A THEOREM OF GLAISHER

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1. Introduction. Let

$$(x-1)(x-2)\ldots(x-p+1) = x^{p-1} - A_1 x^{p-2} + \ldots + A_{p-1}.$$

Then if p is a prime > 3, Glaisher [4] proved

$$(1.1) \qquad \frac{1}{p}A_{2r} \equiv -\frac{1}{2r}B_{2r} \pmod{p},$$

$$(1.2) \qquad \frac{1}{p^2}A_{2r+1} \equiv \frac{2r+1}{4r}B_{2r} \pmod{p},$$

where B_m denotes the mth Bernoulli number in the notation of Nörlund; it had been proved earlier by Nielsen [5] that the left members of (1.1) and (1.2) are integral.

In this paper we first show that for $1 < r < \frac{1}{2}(p-1)$,

$$(1.3) \qquad 2rA_{2r+1} \equiv -\tfrac{1}{2}p^2(p-2r-1)B_{2r} - (2r+1)p^3\sum_{i=1}^{r-1}\frac{1}{4i}B_{2i}B_{2r-2i} \pmod{p^4};$$

indeed, a similar but slightly more complicated congruence (mod p^5) is obtained. Clearly (1.3) is a refinement of (1.2) and in fact of (1.1) also; alternatively, it may be looked on as specifying the residue (mod p) of a certain sum involving Bernoulli numbers.

Glaisher made numerous applications of (1.1) and (1.2); in §§3, 4 we make a few additional applications.

In the remainder of the paper we shall attempt to extend Glaisher's theorem to more general sequences. The generalization depends on the fact that the A_m can be expressed in terms of Bernoulli numbers of higher order, namely [8, p. 148],

$$(1.4) \qquad A_r = (-1)^r\binom{p-1}{r}B_r^{(p)}.$$

Hence if

$$f(x) = \sum_{m=1}^{\infty} c_m x^m/m! \qquad (c_1 = 1),$$

where the c_m are integral (mod p), and we define $\beta_m^{(k)}$ by means of

$$(1.5) \qquad (x/f(x))^k = \sum_{m=0}^{\infty}\beta_m^{(k)}x^m/m!,$$

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it is natural, in view of (1.4), to seek congruences satisfied by $\beta_m^{(p)}$. It will be assumed throughout the paper that p is a fixed prime greater than 3.

As we shall see, it is indeed not difficult to generalize (1.1) and (1.2) from this point of view. Moreover, by introducing coefficients $\eta_m^{(k)}$ defined by

$$\left(\frac{1}{1 + af(x)}\right)^k = \sum_{m=0}^{\infty} \eta_m^{(k)} x^m / m! , \tag{1.6}$$

where a is integral (mod p), we also generalize certain results of Nielsen analogous to (1.1) and (1.2). We remark in this connection that in both (1.5) and (1.6) the case $k = -p$ as well as $k = p$ is of interest.

2. Proof of (1.3). Put

$$S_m = S_m(p) = 1^m + 2^m + \ldots + (p-1)^m. \tag{2.1}$$

Then by Newton's formula we have, for r odd,

$$rA_r = \sum_{i=0}^{r-1} (-1)^i A_i S_{r-i} ,$$

which we write in the form

$$rA_r - S_1 A_{r-1} = S_r - A_1 S_{r-1} + \sum_{i=1}^{\frac{1}{2}(r-3)} A_{2i} S_{r-2i} - \sum_{i=1}^{\frac{1}{2}(r-3)} A_{2i+1} S_{r-2i-1} . \tag{2.2}$$

Now by a familiar formula we have for (2.1)

$$S_m = \frac{1}{m+1} \sum_{i=1}^{m+1} \binom{m+1}{i} B_{m+1-i} p^i , \tag{2.3}$$

and this implies for r odd, $3 < r < p$,

$$\begin{aligned} S_r &\equiv \tfrac{1}{2} r B_{r-1} p^2 && \pmod{p^4}, \\ S_{r-1} &\equiv B_{r-1} p && \pmod{p^3}. \end{aligned} \tag{2.4}$$

