BULLETIN 13 (1), 1970

AN APPLICATION OF RAMSEY'S THEOREM

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By an r-graph, we mean a finite set V of elements called vertices and a collection of some of the r-subsets of V called edges with the property that each vertex is incident with at least one edge. An A-chromatic r-graph is an r-graph all of whose edges are coloured A.

THEOREM. *Let $G_1, \ldots, G_t$ denote r-graphs. There exists a nonempty class of r-graphs $\mathscr{G}(G_1, \ldots, G_t)$ such that for each $G \in \mathscr{G}(G_1, \ldots, G_t)$ if the edges of G are painted arbitrarily in t colours $A_1, \ldots, A_t$, then for at least one i in $\{1, \ldots, t\}$, G has an A_i-chromatic r-subgraph which is isomorphic to G_i.*

Proof. Let C_k denote the complete r-graph on k vertices. Suppose that G_i has q_i vertices, $i=1, \ldots, t$ and that n is greater than or equal to the Ramsey number $N(q_1, \ldots, q_t, r)$ (see [1]). Then by Ramsey's theorem [2], if the edges of C_n are painted arbitrarily in colours $A_1, \ldots, A_t$, for at least one i, C_n has an A_i-chromatic r-subgraph isomorphic to C_{q_i}. But G_i is a subgraph of C_{q_i}. Hence $C_n \in \mathscr{G}(G_1, \ldots, G_t)$.

In terms of this theorem, the Ramsey number $N(q_1, \ldots, q_t, r)$ is the smallest integer n such that $C_n \in \mathscr{G}(C_{q_1}, \ldots, C_{q_t})$. It is then natural to define the Ramsey number $N(G_1, \ldots, G_t)$ of the set of r-graphs $G_1, \ldots, G_t$ as the smallest n for which $C_n \in \mathscr{G}(G_1, \ldots, G_t)$.

We finally state a few simple properties of $\mathscr{G}(G_1, \ldots, G_t)$, $N(G_1, \ldots, G_t)$.

(i) $\mathscr{G}(G_1, \ldots, G_t)$ and $N(G_1, \ldots, G_t)$ are invariant under permutations of the subscripts $1, \ldots, t$.

(ii) If F_i is a subgraph of G_i for each $i=1, \ldots, t$ then $\mathscr{G}(F_1, \ldots, F_t) \subseteq \mathscr{G}(G_1, \ldots, G_t)$ and $N(F_1, \ldots, F_t) \leq N(G_1, \ldots, G_t)$.

(iii) If F is a subgraph of G and $F \in \mathscr{G}(G_1, \ldots, G_t)$ then it follows that $G \in \mathscr{G}(G_1, \ldots, G_t)$. Of particular interest, therefore will be elements of $\mathscr{G}(G_1, \ldots, G_t)$ with some minimal property; e.g. the smallest complete r-graph of $\mathscr{G}(G_1, \ldots, G_t)$ and elements of the class, no proper r-subgraph of which are also elements of the class.

(iv) $\mathscr{G}(G^t)$ will denote $\mathscr{G}(G_1, \ldots, G_t)$ where $G_i = G$ for each $i=1, \ldots, t$. Let X be a 2-graph. The chromatic index of X is defined as the least number of colours required to paint the edges of G so that no two similarly painted edges are incident at a common vertex. In terms of our new notation, the chromatic index of X is equal to the least integer t such that $X \notin \mathscr{G}(G^t)$ where G is the 2-graph having 3 vertices and 2 edges.

(1) The author gratefully acknowledges the support of the Canadian National Research Council Grant A4810.

References

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https://doi.org/10.4153/CMB-1970-032-5 Published online by Cambridge University Press