd(I_1(p\varphi))).$$
(3) *If $p = \Psi(q)$ for q a general point in $\mathbb{P}_k^1$, then $\deg(\gcd(I_1(p\varphi))) = [\mathrm{Quot}(B) : \mathrm{Quot}(A)]$.*

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(4) *All entries of the matrix* φ *have degree at least* $[\mathrm{Quot}(B) : \mathrm{Quot}(A)]$.

Proof. Items (1)–(3) follow from Corollary 3.10. As for item (4), notice that each column of φ contains a nonzero element, hence the linear space

$$V = \{[\beta_1, \ldots, \beta_n] \in k^n \mid \sum \beta_i \varphi_{ij} = 0\}$$

is a proper linear subspace of $\mathbb{P}_k^{n-1}$. Because $g_1, \ldots, g_n$ are k-linearly independent the image of Ψ cannot be contained in V. Thus if q is a general point in $\mathbb{P}_k^1$ every entry in the product $p\varphi = \Psi(q)\varphi$ is nonzero. Now item (4) follows from (3). □

§4. Reparameterization

Data 4.1. (1) Let $R = k[x_1, x_2]$ be a standard-graded polynomial ring over an infinite perfect field k with maximal homogeneous ideal $\mathfrak{m}$, I be a height two ideal of R which is generated by homogeneous forms of the same degree d, A and B be the k-subalgebras $A = k[I_d]$ and $B = k[R_d]$ of R, and r be the degree of the field extension $\mathrm{Quot}(A) \subset \mathrm{Quot}(B)$. View A and B as standard-graded k-algebras with maximal homogeneous ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$, respectively. Let $e = e(A)$.

(2) Let $g_1, \ldots, g_n$ be homogeneous forms in R of degree d which minimally generate the ideal I and consider the morphism $\Psi : \mathbb{P}_k^1 \longrightarrow \mathbb{P}_k^{n-1}$ which is defined by these forms; that is, Ψ sends the point q of $\mathbb{P}_k^1$ to the point $[g_1(q) : \cdots : g_n(q)] \in \mathbb{P}_k^{n-1}$. The corresponding homomorphism of k-algebras is $\psi : S = k[T_1, \ldots, T_n] \to R$, which sends T_i to g_i, for each i.

(3) Let φ be a minimal homogeneous Hilbert–Burch matrix for the row vector $[g_1, \ldots, g_n]$. In other words, there exist shifts D_j and a unit u in k so that

$$0 \to \bigoplus_{j=1}^{n-1} R(d - D_j) \xrightarrow{\varphi} R(-d)^n \xrightarrow{[g_1, \ldots, g_n]} R \tag{4.1.1}$$

is a minimal homogeneous resolution of R/I and g_i is equal to $u(-1)^i$ times the determinant of φ with row i deleted, for $1 \leqslant i \leqslant n$.

In this section, we prove an algebraic analogue of a consequence of Hurwitz' theorem. Let r be the degree of the morphism $\Psi : \mathbb{P}_k^1 \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Im}(\Psi)$,

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which at the level of coordinate rings corresponds to the embedding $A \hookrightarrow k[R_d]$. Thus r is the degree of the field extension $\mathrm{Quot}(A) \subset \mathrm{Quot}(k[R_d])$. We show that there exist homogeneous forms f_1, f_2 of degree r in R such that the entries of the matrix φ are homogeneous polynomials in the variables f_1 and f_2. In particular, the ideal I is extended from an ideal in $k[f_1, f_2]$. Thus, replacing $k[x_1, x_2]$ by $k[f_1, f_2]$, we can reduce the nonbirational case to the birational one. Furthermore, we provide an explicit description of f_1 and f_2 in terms of φ. If q_1 and q_2 are general points in $\mathbb{P}^1_k$, then $f_i = \gcd(I_1(p_i\varphi))$ with $p_i = \Psi(q_i)$.

The first half of the present section is devoted to the proof of the main result, Theorem 4.2. In the second half of the section, beginning with Corollary 4.5, we use Theorem 4.2 to reduce the general (that is, nonbirational) case to the birational one.

THEOREM 4.2. *Adopt the data of 4.1. If q_1 and q_2 are general points in $\mathbb{P}^1_k$, then $A \subset k[f_1, f_2]$, where $f_i = \gcd(I_1(p_i\varphi))$ with $p_i = \Psi(q_i)$.*

The proof of Theorem 4.2 involves a number of steps. Before getting started we record some numerical information. In preparation for the first step in the proof of Theorem 4.2, we record a result from [36, Corollary 5.2]. Recall that a standard-graded Cohen–Macaulay k-algebra C over a field k with homogeneous maximal ideal $\mathfrak{m}_C$ is said to have *minimal multiplicity* if its multiplicity has the smallest possible value, namely, $e(C) = \mu(\mathfrak{m}_C) - \dim(C) + 1$.

PROPOSITION 4.3. *Let k be a perfect field and $A \subset B$ a homogeneous integral extension of standard-graded k-domains. Further assume that A is normal and Cohen–Macaulay. If B has minimal multiplicity, then so does A.*

Claim 4.4. Adopt the data of 4.1. Theorem 4.2 holds if the ring A is normal.

Proof. Apply Proposition 4.3. In the present situation, A and B are normal of dimension 2 and therefore Cohen–Macaulay, and B has minimal multiplicity since $d = e(B) = \mu(\mathfrak{m}_B) - 1$. It follows that A has minimal multiplicity as well, namely,

$$e = \mu(\mathfrak{m}_A) - 1 = \mu(I) - 1 = n - 1.$$

We conclude that the syzygy matrix φ has size $e + 1 \times e$. Each entry in column j of φ is a form of degree D_j, in the language of (4.1.1), and,

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according to Corollary 3.11(4), $D_j \geqslant r$. The g_i have degree d and each one of them is the determinant of an $e \times e$ minor of φ; thus, $d = \sum_{j=1}^{e} D_j \geqslant er = d$, where the last equality follows from Observation 3.2. We conclude that each entry of φ is a homogeneous form of degree r.