Thus by (1.1) and (1.2),

$$\begin{aligned} \sum_{i=1}^{\frac{1}{2}(r-3)} A_{2i} S_{r-2i} &\equiv - \sum_{i=1}^{\frac{1}{2}(r-3)} \frac{1}{2i} B_{2i} p \,.\, \tfrac{1}{2}(r-2i) B_{r-1-2i} p^2 && \pmod{p^5}, \\ \sum_{i=1}^{\frac{1}{2}(r-3)} A_{2i+1} S_{r-2i-1} &\equiv \sum_{i=1}^{\frac{1}{2}(r-3)} \frac{2i+1}{4i} B_{2i} p^2 \,.\, B_{r-2i-1} p && \pmod{p^5}, \end{aligned}$$

so that

$$\begin{aligned} &\sum_{i=1}^{\frac{1}{2}(r-3)} A_{2i} S_{r-2i} - \sum_{i=1}^{\frac{1}{2}(r-3)} A_{2i+1} S_{r-2i-1} \\ &\qquad \equiv -(r+1)p^3 \sum_{i=1}^{\frac{1}{2}(r-3)} \frac{1}{4i} B_{2i} B_{r-2i-1} && \pmod{p^5}. \end{aligned} \tag{2.5}$$

Now by (2.3) we find that

$$(2.6)\quad S_r - A_1 S_{r-1} \equiv \tfrac{1}{2}p^2(r-p+1)B_{r-1} + \tfrac{1}{24}p^4(r-1)(r-2)(r+2)B_{r-3} \pmod{p^5}.$$

Hence combining (2.2), (2.5), (2.6) we get

$$(2.7)\quad \begin{aligned} rA_r - \tfrac{1}{2}p(p-1)A_{r-1} \equiv {} & \tfrac{1}{2}p^2(r-p+1)B_{r-1} \\ & + \tfrac{1}{24}p^4(r-1)(r-2)(r+2)B_{r-3} \\ & - (r+1)p^3 \sum_{i=1}^{\frac{1}{2}(r-3)} \frac{1}{4i} B_{2i} B_{r-2i-1} \pmod{p^5}. \end{aligned}$$

In the next place it follows from [**3**, §19] that for $t > 1$, $p = 2m+1$,

$$(2.8)\quad (m-t)pA_{2t} - A_{2t+1} \equiv \tfrac{1}{6}(p-2t)(m-t)(m-t+1)p^3\sigma_{t-1},$$

where σ_t has the same meaning as in [**3**, §12]. Also

$$\sigma_{t-1} \equiv A_{2t-2} \pmod{p^2}.$$

Consequently (2.8) becomes $(r = 2t+1)$

$$A_r \equiv \tfrac{1}{2}p(p-r)A_{r-1} + \tfrac{1}{24}r(r-1)(r-2)p^3A_{r-3} \pmod{p^5},$$

which by (1.1) yields

$$(2.9)\quad A_r \equiv \tfrac{1}{2}p(p-r)A_{r-1} - \frac{\tfrac{1}{24}r(r-1)(r-2)}{r-3}p^4B_{r-3} \pmod{p^5}.$$

Comparison of (2.7) and (2.9) now gives

$$(2.10)\quad \begin{aligned} (r-1)(p-r-1)A_r \equiv {} & -\tfrac{1}{2}p^2(p-r)(p-r-1)B_{r-1} \\ & - \frac{r(r-1)(r-2)(r^2-r-5)}{24(r-3)}p^4B_{r-3} \\ & - (p-r)(r+1)p^3 \sum_{i=1}^{\frac{1}{2}(r-3)} \frac{1}{4i} B_{2i} B_{r-2i-1} \pmod{p^5}. \end{aligned}$$

In particular (2.10) implies

$$(2.11)\quad (r-1)A_r \equiv -\tfrac{1}{2}p^2(p-r)B_{r-1} - rp^3 \sum_{i=1}^{\frac{1}{2}(r-3)} \frac{1}{4i} B_{2i} B_{r-2i-1} \pmod{p^4}.$$

