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# ON OPTIMAL REINSURANCE IN THE PRESENCE OF PREMIUM BUDGET CONSTRAINT AND REINSURER'S RISK LIMIT

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**Abstract**

In this paper, we propose two new optimal reinsurance models in which both premium budget constraints and the reinsurer's risk limits are taken into account. To be precise, we assume that the reinsurance premium has an upper bound, and that the admissible ceded loss functions have a pre-specified upper limit. Moreover, we assume that the reinsurance premium principle is calculated by the expected value premium principle. Under the optimality criteria of minimizing the value at risk and conditional value at risk of the insurer's total risk exposure, we derive the explicit optimal reinsurance treaties, which are layer reinsurance treaties. A new approach is developed to construct the optimal reinsurance treaties. Comparisons with existing studies are also made. Finally, we provide a numerical study based on real data and an example to illustrate the proposed models and results. Our work provides a novel generalization of several known achievements in the literature.

*Keywords:* Optimal reinsurance; budget constraint; value at risk; conditional value at risk; expected value premium principle

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## 1. Introduction

Reinsurance is an effective risk management tool for insurers, as it can enable an insurer to reduce its underwriting risk. In a typical reinsurance treaty, an insurer cedes part of the loss to a reinsurer, and incurs an additional cost in the form of a reinsurance premium which is payable to the reinsurer. This implies that an insurer must address the classical trade-off between the risk retained and the premium paid to the reinsurer. To develop an optimal reinsurance treaty, the insurer needs to mathematically determine three things: (i) an optimal objective, (ii) a set of admissible ceded loss functions, and (iii) a reinsurance premium principle. Any change in one or more of these three aspects of the problem could potentially lead to a very different optimal solution. In other words, an important issue in developing a reinsurance treaty is to identify the optimal ceded loss functions, or equivalently, the optimal retained functions according to

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certain optimality criteria. In this paper, we aim to study the optimal reinsurance treaties for optimal reinsurance models with premium budget constraints and reinsurer's risk limits.

In relation to the optimal objectives, a seminal work of Borch [5] considers the variance of the insurer's retained risk. Similar studies can also be found in Gajek and Zagrodny [24] and Kaluszka [29]. Another seminal paper by Arrow [3] considered maximizing the expected utility. For more works within Arrow's framework, we refer to Young [40], Deprez and Gerber [21], Promislow and Young [35], and Kaluszka [30]. Over the past 15 years, risk measures such as value at risk (VaR), conditional value at risk (CVaR), distortion risk measures and coherent risk measures have been extensively employed as optimal criteria in actuarial studies. For example, see Gajek and Zagrodny [25], Cai and Tan [8], Cai *et al.* [9], Bernard and Tian [4], Tan *et al.* [37], Weng [39], Chi [14], Chi and Tan [17], Chi and Weng [18], Assa [2], Cheung *et al.* [11], Lu *et al.* [34], Wang and Peng [38], Zhuang *et al.* [44], Cheung and Lo [13], Chi *et al.* [16], Lo [31, 32], Cheung *et al.* [10], Ghossoub [26], Huang and Yin [27], Asimit *et al.* [1], and the references therein.

In relation to admissible ceded loss functions, an important class was first suggested by Huberman *et al.* [28], who required that both the ceded and retained loss functions be increasing in order to preclude the moral hazard from a reinsurance treaty. Mathematically, this is equivalent to requiring that the ceded loss function be increasing and Lipschitz continuous.

Recently, there has been extensive study of optimal reinsurance models with premium budget constraints from a practical point of view, taking into account the various constraints commonly encountered in practice. Weng [39], Zheng and Cui [41] and Ghossoub [26], to name just a few, studied optimal reinsurance policies in the case where the reinsurance premium with an upper bound is calibrated by the expected value premium principle. Optimal reinsurance policies in the case where the reinsurance premium with an upper bound is calculated by a distortion premium principle have been heavily investigated in the literature; for example, see Cheung *et al.* [12], Cui *et al.* [19], Zhuang *et al.* [44], Cheung and Lo [13], Lo [31, 32], Huang and Yin [27], and the references therein. Cheung *et al.* [10] studied optimal reinsurance policies in the case where the reinsurance premium with an upper bound is calibrated by a convex risk measure. Cheung *et al.* [11] studied optimal reinsurance policies in the case where the reinsurance premium with an upper bound is assumed to be a functional of the expectation, VaR, and average value at risk (AVaR) of the insurer's loss variable. More recently, Asimit *et al.* [1] studied Pareto-optimal insurance contracts in the case where the reinsurance premium is assumed to be a constant with lower and upper bounds, and the upper and lower bounds stand for premium budget and minimum charge constraints, respectively.