Let $q_1, \dots, q_{e+1}$ be general points in $\mathbb{P}^1_k$. Hence the points $\Psi(q_1), \dots, \Psi(q_{e+1})$ are general on the curve $\mathcal{C} = \operatorname{Im} \Psi$, and since $\mathcal{C}$ is nondegenerate in $\mathbb{P}^e$, it follows that $\Psi(q_1), \dots, \Psi(q_{e+1})$ span $\mathbb{P}^e$. In other words, the $(e+1) \times (e+1)$ scalar matrix Γ, whose rows are $\Psi(q_i)$, is invertible. We apply Corollary 3.11(3) to see that each $\gcd(I_1(\Psi(q_i)\varphi))$ is a form in R of degree r. Write $f_i = \gcd(I_1(\Psi(q_i)\varphi))$. Degree considerations show that

$$\Psi(q_i)\varphi = f_i \cdot \lambda_i, \tag{4.4.1}$$

where λ_i is a vector of scalars. It follows that

$$\Gamma\varphi = D\Lambda, \tag{4.4.2}$$

where D is the diagonal matrix whose diagonal entries are the forms $f_1, \dots, f_{e+1}$, and Λ is the $(e+1) \times e$ scalar matrix whose rows are the λ_i. We notice that Λ has maximal rank e because φ does.

Let q be another general point in $\mathbb{P}^1$. Write

$$\Psi(q)\varphi = f\lambda, \tag{4.4.3}$$

as in (4.4.1), where f is a form of degree r in R and λ is a row vector of scalars from k. There is an $e \times (e-1)$ matrix of scalars Δ so that $\lambda\Delta = 0$ and Δ has full rank $e-1$. Combine (4.4.3), the definition of Ψ as given in 4.1(2), and (4.4.2) to see that

$$0 = f\lambda\Delta = \Psi(q)\varphi\Delta = [g_1(q), \dots, g_{e+1}(q)]\Gamma^{-1}D\Lambda\Delta.$$

The point q in $\mathbb{P}^1$ is general; so the product $[g_1(q), \dots, g_{e+1}(q)]\Gamma^{-1}$ is a row vector of nonzero scalars and the product $[g_1(q), \dots, g_{e+1}(q)]\Gamma^{-1}D$ is equal to $[f_1, \dots, f_{e+1}]D'$, where D' is an invertible diagonal matrix of scalars. Thus,

$$0 = [f_1, \dots, f_{e+1}]D'\Lambda\Delta,$$

where $D'\Lambda\Delta$ is an $(e+1) \times (e-1)$ matrix of scalars of rank $e-1$. We conclude that the vector space V spanned by the homogeneous forms $f_1, \dots, f_{e+1}$ has dimension at most two. Select a generating set f_i, f_j for V.

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Every entry of the Hilbert–Burch matrix φ is an element of the ring $k[f_i, f_j]$ (as can be seen from (4.4.2)); hence $A = k[I_d] \subset k[f_i, f_j]$ and the proof of Claim 4.4 is complete. □

Next we are ready to prove Theorem 4.2.

Proof. We prove the theorem by reducing to the normal case. The integral closure $\overline{A}$ of A in Quot(A) is a graded k-subalgebra of B, though not necessarily standard-graded. However, $\overline{A}$ is a finitely generated graded module over the standard-graded k-algebra A and hence there exists a positive integer s such that the t-Veronese subring of the integral closure of A, denoted $\overline{A}^{(t)}$, is standard-graded for all $t \geqslant s$. We observe that $\overline{A}^{(t)} = \overline{A^{(t)}}$.

We apply Claim 4.4 to $\overline{A^{(s)}}$. For general points q_1 and q_2 in $\mathbb{P}^1_k$ we obtain forms $\widetilde{f}_1$ and $\widetilde{f}_2$ such that $\overline{A^{(s)}} \subset k[\widetilde{f}_1, \widetilde{f}_2]$. Let $\mathfrak{Q}_\mathfrak{i}$ be the prime ideals in R corresponding to q_i. We have

$$\begin{aligned}\widetilde{f}_i R &= ((\mathfrak{Q}_\mathfrak{i} \cap \overline{A^{(s)}})R) : \mathfrak{m}^\infty \\ &= ((\mathfrak{Q}_\mathfrak{i} \cap A^{(s)})R) : \mathfrak{m}^\infty = ((\mathfrak{Q}_\mathfrak{i} \cap A)R) : \mathfrak{m}^\infty = f_i R,\end{aligned}$$

where the first and last equality follow from Observation 3.9 (and 2.1) and the second and third equality hold due to Remark 3.5. We conclude that $\widetilde{f}_i k = f_i k$ and therefore $\overline{A^{(s)}} \subset k[f_1, f_2]$. In the same manner we obtain $\overline{A^{(s+1)}} \subset k[f_1, f_2]$. Hence for every nonzero homogeneous element α in A we have

$$\alpha = \frac{\alpha^{s+1}}{\alpha^s} \in \text{Quot}(k[f_1, f_2]).$$

On the other hand, α is integral over $A^{(s)}$, hence integral over $k[f_1, f_2]$. We conclude that $\alpha \in k[f_1, f_2]$ since $k[f_1, f_2]$ is normal. □

COROLLARY 4.5. *Adopt the data of 4.1. Let f_1 and f_2 as in Theorem 4.2.*

(1) *The elements f_1, f_2 are a regular sequence of forms of degree r in R.*
(2) *The ring $R' = k[f_1, f_2]$ is a polynomial ring in the variables f_1 and f_2. After regrading, we view R' as a standard-graded k-algebra.*
(3) *The vector space I_d is a subset of R' and the ideal $I' = I_d R'$ is generated by forms of degree d/r in R'.*
(4) *The extension $A = k[I'_{d/r}] \subset k[R'_{d/r}]$ is birational.*
(5) *The ring R is a free R'-module of rank r^2 and $I = I'R$.*
(6) *There exists a minimal homogeneous Hilbert–Burch matrix for the row vector $[g_1, \ldots, g_n]$ whose entries are in the ring $k[f_1, f_2]$.*