In can be verified that

$$(2.12)\quad A_3 \equiv -\tfrac{1}{24}p^2(p-3) - \tfrac{5}{16}p^3 \pmod{p^4}.$$

In view of (2.9) one can specify the residue of A_{2t}, $2 \leqslant 2t \leqslant p-3$, mod p^3. In this connection the related formula [**7**, p. 366]

$$(2.13)\quad \frac{1}{p}(W_p - K_p) \equiv W_p + \sum_{r=1}^{m-1} \frac{1}{2r} B_{2r} B_{2m-2r} \pmod{p},\quad p = 2m+1,$$

where $W_p = (A_{p-1} + 1)/p$, $K_p = k_1 + \ldots + k_{p-1}$, $k(r) = (r^{p-1} - 1)/p$, is of interest.

Another formula of a similar kind is

$$(p+2)B_{p+1} + \tfrac{1}{6}p(p+1)B_{p-1} \equiv 2p \sum_{r=2}^{\frac{1}{2}(p-3)} \frac{1}{2r} B_{2r} B_{p+1-2r} \pmod{p^2},$$

which is an easy consequence of Euler's formula

$$(2m+1)B_{2m} + \sum_{r=1}^{m-1} \binom{2m}{2r} B_{2r} B_{2m-2r} = 0 \qquad (m > 1).$$

3. An application. It follows from the definition of A_m that

$$(x-2)(x-4)\ldots(x-2(p-1)) = x^{p-1} - 2A_1 x^{p-2} + \ldots + 2^{p-1}A_{p-1};$$

if we put $x = p = 2m + 1$ this evidently becomes

$$(-1)^m(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (p-2))^2 \equiv 2^{p-1}A_{p-1} - 2^{p-2}pA_{p-2} + 2^{p-3}p^2A_{p-3} \pmod{p^5}$$

$$\equiv 2^{2m}(2m)! + 2^{2m}(-\tfrac{1}{6} + \tfrac{1}{12})\, B_{p-3} \pmod{p^4}$$

$$\equiv 2^{2m}(2m)!\,(1 + \tfrac{1}{12}p^3B_{p-3}) \pmod{p^4},$$

where we have used

$$A_{p-2} \equiv \tfrac{1}{3}p^3B_{p-3} \pmod{p^4},$$

$$A_{p-3} \equiv \tfrac{1}{3}p^2B_{p-3} \pmod{p^3}.$$

Thus it follows that

$$(3.1) \qquad (-1)^m\binom{2m}{m} \equiv 2^{4m}(1 + \tfrac{1}{12}p^3B_{p-3}) \pmod{p^4}.$$

The weaker form of this congruence

$$(-1)^m\binom{2m}{m} \equiv 2^{4m} \pmod{p^3}$$

is due to F. Morley (for references see [**2**, p. 273]); see also Nielsen [**6**, p. 81] for an equivalent result.

4. Other applications. Let us take next the familiar quotient

$$(4.1) \qquad \frac{(np)!}{n!(p!)^n} = \frac{(p+1)\ldots(2p-1)}{(p-1)!}\,\frac{(2p+1)\ldots(3p-1)}{(p-1)!}\cdots\frac{((n-1)p+1)\ldots(np-1)}{(p-1)!}$$

$$= \binom{2p-1}{p-1}\binom{3p-1}{p-1}\cdots\binom{np-1}{p-1}.$$

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But as Glaisher proved

$$\binom{kp-1}{p-1} \equiv 1 - \tfrac{1}{3}k(k-1)p^3B_{p-3} \pmod{p^4}, \tag{4.2}$$

from which it follows that

$$\frac{(np)!}{n!(p!)^n} \equiv 1 - \tfrac{1}{9}(n^3 - n)p^3B_{p-3} \pmod{p^4}. \tag{4.3}$$

The weaker congruence

$$\frac{(np)!}{n!(p!)^n} \equiv 1 \pmod{p^3}$$

is due to Mason and Child (for references see [**2**, p. 278]).