The premium budget constraints mentioned above can also be understood as a sort of insurer's risk control, in the sense that the insurer would control the premium payoff due to the financial budget. On the other hand, from the perspective of reinsurer's risk control, some optimal reinsurance models with reinsurer's risk limits have been studied in the literature. Two basic types of reinsurer's risk limits have been investigated. One type was proposed by Cummins and Mahul [20], who suggested that the ceded loss function should have a prespecified upper limit constraint, which means that the reinsurer would control his/her maximal underwriting coverage; see also Raviv [36]. Recently, along the lines of Cummins and Mahul [20], Lu *et al.* [34] studied optimal reinsurance treaties minimizing the VaR and CVaR of the insurer's total risk exposure. Another type of reinsurer's risk limit was proposed by Zhou and Wu [42], who imposed some constraints on the amount of the ceded loss minus premium, which means that the reinsurer would control the risk of his/her net loss. For more studies along these lines, we refer to Zhou *et al.* [43], Cheung *et al.* [12], Chi and Lin [15], Lu *et al.* [34], and Lo [32].

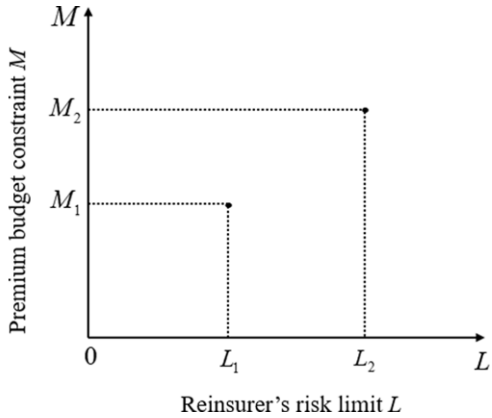


FIGURE 1. Relationship between reinsurance's risk limit $L$ and premium budget constraint $M$.

While premium budget constraints and reinsurer's risk limits are prevalent both in academia and in the insurance industry, they are not well studied in the literature on optimal reinsurance models. Therefore, from both the theoretical and the practical point of view, it is natural and reasonable to expect optimal reinsurance models that can simultaneously take into account premium budget constraints and reinsurer's risk limits. Such optimal reinsurance models are worth studying because, compared to models involving either premium budget constraints or reinsurer's risk limits alone, they are apparently more suitable for modeling realistic insurance markets. Motivated by this observation, in this paper we strive to interweave the aforementioned lines of research on premium budget constraints and reinsurer's risk limits. It should be noted that the optimal reinsurance models that arise here are far from simple hybrids of models involving either premium budget constraints or reinsurer's risk limits alone. Indeed, it turns out not only that it is a delicate problem to develop an appropriate method for deriving the optimal reinsurance treaties, but also that the solutions to such models are also more complicated; see especially the preparatory results given in Lemmas 3.1–3.4 (Subsection 3.1), Theorem 3.2, and Remark 3.2(1) below.

In the present paper, we assume that both the reinsurance premium and the ceded loss function have pre-specified upper limits, which respectively represent the premium budget constraint and the Cummins–Mahul-type reinsurer's risk limit constraint. Moreover, in order to exclude the moral hazard, we assume that the ceded loss functions are increasing and Lipschitz continuous, as there is no incentive for the insurer to go for large risk while they know their loss is close to the reinsurer's risk limit. Furthermore, we suppose that the reinsurance premium is calculated by the expected value premium principle. Under the optimality criteria of minimizing the VaR and CVaR of the insurer's total risk exposure, we construct explicit optimal reinsurance treaties, which are layer reinsurance treaties. We compare our results with those of existing studies. Finally, we provide a numerical study based on real data and an example to illustrate the proposed models and results. Our work provides a novel generalization of several known achievements in the literature.