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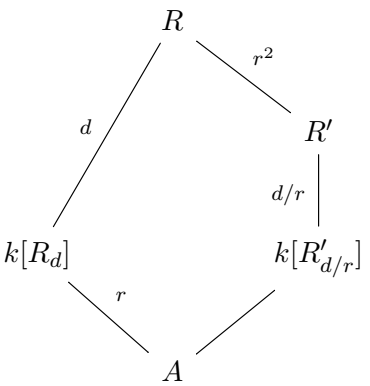
Proof. The forms f_1 and f_2 have degree r according to Corollary 3.11(3). From Theorem 4.2 we obtain $I_d \subset R' = k[f_1, f_2]$. In particular, $I \subset (f_1, f_2)R$ because $I = I_d R$. It follows that f_1, f_2 form a regular sequence in R and hence R is a free module over $R' = k[f_1, f_2]$, necessarily of rank r^2. Parts (1) and (5) are established. At this point, items (2), (3), and (6) are clear.

To prove (4) we use the following diagram, where the relevant field degrees are displayed:



The diagram shows $[k[R'_{d/r}] : A] = 1$. □

In the following corollary we gather important numerical consequences of Theorem 4.2.

COROLLARY 4.6. *Adopt the data of 4.1.*

(1) $[B : A] \mid \deg[\varphi]_{ij}$.
(2) $[B : A] \mid \gcd(\textit{column degrees of } \varphi)$.
(3) *If* $\gcd(\textit{column degrees of } \varphi) = 1$, *then* $A \subset B$ *is a birational extension; in other words,* Ψ *is birational onto its image.*
(4) *If every entry of* φ *has the same degree and that degree is a prime integer, then the morphism* Ψ *is birational onto its image if and only if* $\mu(I_1(\varphi)) \geqslant 3$.

Proof. Items (1) and (2) follow from items (1)–(6) of Corollary 4.5. Item (3) is now clear. Item (4) is [16, 0.11]. □

Observation 4.7. Adopt the data of 4.1 and assume that I is a monomial ideal. Then

$$[B : A] = \gcd(\text{column degrees of } \varphi).$$

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Proof. We may assume that $g_i = x^{a_i}y^{d-a_i}$ with $0 = a_1 < a_2 < \cdots < a_n = d$. Hence in the language of (4.1.1),

$$D_j = a_{j+1} - a_j \quad \text{for } 1 \leqslant j \leqslant n-1.$$

On the other hand, as is well known,

$$[R : A]\mathbb{Z} = I_2 \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ d-a_1 & d-a_2 & \dots & d-a_n \end{pmatrix}.$$

After row and column operations on the matrix, this determinantal ideal becomes

$$d \cdot I_2 \begin{pmatrix} 0 & a_2 - a_1 & \dots & a_n - a_{n-1} \\ 1 & 0 & \dots & 0 \end{pmatrix} = d \cdot \gcd(D_1, \dots, D_{n-1})\mathbb{Z}.$$

It follows that

$$[B : A] = \frac{[R : A]}{d} = \gcd(D_1, \dots, D_{n-1}) = \gcd(\text{column degrees of } \varphi). \quad \square$$

REMARK 4.8. Adopt the data of 4.1. If I is a monomial ideal

$$\gcd(\text{column degrees of } \varphi) = 1 \iff A \subset B \text{ is a birational extension.}$$

However, the direction ($\Leftarrow$) is far from true if the ideal I is not monomial; see, for example, Corollary 4.6(4) or all of [16].

The following theorem summarizes some of the results of the present section that were used in [16]. It was recorded as [16, Theorem 0.10]. (The present statement is the correct one.)

THEOREM 4.9. *Let R be the standard-graded polynomial ring $k[x, y]$, with k an infinite perfect field, I be a height two ideal of R generated by forms $g_1, \dots, g_n$ of degree d, and φ be a homogeneous Hilbert–Burch matrix for the row vector $[g_1, \dots, g_n]$. If A and B are the standard-graded k-algebras $A = k[I_d]$ and $B = k[R_d]$, r is the degree of the field extension $[\mathrm{Quot}(B) : \mathrm{Quot}(A)]$, and e is the multiplicity of A, then the following statements hold.*

(1) $re = d$.
(2) *The morphism $\mathbb{P}^1_k \to \mathbb{P}^{n-1}_k$, which is given by $q \mapsto [g_1(q) : \cdots : g_n(q)]$, is birational onto its image if and only if $e = d$.*

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(3) *There exist forms f_1 and f_2 of degree r in R such that $A \subset k[f_1, f_2]$. In particular, I is extended from an ideal in $k[f_1, f_2]$ in the sense that $I = (I \cap k[f_1, f_2])R$. Furthermore, we can choose $f_i = \gcd(I_1(p_i\varphi))$ where $p_i = [g_1(q_i) : \cdots : g_n(q_i)]$ for general points q_1 and q_2 in $\mathbb{P}^1_k$.*

Proof. Item (1) is Observation 3.2; (2) follows from (1) and the definition of birationality; and (3) is Theorem 4.2 and Corollary 4.5. □

§5. Rees rings

Data 5.1. Let R be a standard-graded domain over a field k, of dimension s, with maximal homogeneous ideal $\mathfrak{m}$. Let I be a homogeneous ideal in R generated by forms of degree d, A be the ring $k[I_d]$, and B be the Veronese ring $k[R_d]$. After regrading, we view $A \subset B$ as standard-graded k-algebras.

In this section, we relate properties of the extension $A \subset B$ to properties of the corresponding inclusions of Rees algebras $\mathcal{R}(I) \subset \mathcal{R}(\mathfrak{m}^d)$. The extension of Rees algebras has the advantage that it is automatically birational.