We can generalize (4.3) without much trouble. To begin with we replace (4.1) by

$$\frac{(np^r)!}{n!(p^r!)^n} = \prod_{k=1}^{n} \binom{kp^r-1}{p^r-1}, \tag{4.4}$$

which is easily verified. Secondly, for $r \geqslant 2$,

$$Q_r = \binom{kp^r-1}{p^r-1} = Q_{r-1} \prod_{j=1}^{p^{r-1}} \binom{(k-1)p^r + jp - 1}{p-1} \Big/ \binom{jp-1}{p-1}. \tag{4.5}$$

But by (4.2) it is clear that (4.5) implies

$$Q_r \equiv Q_{r-1} \pmod{p^4}. \tag{4.6}$$

Thus comparison with (4.4) and (4.3) yields

$$\frac{(np^r)!}{n!(p^r!)^n} \equiv 1 - \tfrac{1}{9}(n^3 - n)p^3B_{p-1} \pmod{p^4}, \tag{4.7}$$

which is valid for all $r \geqslant 1$.

We remark that for $n \equiv 0, \pm 1 \pmod{p}$, (4.7) becomes

$$\frac{(np^r)!}{n!(p^r!)^n} \equiv 1 \pmod{p^4},$$

while for $m \equiv n \pmod{p}$,

$$\frac{(mp^r)!}{m!(p^r!)^m} \equiv \frac{(np^r)!}{n!(p^r!)^n} \pmod{p^4}.$$

5. General sequences. In order to generalize Glaisher's theorem we take

$$f = f(x) = \sum_{m=1}^{\infty} \frac{c_m x^m}{m!} \qquad (c_1 = 1), \tag{5.1}$$

where the rational numbers c_m are integral (mod p). Now put

$$\frac{x}{f} = \sum_{m=0}^{\infty} \frac{\beta_m x^m}{m!} \qquad (\beta_0 = 1), \tag{5.2}$$

or what is the same thing

$$\sum_{r=1}^{m} \binom{m}{r} c_r \beta_{m-r} = \begin{cases} 1 & (m = 1), \\ 0 & (m > 1), \end{cases} \tag{5.3}$$

thus recursively defining the β_m. Moreover, it is evident from (5.3) that β_m is integral (mod p) for $m < p - 1$. On the other hand,

$$p\beta_{p-1} + c_p \equiv 0 \qquad (\text{mod } p); \tag{5.4}$$

a somewhat sharper result is

$$p\beta_{p-1} - p\sum_{r=2}^{p-1} \frac{(-1)^r}{r} c_r \beta_{p-r} + c_p \equiv 0 \qquad (\text{mod } p^2). \tag{5.5}$$

In the next place, for $k \geqslant 1$, define

$$\left(\frac{x}{f}\right)^k = \sum_{m=0}^{\infty} \frac{\beta_m^{(k)} x^m}{m!} \qquad (\beta_0^{(k)} = 1), \tag{5.6}$$

so that $\beta_m^{(1)} = \beta_m$. It will also be convenient to define δ_m by means of

$$\frac{xf'}{f} = \sum_{m=0}^{\infty} \frac{\delta_m x^m}{m!} \qquad (\delta_0 = 1). \tag{5.7}$$

By (5.1) and (5.2), (5.7) implies

$$\delta_m = \sum_{r=0}^{m} \binom{m}{r} c_{r+1} \beta_{m-r}. \tag{5.8}$$

Thus δ_m is integral (mod p) for $m < p - 1$, while by (5.4) and (5.8)

$$p\delta_{p-1} + c_p \equiv 0 \qquad (\text{mod } p). \tag{5.9}$$

Indeed (5.8) implies the sharper result

$$\delta_{p-1} - \beta_{p-1} \equiv \sum_{r=1}^{p-1} (-1)^r c_{r+1} \beta_{p-1-r} \qquad (\text{mod } p). \tag{5.10}$$

We remark that for $m = p$, (5.8) implies

$$\delta_p - \beta_p \equiv c_{p+1} + pc_p \beta_{p-1} - p\sum_{r=2}^{p-1} \frac{(-1)^r}{r} c_{r+1} \beta_{p-r} \qquad (\text{mod } p^2); \tag{5.11}$$

that β_p is integral (mod p) is clear from (5.3). In fact (5.3) implies, for $m = p + 1$,

$$\beta_p + \tfrac{1}{2}c_2 p\beta_{p-1} + \frac{c_{p+1}}{p+1} + p\sum_{r=3}^{p} \frac{(-1)^r}{r(r-1)} c_r \beta_{p+1-r} \equiv 0 \qquad (\text{mod } p^2). \tag{5.12}$$

For a generalization of the von Staudt-Clausen theorem for the numbers β_m see **[1]**; the same result applies to δ_m also.