Figure 1 shows the intuitive relation between the Cummins–Mahul-type reinsurer's risk limit and the reinsurance premium budget. In Figure 1, the premium budget constraint $M_i$ corresponds to the reinsurer's risk limit $L_i$, $i$=1, 2. Intuitively, compared with a lower reinsurer's risk limit $L_1$, if the reinsurer sets a higher risk limit $L_2$, then it means that the reinsurer is willing

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to undertake higher coverage, and thus the reinsurer will most likely require a higher premium. In this situation, the insurer is supposed to make the premium budget as high as possible so that he/she can purchase more expensive reinsurance from the reinsurer. In other words, in this situation, the reinsurance premium budget $M_2$ should not be less than $M_1$. Alternatively, if the insurer has a high enough premium budget $M$, then the reinsurer's risk limit $L$ should significantly affect the insurer's optimal reinsurance treaty. This consideration will also be demonstrated in the real-data-based numerical study in the sequel.

It might be helpful for us to briefly comment on the main contributions of the present paper. First, we propose two new optimal reinsurance models that simultaneously take into account premium budget constraints and reinsurer's risk limits, which are not well studied in the literature. Explicit optimal reinsurance treaties are obtained. This paper significantly generalizes recent works of Lu *et al.* [34] by imposing budget constraint on the reinsurance premium; see Remark 2.1, Theorems 3.1 and 3.2, Remarks 3.1 and 3.2, and Example 3.1 below. From the viewpoint of behavioral finance and in practice, considering the perspective of the insurer, we believe that it is also reasonable to impose a pre-specified upper limit on the reinsurance premium, and this consideration is the starting point of the present study. Note that such a budget constraint on the reinsurance premium was previously studied by Weng [39, Chapters 3 and 4], where it was interpreted in the context of insurance economics. Second, compared with the relevant works of Lu *et al.* [34], the generalizations achieved in the present paper are non-trivial, because new arguments need to be developed to construct the optimal reinsurance treaties, which are two-layer reinsurance treaties. Although the optimal reinsurance contracts in both the present paper and that of Lu *et al.* [34] are two-layer reinsurance treaties, the imposition of a premium budget constraint on the reinsurance premium makes it more difficult and complicated to correctly obtain the two-layer reinsurance treaties. Inspired by Lu *et al.* [34], we develop a new approach to showing the existence of the optimal two-layer reinsurance treaties. These newly developed arguments are far more delicate and complicated; see the preparation lemmas in Subsection 3.1, Theorems 3.1 and 3.2, and Remarks 3.1 and 3.2 below. In particular, for the CVaR-based optimal reinsurance model, compared with the relevant model of Lu *et al.* [34], one more new case needs to be discussed; see Theorem 3.2 and Remark 3.2(1) below. We believe that these newly developed arguments are also interesting in their own right.

It should be mentioned that our optimality criterion of minimizing the VaR of the insurer's total risk exposure is closely related to the optimality criterion of maximizing the insurer's survival probability, which was studied by Gajek and Zagrodny [25]. Indeed, the two criteria are equivalent, in the sense that they have the same optimal solutions. Starting from the consideration that the insurer has enough money to purchase the stop-loss reinsurance contract, and under the assumption that the reinsurance premium does not exceed some pre-specified upper limit, Gajek and Zagrodny [25] found the deductible which is optimal in the sense that the resulting stop-loss reinsurance contract is the least expensive reinsurance arrangement with the insurer's insolvency probability equal to zero. Gajek and Zagrodny [25] classified this case as optimal full protection against ruin. Furthermore, they also studied the other case of optimal partial protection against ruin, in which the reinsurance premium with respect to a pre-specified stop-loss contract is assumed to have a lower bound. They found that in this case, a kind of truncated stop-loss reinsurance contract is optimal. In the present paper, we employ a special class of bounded ceded loss functions, namely the class of admissible ceded loss functions, which excludes the stop-loss contract. In addition, the optimal reinsurance treaty obtained in the present paper is a two-layer reinsurance treaty, which is different from both the stop-loss and the truncated stop-loss contracts of Gajek and Zagrodny [25]. Taking the above considerations into account, the present study of the VaR-based optimal model can be viewed as a meaningful complement to the study of Gajek and Zagrodny [25].