Remark 5.2. Adopt the data of 5.1. The injective map of graded k-algebras

$$A \cong k[I_d t] \hookrightarrow R[It] = \mathcal{R}(I)$$

induces an isomorphism between the ring A and the special fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$; see, for example, (2.12). The ring A is a domain; hence, $\mathfrak{m}\mathcal{R}(I)$ is a prime ideal. Since $\dim \mathcal{R}(I)$ is equal to $s+1$ (see, for example, [51, 5.1.4]), we obtain

$$\dim A = s + 1 - \dim \mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)}.$$

In the next result we show that the degree of the morphism $\Psi : \mathbb{P}^{s-1}_k \longrightarrow \operatorname{Im}\Psi$ is equal to the multiplicity of the local ring $\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)}$. In addition, we relate the degree of the morphism Ψ, the degree of the image, and the j-multiplicity of the ideal I. The notion of j-multiplicity and the importance of this result is discussed in the introduction and in Corollary 3.8.

Theorem 5.3. *Adopt the data of 5.1. If the Krull dimension of A is equal to s, then*

$$e(\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)}) = [B : A] \quad \text{and} \quad j(I) = d \cdot [B : A] \cdot e(A).$$

Proof. Inside the Rees ring $\mathcal{R} := \mathcal{R}(I) = R[It] \subset R[t]$ we consider the k-subalgebra $A' = k[I_d t]$, which is isomorphic to A. Write K for the quotient

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field of A'. We notice that $\mathcal{R} \otimes_{A'} K$ is a standard-graded K-algebra with maximal homogeneous ideal $\mathfrak{m}(\mathcal{R} \otimes_{A'} K)$ and $(\mathcal{R} \otimes_{A'} K)_{\mathfrak{m}(\mathcal{R} \otimes_{A'} K)}$ is equal to $\mathcal{R}_{\mathfrak{m}\mathcal{R}}$. Therefore, the local ring $\mathcal{R}_{\mathfrak{m}\mathcal{R}}$ and the standard-graded K-algebra $\mathcal{R} \otimes_{A'} K$ have the same multiplicity and the same Krull dimension. This dimension is one by Remark 5.2.

Let g be a nonzero homogeneous element of I_d. Notice that g is a homogeneous nonzerodivisor of degree d in the one-dimensional standard-graded K-algebra $\mathcal{R} \otimes_{A'} K$. We obtain

$$\begin{aligned} e(\mathcal{R} \otimes_{A'} K) = e((\mathcal{R} \otimes_{A'} K)^{(d)}) &= [(\mathcal{R} \otimes_{A'} K)^{(d)} : K[g]] \\ &= \frac{1}{d}[(\mathcal{R} \otimes_{A'} K) : K[g]] \\ &= \frac{1}{d}[R[t] : A[t]]. \end{aligned}$$

The last equality holds because

$$\operatorname{Quot}(K[g]) = \operatorname{Quot}(k[I_d t, g]) = \operatorname{Quot}(k[I_d, t]) = \operatorname{Quot}(A[t])$$

and $\operatorname{Quot}(\mathcal{R} \otimes_{A'} K) = \operatorname{Quot}(R[t])$. Now we conclude that

$$e(\mathcal{R}_{\ \mathfrak{m}\mathcal{R}}) = e(\mathcal{R} \otimes_{A'} K) = \frac{1}{d}[R : A] = [B : A].$$

This concludes the proof of the first equality.

The second equality follows from [49, 6.1(a) and 3.4]. Here we give a self-contained proof. We write $\mathcal{G} := \operatorname{gr}_I(R) = \mathcal{R}/I\mathcal{R}$ for the associated graded ring of I. Notice that $\mathcal{G}/\mathfrak{m}\mathcal{G} \cong \mathcal{R}/\mathfrak{m}\mathcal{R} \cong A$. In particular, $\mathfrak{m}\mathcal{G}$ is a prime ideal in $\mathcal{G}$ and $\mathcal{G}_{\mathfrak{m}\mathcal{G}}$ is Artinian, because A is a domain and $\dim A = s = \dim \mathcal{G}$.

The j-multiplicity of I is

$$j(I) = e(0 :_{\mathcal{G}} \mathfrak{m}^\infty).$$

Since $\operatorname{Supp}_{\mathcal{G}}(0 :_{\mathcal{G}} \mathfrak{m}^\infty) = V(\mathfrak{m}\mathcal{G})$, the associativity formula for multiplicity 2.7 gives

$$e(0 :_{\mathcal{G}} \mathfrak{m}^\infty) = \lambda_{\mathcal{G}_{\mathfrak{m}\mathcal{G}}}((0 :_{\mathcal{G}} \mathfrak{m}^\infty)_{\mathfrak{m}\mathcal{G}}) \cdot e(\mathcal{G}/\mathfrak{m}\mathcal{G}) = \lambda(\mathcal{G}_{\mathfrak{m}\mathcal{G}}) \cdot e(A).$$

We conclude that

$$j(I) = \lambda(\mathcal{G}_{\mathfrak{m}\mathcal{G}}) \cdot e(A).$$

(The same equality was proved in [32, the proof of 3.1].)

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If $g \neq 0$ is an element of I_d as above, then $gt \in \mathcal{R} \setminus \mathfrak{m}\mathcal{R}$, hence $I/g = It/gt \subset \mathcal{R}$. It follows that $I\mathcal{R}_{\mathfrak{m}\mathcal{R}} = g\mathcal{R}_{\mathfrak{m}\mathcal{R}}$, which gives

$$\mathcal{G}_{\mathfrak{m}\mathcal{G}} \cong \mathcal{R}_{\mathfrak{m}\mathcal{G}}/(g).$$

Now

$$\begin{aligned}\lambda(G_{\mathfrak{m}\mathcal{G}}) &= \lambda(\mathcal{R}_{\mathfrak{m}\mathcal{G}}/(g)) = e((\mathcal{R} \otimes_{A'} K)/(g)) = d \cdot e(\mathcal{R} \otimes_{A'} K) \\ &= d \cdot e(\mathcal{R}_{\mathfrak{m}\mathcal{R}}) = d \cdot [B : A].\end{aligned}$$

We conclude that $j(I) = d \cdot [B : A] \cdot e(A)$. □

In the next corollary we express the degree of certain dual varieties in terms of the j-multiplicity of Jacobian ideals.