6. Generalization of Glaisher's theorem. Differentiation of (5.6) yields

$$k\left(\frac{x}{f}\right)^k - k\left(\frac{x}{f}\right)^k \frac{xf'}{f} = \sum_{m=0}^{\infty} \frac{m\beta_m^{(k)} x^m}{m!},$$

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and thus by (5.7) we get

$$\sum_0^\infty \frac{(k-m)\beta_m^{(k)} x^m}{m!} = k\sum_0^\infty \frac{\beta_m^{(k)} x^m}{m!}\sum_0^\infty \frac{\delta_n x^n}{n!}.$$

This identity is equivalent to

$$m\beta_m^{(k)} + k\sum_{r=1}^m \binom{m}{r}\delta_r\,\beta_{m-r}^{(k)} = 0. \tag{6.1}$$

We take $k = p$ in (6.1) and suppose $m < p$. It follows at once that

$$\beta_m^{(p)} \equiv 0 \qquad (\bmod p),\ 1 \leqslant m < p-1, \tag{6.2}$$

while for $m = p - 1$,

$$(p-1)\,\beta_{p-1}^{(p)} \equiv -p\delta_{p-1} \equiv c_p \qquad (\bmod p). \tag{6.3}$$

We shall now sharpen (6.2) and (6.3).

In the first place (6.1) becomes, for $m = p - 1$,

$$(p-1)\,\beta_{p-1}^{(p)} + p\delta_{p-1} = -p\sum_{r=1}^{p-2}\binom{p-1}{r}\delta_r\,\beta_{p-1-r}^{(p)}. \tag{6.4}$$

For $1 \leqslant m < p - 1$, (6.1) implies, using (6.2),

$$m\beta_m^{(p)} + p\delta_m \equiv 0 \qquad (\bmod p^2). \tag{6.5}$$

If we substitute from (6.5) in the right member of (6.4), we get

$$\begin{aligned}(p-1)\,\beta_{p-1}^{(p)} + p\delta_{p-1} &\equiv -p^2\sum_{r=1}^{p-2}\frac{(-1)^r}{r+1}\delta_r\,\delta_{p-1-r}\\ &\equiv p^2\sum_{r=1}^{p-2}\frac{(-1)^r}{r}\delta_r\,\delta_{p-1-r} \qquad (\bmod p^3).\end{aligned} \tag{6.6}$$

Similarly if $m < p - 1$, (6.1) yields

$$m\beta_m^{(p)} + p\delta_m \equiv p^2\sum_{r=1}^{m-1}\binom{m}{r}\frac{\delta_r\,\delta_{m-r}}{r} \qquad (\bmod p^3). \tag{6.7}$$

If we substitute from (6.7) in (6.1) we get even stronger (but rather complicated) results. For example (6.4) becomes

$$\begin{aligned}(p-1)\,\beta_{p-1}^{(p)} + p\delta_{p-1} &\equiv p^2\sum_{r=1}^{p-2}\binom{p-1}{r}\frac{\delta_r\,\delta_{p-1-r}}{r}\\ &\quad - p^3\sum_{r=1}^{p-2}\sum_{s=1}^{r-1}\binom{p-1}{r}\frac{\delta_{p-1-r}\,\delta_s\,\delta_{r-s}}{s} \qquad (\bmod p^4).\end{aligned} \tag{6.4'}$$