It should also be mentioned again that the optimal reinsurance models with premium budget constraints were previously studied by Weng [39, Chapters 3 and 4], who also assumed the reinsurance premium principle to be the expected value premium principle. Nevertheless, there are significant differences between this paper and [39, Chapters 3 and 4], which we explain briefly here. For the VaR-based optimal reinsurance model, Weng [39, Chapter 3] employed increasing convex loss functions as the class of admissible ceded loss functions. This excludes the case considered in the present paper, because the admissible ceded loss functions employed in this paper are bounded from above by a pre-specified constant. There are also differences between the mathematical techniques used in this paper and those of Weng [39, Chapter 3]. For the CVaR-based optimal reinsurance model, Weng [39, Chapter 4] employed a general class of admissible ceded loss functions such that the ceded loss has finite first and second moments; this includes the class considered in the present paper as a subclass. However, it turns out that there is still something new in the present paper. On the one hand, the results obtained below will show how the pre-specified upper limit imposed on the ceded loss functions in this paper influences the optimal reinsurance treaties, which has a significant interpretation in terms of insurance economics; see Theorem 3.2 and Remark 3.2 below. On the other hand, both the results obtained and the methods used in this paper are somewhat different from those of Weng [39, Chapter 4], as exemplified by the process of proving Theorem 3.2; see Subsection 3.2 below. Thus, the present study can also be viewed as a meaningful complement to that of Weng [39, Chapters 3 and 4].

The rest of this paper is organized as follows. In Section 2 we give some preliminaries, including a description of the optimal reinsurance models. In Section 3 we study the optimal reinsurance problems under the VaR and CVaR optimality criteria. The explicit optimal reinsurance treaties are provided. An example is also given to illustrate the proposed models and the results obtained.

## 2. Preliminaries

Let $X$ be a random loss initially faced by an insurer. (For example, $X$ could be a claim or an aggregate of claims.) We assume that $X$ is a non-negative random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with positive and finite expectation $\mathbb{E}(X)$. Denote by $F_X(x) := \mathbb{P}(X \le x)$, $x \in \mathrm{R}$, the distribution function of $X$, and by $S_X(x) := 1 - F_X(x)$ the survival function of $X$. Denote by $\mathscr{X}$ the class of non-negative random variables with finite expectation and $\mathbb{P}(X > 0) > 0$. In a classical reinsurance treaty, the insurer would cede part of the loss $X$, say $f(X)$, to a reinsurer, and retain the rest of the loss $X$, say $R_f(X) := X - f(X)$. We call the function $f(x) : [0, +\infty) \to [0, +\infty)$ the ceded loss function, and $R_f(x) : [0, +\infty) \to [0, +\infty)$ the retained loss function. When an insurer cedes part of the loss to a reinsurer, the insurer incurs an additional cost in the form of a reinsurance premium which is payable to the reinsurer. Let $\pi(\cdot)$ be the reinsurance premium principle, which is a mapping from $\mathscr{X}$ to $\mathbf{R}_+ := [0, +\infty)$. In the presence of reinsurance, the liability of the insurer is the total risk exposure,

$$T_f(X) := R_f(X) + \pi[f(X)].$$

As pointed out by Huberman *et al.* [28], in order to reduce the moral hazard, a feasible reinsurance treaty should be designed so that both the ceded and retained loss functions are

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increasing. We denote by $\mathscr{F}$ the class of such ceded loss functions; that is,

$$\mathscr{F} := \left\{ f : \mathbf{R}_+ \to \mathbf{R}_+,\ 0 \leq f(x) \leq x,\ f(x) \text{ and } R_f(x) \text{ are increasing in } x \right\}.$$

Not that the property that $R_f(x)$ is increasing is equivalent to the Lipschitz continuity of $f(x)$; that is, for any $0 \leq x_1 \leq x_2$,

$$f(x_2) - f(x_1) \leq x_2 - x_1. \tag{2.1}$$

In this paper, the class $\mathscr{F}$ will serve as a starting point for determining the admissible ceded loss functions.

We now introduce the reinsurer's risk limit constraint which we will consider. The constraint is that the ceded loss function is bounded from above by a pre-specified upper limit $L > 0$, as suggested by Cummins and Mahul [20] from the practical point of view. We denote by $\mathscr{F}_1$ the subclass of $\mathscr{F}$ defined by

$$\mathscr{F}_1 := \{ f \in \mathscr{F} : f(x) \leq L \quad \text{for} \quad x \in \mathbf{R}_+ \}.$$

Throughout this paper, we assume that the reinsurance premium principle is calculated by the common expected value premium principle; that is,

$$\pi(Y) := (1 + \rho)\mathbb{E}[Y], \qquad Y \in \mathscr{X},$$

where $\rho > 0$ is the safety loading factor. Write $\rho^* := \dfrac{1}{1+\rho}$.