COROLLARY 5.4. *Let k be an algebraically closed field of characteristic zero, let $X = V(f) \subset \mathbb{P}_k^{n-1}$ be a reduced and irreducible hypersurface of degree $d > 1$, and assume that the dual variety $X' \subset \mathbb{P}_k^{n-1'}$ is nondeficient, that is, is a hypersurface as well. Let R denote the homogeneous coordinate ring of $X \subset \mathbb{P}_k^{n-1}$, and $I = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)R$ its Jacobian dual. Then*

$$\deg X' = j(I)/(d-1).$$

Proof. We apply the notation of Data 5.1 to the present setting. The ideal I is generated by forms of degree $d-1$, and the algebra A is the homogeneous coordinate ring of $X' \subset \mathbb{P}_k^{n-1'}$. According to [33, Theorem 4] or [43, Proposition 3.3], the extension $A \subset B$ is birational, that is, $[B : A] = 1$. Now Theorem 5.3 shows that $j(I) = (d-1)\deg X'$. □

COROLLARY 5.5. *Adopt the data of 5.1. Let J be a reduction of I generated by s forms of degree d. If the Krull dimension of A is equal to s, then*

$$e(A) = \frac{e(\mathcal{R}(J)_{\mathfrak{m}\mathcal{R}(J)})}{e(\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)})}.$$

REMARK. If k is infinite, then I has a reduction generated by s forms of degree d.

Proof. Notice that $k[J_d]$ is a homogeneous Noether normalization of $A = k[I_d]$. Therefore, $e(A) = [k[I_d] : k[J_d]]$. On the other hand,

$$[k[I_d] : k[J_d]] = \frac{[B : k[J_d]]}{[B : k[I_d]]} = \frac{e(\mathcal{R}(J)_{\mathfrak{m}\mathcal{R}(J)})}{e(\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)})},$$

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where the last equality follows from Theorem 5.3 applied to the two ideals J and I, respectively. □

As a consequence we obtain the following numerical criterion for birationality in terms of the Rees ring or the special fiber ring, respectively.

COROLLARY 5.6. *Adopt the data of 5.1. The following statements are equivalent.*

(1) *The ring extension $A \subset B$ is birational, that is, $[B : A] = 1$.*
(2) *The ring $\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)}$ is a discrete valuation ring.*
(3) *The ring A has dimension s and $e(A) = e(\mathcal{R}(J)_{\mathfrak{m}\mathcal{R}(J)})$, where J is a reduction of I generated by s forms of degree d.*

Proof. Notice that if the ring extension $A \subset B$ is birational, then A and B have the same transcendence degree over k and therefore, $\dim A = \dim B = s$. Furthermore, $\dim \mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)} = 1$ if and only if $\dim A = s$ according to Remark 5.2. Now the asserted equivalences follow from Theorem 5.3 and Corollary 5.5. □

More can be said if the rational map Ψ of data of 3.1 is a morphism, or equivalently, if the ideal I is $\mathfrak{m}$-primary. A crucial statement in the following corollary is the statement about canonical modules. Indeed, cores, adjoints, and S_2-fications are all constructed using the canonical module of the Rees ring.

REMARK 5.7. If I is an $\mathfrak{m}$-primary ideal generated by forms of degree d, then the ideal $\mathfrak{m}^d$ is integral over I. Hence the ring extension

$$\mathcal{R}(I) \subset \mathcal{R}(\mathfrak{m}^d)$$

is not only birational, but also module-finite. Thus applying the functor $-^\vee = \operatorname{Hom}_{\mathcal{R}(I)}(-, \omega_{\mathcal{R}(I)})$ induces a containment of the canonical modules

$$\omega_{\mathcal{R}(\mathfrak{m}^d)} \subset \omega_{\mathcal{R}(I)},$$

and evaluating $-^\vee$ twice yields an inclusion of the S_2-ifications

$$\operatorname{End}_{\mathcal{R}(I)}(\omega_{\mathcal{R}(I)}) \subset \operatorname{End}_{\mathcal{R}(\mathfrak{m}^d)}(\omega_{\mathcal{R}(\mathfrak{m}^d)}).$$

Applying $-^\vee$ once more to last inclusion returns the previous one. In particular, one of the two last inclusions is an equality if and only if the other is.

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COROLLARY 5.8. *Adopt the data of 3.1. In addition, assume that I is $\mathfrak{m}$-primary and the field k is infinite. Statements (1)–(5) are equivalent.*

(1) *The morphism Ψ is birational onto its image, that is, $[B : A] = 1$.*
(2) *The Rees ring $\mathcal{R}(I)$ satisfies Serre's condition (R_1).*
(3) $\omega_{\mathcal{R}(I)} = \omega_{\mathcal{R}(\mathfrak{m}^d)}$.
(4) $\operatorname{End}_{\mathcal{R}(I)}(\omega_{\mathcal{R}(I)}) = \operatorname{End}_{\mathcal{R}(\mathfrak{m}^d)}(\omega_{\mathcal{R}(\mathfrak{m}^d)})$.
(5) $e(A) = d^{s-1}$.

Furthermore, statements (1)–(5) all imply

(6) $\operatorname{gradedcore}(I) = \operatorname{core}(I) = \operatorname{core}(\mathfrak{m}^d) = \mathfrak{m}^{(d-1)s+1} = \operatorname{adj}(I^s)$.
(7) *If $\mathcal{R}(I)$ satisfies Serre's property (S_2), then $\mathcal{R}(I)$ is Cohen–Macaulay.*

Proof. Since I is $\mathfrak{m}$-primary the ring A has Krull dimension s. In particular, $\mathfrak{m}\mathcal{R}(I)$ is a prime ideal of height one. If Q is any other height one prime ideal of $\mathcal{R}(I)$, then Q does not contain $\mathfrak{m}$ and hence does not contain I. It follows that $\mathcal{R}(I)_Q$ is regular. Thus $\mathcal{R}(I)$ satisfies Serre's condition (R_1) if and only if $\mathcal{R}(I)_{\mathfrak{m}\mathcal{R}(I)}$ is a discrete valuation ring. Now the equivalence of (1) and (2) follows from Corollary 5.6.