We remark that

$$\beta_p^{(p)} \equiv -\delta_p \qquad (\bmod p^2).$$

7. Special cases. It is of interest to see what some of the above formulae reduce to when $c_m = 1$ for all $m \geqslant 1$ in (5.1). Then in the first place $\beta_m = B_m$, the mth Bernoulli number in Nörlund's notation. In the second place, by (5.7),

$$\delta_1 = \tfrac{1}{2}, \quad \delta_m = B_m \qquad (m > 1). \tag{7.1}$$

In particular $\beta_{2m+1} = \delta_{2m+1} = 0$ for $m \geqslant 1$. It is also clear from (5.6) that

$$\beta_m^{(k)} = B_m^{(k)}.$$

In the next place (6.1) reduces to

$$mB_m^{(k)} + k\sum_{r=1}^{m} (-1)^r \binom{m}{r} B_r B_{m-r}^{(k)} = 0,$$

which is identical with [**8**, p. 146 (83)]. Now in view of

$$(-1)^r \binom{p-1}{r} B_r^{(p)} = A_r,$$

we see that (6.2) and (6.3) become

$$A_m \equiv 0 \ (1 \leqslant m < p-1), \quad A_{p-1} \equiv -1 \qquad (\text{mod } p).$$

Next (6.5) implies for m odd, $1 < m < p$, $A_m \equiv 0$ (mod p_s), while (6.4) yields

$$(p-1) A_{p-1} + pB_{p-1} \equiv 0 \qquad (\text{mod } p^2),$$

another theorem due to Glaisher [**4**, p. 325]. We have also from (6.5) for m even, $2 \leqslant m < p-1$,

$$\frac{1}{p} A_m \equiv -\frac{1}{m} B_m \qquad (\text{mod } p),$$

which is the same as (1.1). As for (6.6), it evidently implies

$$(p-1) A_{p-1} + pB_{p-1} \equiv p^2 \sum_{r=1}^{\frac{1}{2}(p-3)} \frac{1}{2r} B_{2r} B_{p-1-2r} \qquad (\text{mod } p^3) \tag{7.2}$$

which is equivalent to a result of Nielsen already referred to (see (2.13) above).

Finally (6.7) yields for m odd, $3 < m < p$,

$$\frac{1}{p^2} A_m \equiv \frac{m}{2(m-1)} B_{m-1} \qquad (\text{mod } p)$$

which is the same as (1.2). For m even, $2 < m < p-1$, we get

$$\frac{1}{p^2} (mA_m + pB_m) \equiv \sum_{r=1}^{\frac{1}{2}m-1} \frac{1}{2r} \binom{m}{2r} B_{2r} B_{m-2r} \qquad (\text{mod } p), \tag{7.3}$$

which seems to be new. For $m = p - 1$, (7.3) coincides with (7.2).

8. The case k negative. In (5.6) we assumed $k \geqslant 1$. However the definition is valid for negative k also and it is of some interest to consider an application for such k. If then we take $k = -p$, (6.1) implies

$$m\beta_m^{(-p)} = p\sum_{r=1}^{m} \binom{m}{r} \delta_r \beta_{m-r}^{(-p)}. \tag{8.1}$$

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Thus corresponding to (6.2) and (6.3) we get

(8.2) $$\beta_m^{(-p)} \equiv 0 \qquad (\text{mod } p),\ 1 \leqslant m < p-1,$$

(8.3) $$(p-1)\,\beta_{p-1}^{(-p)} \equiv p\delta_{p-1} \equiv -c_p \qquad (\text{mod } p).$$

In the next place we have

(8.4) $$(p-1)\,\beta_{p-1}^{(-p)} - p\delta_{p-1} = p\sum_{r=1}^{p-2}\binom{p-1}{r}\delta_r\,\beta_{p-1-r}^{(-p)}$$

and

(8.5) $$m\beta_m^{(-p)} - p\delta_m \equiv 0 \qquad (\text{mod } p^2),\ 1 \leqslant m < p-1.$$