From the viewpoint of behavioral finance and in practice, considering the perspective of the insurer, when the ceded loss function is assumed to be bounded from above by a pre-specified upper limit, we argue that it is also reasonable to assume that the insurer will require a pre-specified upper limit on the reinsurance premium payable to the reinsurer. This is why we believe that optimal reinsurance models with premium budget and reinsurer's risk limit constraints are worth studying. Such optimal reinsurance models are generalizations of the relevant models studied by Lu *et al.* [34].

We now introduce the premium budget constraint on the reinsurance premium principle which we will consider. Define

$$\mathscr{H} := \{ f \in \mathscr{F}_1 : \pi[f(X)] \leq M \} = \{ f \in \mathscr{F}_1 : (1+\rho)\mathbb{E}[f(X)] \leq M \},$$

where $M > 0$ is a pre-specified upper limit on the reinsurance premium.

In light of the popularity of the VaR and CVaR risk measures among banks and companies for risk management and setting regulatory capital, we will use VaR and CVaR to evaluate the insurer's liability.

Next, we introduce the definitions of VaR and CVaR.

**Definition 2.1.** Let $\alpha \in (0, 1)$ and $X \in \mathscr{X}$.

(i) The VaR of $X$ at confidence level $(1 - \alpha)$ is defined as

$$\mathrm{VaR}_\alpha(X) := \inf\{ x \geq 0 : \mathbb{P}(X > x) \leq \alpha \}.$$

(ii) The CVaR of $X$ at confidence level $(1 - \alpha)$ is defined as

$$\mathrm{CVaR}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\theta(X) \mathrm{d}\theta.$$

In the next lemma, we collect some properties of VaR and CVaR.

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**Lemma 2.1.** *Let $X \in \mathscr{X}$. Then VaR and CVaR have the following properties:*

*(a) For any* $\alpha \in (0, 1)$, $\mathrm{VaR}_\alpha(X) \in [0, +\infty)$.

*(b) There exists* $\alpha_0 \in (0, 1)$ *such that* $\mathrm{VaR}_{\alpha_0}(X) > 0$.

*(c)* $\mathrm{VaR}_\alpha(X)$ *is decreasing in* $\alpha$ *on (0,1); that is,* $\mathrm{VaR}_\alpha(X) \le \mathrm{VaR}_\beta(X)$ *if* $0 < \beta \le \alpha < 1$.

*(d) Translation-invariance: for* $0 < \alpha < 1$ *and any* $c \in \mathrm{R}$,

$$\mathrm{VaR}_\alpha(X + c) = \mathrm{VaR}_\alpha(X) + c$$

*and*

$$\mathrm{CVaR}_\alpha(X + c) = \mathrm{CVaR}_\alpha(X) + c.$$

*(e) For* $0 < \alpha < 1$ *and* $x \ge 0$,

$$\mathrm{VaR}_\alpha(X) \le x \quad \textit{if and only if} \quad S_X(x) \le \alpha.$$

*(f) For any increasing continuous function* $\phi$, *we have*

$$\mathrm{VaR}_\alpha(\phi(X)) = \phi(\mathrm{VaR}_\alpha(X)), \qquad 0 < \alpha < 1.$$

Note that Part (e) is from (11) of Dhaene *et al.* [22]. Part (f) is a consequence of Dhaene *et al.* [22, (15)], where the function $\phi$ is only required to be non-decreasing and left-continuous. The other properties are obvious. In particular, $\mathrm{VaR}_0(X) := \mathrm{esssup} X$ and $\mathrm{VaR}_1(X) := 0$.

Next, we introduce the optimal reinsurance models we will consider in the present paper, which are described as follows:

- VaR-based optimal reinsurance model with limits on coverage:

$$\mathrm{VaR}_\alpha[T_{f^*}(X)] = \min_{f \in \mathscr{H}} \mathrm{VaR}_\alpha[T_f(X)] \tag{2.2}$$

- CVaR-based optimal reinsurance model with limits on coverage:

$$\mathrm{CVaR}_\alpha[T_{f^*}(X)] = \min_{f \in \mathscr{H}} \mathrm{CVaR}_\alpha[T_f(X)] \tag{2.3}$$

In both cases, $f^*$ denotes the resulting optimal ceded loss function.