To see the equivalence of (2) and (3) notice that $\mathcal{R}(\mathfrak{m}^d)$ is the integral closure of $\mathcal{R}(I)$. Hence $\mathcal{R}(I)$ satisfies Serre's condition (R_1) if and only if the $\mathcal{R}(I)$-module $\mathcal{R}(\mathfrak{m}^d)/\mathcal{R}(I)$ has codimension at least two, which in turn is equivalent to the equality $\omega_{\mathcal{R}(I)} = \omega_{\mathcal{R}(\mathfrak{m}^d)}$. The second equivalence is obtained by analyzing the long exact sequence of $\operatorname{Ext}^{\bullet}_{R\otimes_k S}(-, \omega_{R\otimes_k S})$ associated to the short exact sequence

$$0 \longrightarrow \mathcal{R}(I) \longrightarrow \mathcal{R}(\mathfrak{m}^d) \longrightarrow \mathcal{R}(\mathfrak{m}^d)/\mathcal{R}(I) \longrightarrow 0$$

and bearing in mind that $\mathcal{R}(\mathfrak{m}^d)$ is Cohen–Macaulay. The equivalence of (3) and (4) is explained in Remark 5.7.

To see that (5) is equivalent to the other statements we apply Corollary 5.6. We may assume that k is infinite. It suffices to show the equality $e(\mathcal{R}(J)_{\mathfrak{m}\mathcal{R}(J)}) = d^{s-1}$ for the ideal J of Corollary 5.6. Since I is $\mathfrak{m}$-primary, the ideal J is generated by a regular sequence of s forms of degree d. Therefore, $[R : k[J_d]] = d^s$ and then $e(\mathcal{R}(J)_{\mathfrak{m}\mathcal{R}(J)}) = [B : k[J_d]] = d^{s-1}$ according to Theorem 5.3.

We now show that (2) implies (6). We will repeatedly use the fact from [45, 2.1] that the core of $\mathfrak{m}$-primary ideals localizes. The first equality follows from [12, 4.5] as I is generated by forms of the same degree. Recall that the Rees ring $R[It]$ and the extended Rees ring $R[It, t^{-1}]$ have isomorphic projective spectra. Hence if the Rees ring satisfies R_1, then so

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does $\operatorname{Proj}(R[It, t^{-1}])$. It follows that $R[It, t^{-1}]$ satisfies R_1 as well because it suffices to consider prime ideals containing t^{-1} and because the ideal $(It, t^{-1})R[It, t^{-1}]$ has height $s + 1 > 1$. One uses the argument that (2) implies (3) to see that the R_1 property of the extended Rees ring implies the bigraded isomorphism

$$\omega_{R[It,t^{-1}]} \cong \omega_{R[\mathfrak{m}^d t,t^{-1}]}.$$

By [44, 2.2] and [20, 1.2] the core can be recovered from the canonical module of the extended Rees ring. Therefore, $\operatorname{core}(I) = \operatorname{core}(\mathfrak{m}^d)$, which is the second equality in (6). The third equality follows from [13, 4.2]. Finally, one has $\mathfrak{m}^{(d-1)s+1} = \operatorname{adj}(\mathfrak{m}^{ds})$ by [40, 1.3.2(c)] and $\operatorname{adj}(\mathfrak{m}^{ds}) = \operatorname{adj}(I^s)$ because $\mathfrak{m}^{ds}$ is integral over I^s.

To see (7), notice that $\mathcal{R}(I)$ is normal, hence $\mathcal{R}(I) = \mathcal{R}(\mathfrak{m}^d)$. The last ring is Cohen–Macaulay because it is a direct summand of $\mathcal{R}(\mathfrak{m})$, which is a Cohen–Macaulay ring and a finitely generated module over $\mathcal{R}(\mathfrak{m}^d)$. □

The canonical module and the S_2-ification of $\mathcal{R}(m^d)$ that appear in Corollary 5.8 are known explicitly.

REMARK 5.9. Let $R = k[x_1, \dots, x_s]$ be a standard-graded polynomial ring in s variables over a field k, with maximal homogeneous ideal $\mathfrak{m}$. Let d be a positive integer and write $s - 1 = qd + r$ where q and r are nonnegative integers with $r < d$. The bigraded canonical module of the Rees ring of $\mathfrak{m}^d$ is

$$\omega_{\mathcal{R}(\mathfrak{m}^d)} = \begin{cases} x_1^s t\mathfrak{m}^{d-s+1}\mathcal{R}(\mathfrak{m}^d) & \text{if } s \leqslant d, \\ x_1^s t((1,t)^{q-1}, \mathfrak{m}^{d-r}t^q)\mathcal{R}(\mathfrak{m}^d) & \text{if } d < s. \end{cases}$$

Furthermore, the natural inclusion $\mathcal{R}(m^d) \subset \operatorname{End}_{\mathcal{R}(m^d)}(\omega_{\mathcal{R}(m^d)})$ is an equality.

Proof. From the presentation of the Rees ring

$$\mathcal{R}(\mathfrak{m}) = \frac{R[T_1, \dots, T_s]}{I_2\left(\begin{bmatrix} x_1 & \dots & x_s \\ T_1 & \dots & T_s \end{bmatrix}\right)}$$

one computes

$$\omega_{\mathcal{R}(\mathfrak{m})} \cong x_1^2 t(x_1, x_1 t)^{s-2}\mathcal{R}(\mathfrak{m}),$$

where the factor $x_1^2 t$ is required to make the isomorphism bihomogeneous. The Rees ring $\mathcal{R}(\mathfrak{m}^d)$ is obtained by taking the d-Veronese subring of $\mathcal{R}(\mathfrak{m})$

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with respect to the grading given by the T-variables and then rescaling the grading. Since the Veronese functor commutes with taking canonical modules, we obtain the desired statement.

The natural inclusion $\mathcal{R}(m^d) \subset \mathrm{End}_{\mathcal{R}(m^d)}(\omega_{\mathcal{R}(m^d)})$ is an equality because $\mathcal{R}(m^d)$ is Cohen–Macaulay and $\mathrm{End}_{\mathcal{R}(m^d)}(\omega_{\mathcal{R}(m^d)})$ is its S_2-ification. □

In Corollaries 5.11 and 5.12 we extend the equivalence (1) $\Longleftrightarrow$ (2) from Corollary 5.8. We begin by comparing the heights of conductors for certain extensions of Rees rings and special fiber rings.