Substitution in (8.4) yields

(8.6) $$(p-1)\,\beta_{p-1}^{(-p)} - p\delta_{p-1} \equiv p^2\sum_{r=1}^{p-2}\frac{(-1)^r}{r}\delta_r\,\delta_{p-1-r} \qquad (\text{mod } p^3);$$

similarly, for $m < p-1$,

(8.7) $$m\beta_m^{(-p)} - p\delta_m = p^2\sum_{r=1}^{m-1}\frac{1}{r}\binom{m}{r}\delta_r\,\delta_{m-r} \qquad (\text{mod } p^3).$$

Comparison with (6.6) and (6.8) gives

(8.8) $$\frac{1}{p^2}(m\beta_m^{(p)} + p\delta_m) \equiv \frac{1}{p^2}(m\beta_m^{(-p)} - p\delta_m) \qquad (\text{mod } p).$$

If we now specialize as in §7, and recall that

$$\left(\frac{e^x-1}{x}\right)^k = \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{m=0}^{\infty}\frac{r^{m+k}\,x^m}{(m+k)!}$$

we see that

$$\beta_m^{(-k)} \to B_m^{(-k)} = \frac{m!}{(m+k)!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}r^{m+k} = m!\,\mathfrak{S}_k^{m+k},$$

so that $B_m^{(-k)}/m!$ is a Stirling number of the second kind. We now have at once

(8.5′) $$B_{2r+1}^{(-p)} \equiv 0 \qquad (\text{mod } p^2),\ 1 < 2r+1 < p-1$$

$$\frac{1}{p}B_{2r}^{(-p)} \equiv \frac{1}{2r}B_{2r} \qquad (\text{mod } p),\ 1 < 2r < p-1,$$

(8.6′) $$(p-1)\,B_{p-1}^{(-p)} - pB_{p-1} \equiv p^2\sum_{r=1}^{\frac{1}{2}(p-3)}\frac{1}{2r}B_{2r}\,B_{p-1-2r} \qquad (\text{mod } p^3),$$

(8.7′) $$\frac{1}{p^2}B_{2r+1}^{(-p)} \equiv \frac{2r+1}{4r}B_{2r} \quad (\text{mod } p),\ 1 < 2r+1 < p-1.$$

Formulae (8.5′) and (8.7′) are due to Nielsen [**7**, p. 338].

9. Generalized Euler numbers. We now briefly consider sequences related to the Euler numbers of higher order. Let a be a fixed rational number which is integral (mod p) and put

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$$(9.1) \qquad (1 + af)^{-k} = \sum_{m=0}^{\infty} \eta_m^{(k)} \frac{x^m}{m!}, \qquad \frac{af'}{1 + af} = \sum_{m=0}^{\infty} \zeta_m \frac{x^m}{m!},$$

where $f = f(x)$ has the same meaning as in (5.1). The coefficients $\eta_m^{(k)}$ and ζ_m are evidently integral (mod p).

If we differentiate the first of (9.1), we get

$$(9.2) \qquad \eta_{m+1}^{(k)} = -k \sum_{s=0}^{m} \binom{m}{s} \zeta_s \eta_{m-s}^{(k)},$$

which is analogous to (6.1). In particular for $k = p$, (9.2) implies

$$(9.3) \qquad \frac{1}{p} \eta_{m+1}^{(p)} - \zeta_m = - \sum_{s=1}^{m} \binom{m}{s} \zeta_{m-s} \eta_s^{(p)},$$

so that

$$(9.4) \qquad \frac{1}{p} \eta_{m+1}^{(p)} \equiv - \zeta_m \pmod{p}.$$

Substitution of (9.4) in (9.3) now yields

$$(9.5) \qquad \frac{1}{p}\left(\frac{1}{p} \eta_{m+1}^{(p)} - \zeta_m\right) \equiv \sum_{s=1}^{m} \binom{m}{s} \zeta_{m-s} \zeta_{s-1} \pmod{p}.$$

Now for $a = \frac{1}{2}$ we have [**8**, p. 143]

$$\left(\frac{2}{e^x + 1}\right)^k = \sum_{m=0}^{\infty} \frac{C_m^{(k)}}{2^m} \frac{x^m}{m!},$$

so that $\eta_m^{(k)} = 2^{-m} C_m^{(k)}$. Also $\zeta_m = -2^{-m-1} C_m$ for $m > 0$, where $C_m = C_m^{(1)}$; we recall that $C_{2r} = 0$ for $r > 0$. We can therefore state the following results as special cases of (9.4) and (9.5):

$$(9.6) \qquad \frac{1}{p} C_{2r}^{(p)} \equiv C_{2r-1}, \qquad \frac{1}{p} C_{2r+1}^{(p)} \equiv 0 \pmod{p};$$

$$(9.7) \qquad \frac{1}{p}\left(\frac{1}{p} C_{2r}^{(p)} - C_{2r-1}\right) \equiv \sum_{s=1}^{r-1} \binom{2r-1}{2s} C_{2r-2s-1} C_{2s-1} \pmod{p},$$

$$(9.8) \qquad \frac{1}{p^2} C_{2r+1}^{(p)} \equiv -(2r+1) C_{2r-1} \pmod{p}.$$

These congruences are evidently analogous to Glaisher's theorem for A_{2r}, A_{2r+1}.

Finally if we take k in (9.1) negative we get results similar to those above. In particular for $k = -p$, we have

$$(9.3') \qquad \frac{1}{p} \eta_{m+1}^{(-p)} - \zeta_m = \sum_{s=1}^{m} \binom{m}{s} \zeta_{m-s} \eta_s^{(-p)},$$

$$(9.4') \qquad \frac{1}{p} \eta_{m+1}^{(-p)} \equiv \zeta_m \pmod{p},$$

$$(9.5') \qquad \frac{1}{p}\left(\frac{1}{p} \eta_{m+1}^{(-p)} + \zeta_m\right) \equiv \sum_{s=1}^{m} \binom{m}{s} \zeta_{m-s} \zeta_{s-1} \pmod{p}.$$

Comparison with (9.5) gives

$$\frac{1}{p}\left(\frac{1}{p}\,\eta_{m+1}^{(p)} - \zeta_m\right) \equiv \frac{1}{p}\left(\frac{1}{p}\,\eta_{m+1}^{(-p)} - \zeta_m\right) \pmod{p}. \tag{9.9}$$

Then if $a = \frac{1}{2}$ we get the special formulae

$$\frac{1}{p}\,C_{2r}^{(-p)} \equiv -\,C_{2r-1} \pmod{p}, \tag{9.6'}$$

$$\frac{1}{p}\left(\frac{1}{p}\,C_{2r}^{(-p)} - C_{2r-1}\right) \equiv \sum_{s=1}^{r-1}\binom{2r-1}{2s} C_{2r-2s-1}\,C_{2s-1} \pmod{p}, \tag{9.7'}$$

$$\frac{1}{p^2}\,C_{2r+1}^{(-p)} \equiv -\,(2r+1)\,C_{2r-1} \pmod{p}. \tag{9.8'}$$

Formulae (9.6′) and (9.8′) are proved by Nielsen [**7**, p. 292]; to facilitate comparison we note that

$$2^{k-m}\,C_m^{(-k)} = \sum_{s=0}^{k}\binom{k}{s} s^m.$$

References

1. L. Carlitz, *The coefficients of the reciprocal of a series*, Duke Math. J., *8* (1941), 689–700.

2. L. E. Dickson, *History of the theory of numbers*, vol. 1 (Washington, 1919).

3. J. W. L. Glaisher, *Congruences relating to the sums of products of the first n numbers and to other sums of products*, Quarterly J. Math., *31* (1900), 1–35.

4. ——— *On the residues of the sums of products of the first p − 1 numbers, and their powers, to modulus p^2 or p^3*, Quarterly J. Math., *31* (1900), 321–353.

5. N. Nielsen, *Om Potenssummer of hele Tal*, Nyt Tidsskrift for Mathematik, *4B* (1893), 1–10.

6. ——— *Recherches sur les suites régulières et les nombres de Bernoulli et d'Euler*, Annali di matematica (3), *22* (1914), 71–115.

7. ——— *Traité élémentaire des nombres de Bernoulli* (Paris, 1923).

8. N. E. Nörlund, *Vorlesungen über Differenzenrechnung* (Berlin, 1924).

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