PROPOSITION 5.10. *Adopt the data of 5.1. In addition, assume that I is $\mathfrak{m}$-primary. Let K be an ideal of R which is generated by forms of degree d and which contains I. Write C for $k[K_d]$; so, $A = k[I_d] \subset C = k[K_d]$. Then*

$$\mathrm{ht}(\mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K)) = 1 + \mathrm{ht}(A :_A C) = 1 + \mathrm{ht}(\mathcal{F}(I) :_{\mathcal{F}(I)} \mathcal{F}(K)).$$

Proof. Let A' be the subring $k[I_d t]$ of $\mathcal{R}(I) = R[I_d t]$ and C' be the subring $k[K_d t]$ of $\mathcal{R}(K) = R[K_d t]$. We first prove that the ideals

$$\mathrm{rad}(\mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K)) \quad \text{and} \quad \mathrm{rad}((\mathfrak{m}, A' :_{A'} C')\mathcal{R}(I)) \tag{5.10.1}$$

of $\mathcal{R}(I)$ are equal.

We prove the inclusion "$\supset$" for the ideals of (5.10.1). The Rees algebra $\mathcal{R}(K)$ is a module-finite extension of $\mathcal{R}(I)$ because I is $\mathfrak{m}$-primary; hence, I is a reduction of K and there is a positive integer t with $K^t K^n \subset I^n$ for all nonnegative integers n. It follows that $K^t \subset \mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K)$; and therefore, $\mathfrak{m} \subset \mathrm{rad}(\mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K))$. Clearly, $A' :_{A'} C' \subset \mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K)$ because $\mathcal{R}(I) = R[A']$ and $\mathcal{R}(K) = R[C']$.

We prove the inclusion "$\subset$" for the ideals of (5.10.1). Let α be a bihomogeneous element of $\mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K)$. In particular, $\alpha = f t^j$ for some homogeneous form f in I and some nonnegative integer j. If the homogeneous form f has degree $i + dj$ as a polynomial in $R = k[x_1, \ldots, x_s]$, then the bi-degree of α in $\mathcal{R}(I)$ is (i, j). If i is positive, then α is in $\mathfrak{m}\mathcal{R}(I)$. If $i = 0$, then α is in A' and $\alpha \cdot C' \subset \alpha \cdot \mathcal{R}(K) \subset \mathcal{R}(I)$. One may compute x-degree to see that $\alpha \cdot C'$ is actually contained in A'. In either case, $\alpha \in (\mathfrak{m}, A' :_{A'} C')\mathcal{R}(I)$.

Now that the assertion of (5.10.1) has been established, we complete the proof of the result by applying the natural quotient homomorphism

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$\mathcal{R}(I) \to \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ to the ideals of (5.10.1). The ring $\mathcal{R}(I)$ is a domain and a finitely generated algebra over a field, and the ideal $\mathfrak{m}\mathcal{R}(I)$ of $\mathcal{R}(I)$ has height 1. Thus,

$$\begin{aligned}\operatorname{ht}(\mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(K)) - 1 &= \operatorname{ht} \tfrac{\operatorname{rad}(\mathcal{R}(I):_{\mathcal{R}(I)}\mathcal{R}(K))}{\mathfrak{m}\mathcal{R}(I)} = \operatorname{ht}(\mathfrak{m}, A' :_{A'} C')\tfrac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} \\ &= \operatorname{ht}(\mathcal{F}(I) :_{\mathcal{F}(I)} \mathcal{F}(K)) = \operatorname{ht}(A :_A C). \qquad \square\end{aligned}$$

COROLLARY 5.11. *Adopt the data of 5.1. In addition, assume that* $R = k[x_1, \dots, x_s]$ *and* I *is* $\mathfrak{m}$*-primary. Let* $i \leqslant s$ *be a fixed positive integer. Then, the following conditions are equivalent.*

(a) $\mathcal{R}(I)$ *satisfies the Serre condition* (R_i).
(b) A *satisfies the Serre condition* (R_{i-1}) *and* $A \subset B$ *is a birational extension.*

Proof. When $i = 1$, the statement is (1) $\iff$ (2) from Corollary 5.8. Henceforth, we assume $2 \leqslant i$ and that $A \subset B$ is a birational extension. Thus, both extensions $\mathcal{R}(I) \subset \mathcal{R}(\mathfrak{m}^d)$ and $A \subset B$ are module-finite and birational.

Moreover, $\dim \mathcal{R}(I) = s + 1$ and $\mathcal{R}(\mathfrak{m}^d)$ has an isolated singularity. Hence for any prime ideal Q of $\mathcal{R}(I)$ with $\dim \mathcal{R}(I)_Q \leqslant s$, the ring $\mathcal{R}(I)_Q$ is regular if and only if $\mathcal{R}(I)_Q = \mathcal{R}(\mathfrak{m}^d)_Q$. This shows that $\mathcal{R}(I)$ satisfies (R_i) if and only if $i < \operatorname{ht}(\mathcal{R}(I) :_{\mathcal{R}(I)} \mathcal{R}(\mathfrak{m}^d))$.

Similarly, $\dim A = s$ and B has an isolated singularity. Hence again, A_q is regular if and only if $A_q = B_q$ for any prime ideal q of A with $\dim A_q \leqslant s - 1$. We conclude that A satisfies (R_{i-1}) if and only if $i - 1 < \operatorname{ht}(A :_A B)$.

Apply Proposition 5.10, with $K = \mathfrak{m}^d$ to complete the proof. $\square$

In the situation of Data 3.1, A is the homogeneous coordinate ring of the image of Ψ. The following geometric consequence of Corollary 5.11 is now immediate.

COROLLARY 5.12. *Adopt the data of 3.1. In addition assume that* I *is* $\mathfrak{m}$*-primary. Then the Rees ring* $\mathcal{R}(I)$ *has an isolated singularity if and only if the image of* Ψ *is smooth and* Ψ *is birational onto its image.*

For $s = 2$ we harvest consequences about canonical modules, S_2-fications, and cores for any ideal generated by forms of the same degree. Indeed, when $s = 2$ we can pass by way of a faithfully flat descent to the birational situation according to Corollary 4.5. We compute canonical modules, S_2-fications, and cores in the birational situation using Corollary 5.8 and

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Remark 5.9. We can pull these objects back because they are preserved under faithfully flat extensions, whereas adjoint ideals are not preserved under such extensions.

THEOREM 5.13. *Adopt the data of 5.1. In addition assume that* $R = k[x_1, x_2]$ *is a polynomial ring in two variables over an infinite field and* I *is* $\mathfrak{m}$*-primary. Write* $r = [B : A]$ *and let* f_1 *and* f_2 *be the forms in* R *of degree* r *of Corollary 4.5. The following statements hold.*

(1) $\omega_{\mathcal{R}(I)} = \omega_{\mathcal{R}((f_1,f_2)^{d/r})} = f_1^2 t(f_1, f_2)^{(d/r)-1}\mathcal{R}((f_1, f_2)^{d/r})$.
(2) *The* S_2*-fication of the Rees ring* $\mathcal{R}(I)$ *is*

$$\operatorname{End}(\omega_{\mathcal{R}(I)}) = \mathcal{R}((f_1, f_2)^{d/r}),$$

which is a Cohen–Macaulay ring.
(3) $e(A) = d/r$.
(4) $\operatorname{gradedcore}(I) = \operatorname{core}(I) = (f_1, f_2)^{2(d/r)-1}$.
(5) *If* $\mathcal{R}(I)$ *satisfies Serre's property* (S_2)*, then* $\mathcal{R}(I)$ *is Cohen–Macaulay.*

Proof. We write $R' = k[f_1, f_2]$ and $I' = I \cap R'$. By Corollary 4.5 the ideal I is extended from I', that is, $I = I'R$. We think of R' as a standard-graded polynomial ring in the variables f_1 and f_2 with maximal homogeneous ideal $\mathfrak{m}'$. The ideal I' is generated by forms of degree d/r in this ring. From Corollary 4.5, we know that the extension $k[I'_{d/r}] \subset k[R'_{d/r}]$ is birational. Now Corollary 5.8 and Remark 5.9 imply the asserted statements for the ideals $I' \subset (\mathfrak{m}')^{d/r} = (f_1, f_2)^{d/r}R'$ in R'. To pass back to the ideals $I = I'R \subset (f_1, f_2)^{d/r}R$ in R we use the fact that the map $R' \subset R$ is flat with Gorenstein fibers. We remind the reader that

$$\operatorname{core}(I) = \operatorname{core}(I'R) = \operatorname{core}(I')R,$$

according to [45, 2.1] and [12, 4.8]. □

Remarkably, if $s = 2$ then all the statements of Corollary 5.8 are equivalent. More specifically, the integral closedness of the core, which is a single graded component of the canonical module of the Rees ring, forces the shape of the entire canonical module.

COROLLARY 5.14. *Adopt the data of 3.1. In addition assume that* I *is* $\mathfrak{m}$*-primary,* $s = 2$*, and the field* k *is infinite. Statements (1)–(8) are equivalent.*

(1) *The morphism* Ψ *is birational onto its image, that is* $[B : A] = 1$.
(2) *The Rees ring* $\mathcal{R}(I)$ *satisfies Serre's condition* (R_1).

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(3) $\omega_{\mathcal{R}(I)} = \omega_{\mathcal{R}(\mathfrak{m}^d)}$.
(4) $\mathrm{End}_{\mathcal{R}(I)}(\omega_{\mathcal{R}(I)}) = \mathrm{End}_{\mathcal{R}(\mathfrak{m}^d)}(\omega_{\mathcal{R}(\mathfrak{m}^d)})$.
(5) $e(A) = d$.
(6) $\mathrm{core}(I) = \mathfrak{m}^{2d-1}$.
(7) $\mathrm{core}(I) = \mathrm{adj}(I^2)$.
(8) *The ideal* $\mathrm{core}(I)$ *is integrally closed.*
Furthermore, statements (1)–(8) are all implied by
(9) $\gcd$ (*column degrees of* φ) $= 1$.

Proof. Corollary 5.8 shows that items (1)–(5) are equivalent and that they imply (6) and (7). Item (8) follows immediately from (6) or (7). We prove that (8) implies (1). From Theorem 5.13, we have $\mathrm{core}(I) = (f_1, f_2)^{2(d/r)-1}$. Hence $\mathrm{core}(I)$ is an $\mathfrak{m}$-primary ideal generated by $2d/r$ forms of degree $2d - r$. If such an ideal is integrally closed, it would have to be $\mathfrak{m}^{2d-r}$, which is minimally generated by $2d - r + 1$ forms. This forces $r = 1$. Finally we appeal to Corollary 4.6(3) to see that (9) implies (1). □

Remarks 5.15.

(a) If I is generated by monomials then statements (1)–(9) of Corollary 5.14 are all equivalent according to Observation 4.7.
(b) When statements (1)–(8) of Corollary 5.14 hold, then it follows from Corollary 5.8 that all of the equalities

$$\mathrm{gradedcore}(I) = \mathrm{core}(I) = \mathrm{core}(\mathfrak{m}^d) = \mathfrak{m}^{2d-1} = \mathrm{adj}(I^2)$$

hold.
(c) Kohlhaas [34] has proven that if $I \subset k[x_1, \ldots, x_s]$ is an $\mathfrak{m}$-primary monomial ideal which has a reduction generated by s monomials, then the following three conditions are equivalent.
(2) $R[It]$ satisfies Serre's condition R_1.
(7) $\mathrm{core}(I) = \mathrm{adj}(I^s)$.
(8) $\mathrm{core}(I)$ is integrally closed.

Acknowledgments. This paper was written at Centre International de Rencontres Mathématiques (CIRM) in Luminy, France, while the authors participated in a Petit Groupe de Travail. The authors are very appreciative of the hospitality offered by the Société Mathématique de France.

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