**Remark 2.1.** If $M \ge (1 + \rho)\mathbb{E}[X]$, then the constraint $(1 + \rho)\mathbb{E}[f(X)] \le M$ in the models (2.2) and (2.3) is redundant. In this case, the optimal reinsurance models (2.2) and (2.3) are reduced to the optimal reinsurance models

$$\min_{f \in \mathscr{F}_1} \mathrm{VaR}_\alpha[T_f(X)] \quad \text{and} \quad \min_{f \in \mathscr{F}_1} \mathrm{CVaR}_\alpha[T_f(X)],$$

respectively, which were studied by Lu *et al.* [34]. For the links between the main results of the present paper and those of Lu *et al.* [34], see Remark 3.1(2) and Remark 3.2(1) below.

## 3. Optimal reinsurance with limits on coverage

In this section we discuss the optimal reinsurance treaties for the optimization models (2.2) and (2.3). From now on, let $X \in \mathscr{X}$ be fixed. Throughout this paper, we assume that the survival

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function $S_X(x)$ of $X$ is absolutely continuous and strictly decreasing on $(0, +\infty)$ but with a possible jump at 0 with $p_0 := P(X = 0)$. We further assume that $X$ is of a density function $f_X(x)$ in the sense that for any $x \geq 0$,

$$F_X(x) = p_0 + \int_0^x f_X(y)\mathrm{d}y.$$

Note that these assumptions imposed on the random loss $X$ could enable one to work with somewhat more general loss distribution functions, which are the same as those employed by Lu *et al.* [34].

### 3.1. Preparations

In this subsection we present some preparatory results, which will play an important role in our discussion later.

Given $0 \leq a \leq b < +\infty$, we define a non-negative function $g(x; a, b)$ on $[0, +\infty)$ by

$$g(x; a, b) := (x-a)_+ - (x-b)_+. \tag{3.4}$$

Given $\alpha \in (0, 1)$ and $0 \leq a \leq \mathrm{VaR}_\alpha(X)$, we define a non-negative function $g(x; a)$ on $[0, +\infty)$ by

$$g(x; a) := g(x; a, \mathrm{VaR}_\alpha(X)) = (x-a)_+ - (x - \mathrm{VaR}_\alpha(X))_+. \tag{3.5}$$

Furthermore, we denote by $\mathscr{H}_1$ the class of the functions $g(x; a)$ satisfying $a \in \mathscr{D}_1$, where $\mathscr{D}_1$ is defined by

$$\mathscr{D}_1 := \left\{ a : [\mathrm{VaR}_\alpha(X) - L]_+ \leq a \leq \mathrm{VaR}_\alpha(X),\ (1+\rho)\int_a^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x \leq M \right\}.$$

Obviously, $\mathscr{D}_1 \neq \emptyset$ and $\emptyset \neq \mathscr{H}_1 \subseteq \mathscr{H}$. Write

$$a^* := \inf_{a \in \mathscr{D}_1} a; \tag{3.6}$$

then $a^*$ is either $[\mathrm{VaR}_\alpha(X) - L]_+$ or the unique solution to the equation

$$(1+\rho)\int_a^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x = M$$

with respect to $a \in ([\mathrm{VaR}_\alpha(X) - L]_+, \mathrm{VaR}_\alpha(X))$.

By the above analysis, we know that the set $\mathscr{D}_1$ can be rewritten as

$$\mathscr{D}_1 = \{a : a^* \leq a \leq \mathrm{VaR}_\alpha(X)\}. \tag{3.7}$$

Given $\alpha \in (0, 1)$, we denote by $\mathscr{H}_2$ the class of the functions $g(x; a, b)$ defined by (3.4) with $(a, b) \in \mathscr{D}_2$, where $\mathscr{D}_2$ is defined by

$$\mathscr{D}_2 := \left\{ (a, b) : 0 \leq a \leq \mathrm{VaR}_\alpha(X) \leq b \leq a + L,\ (1+\rho)\int_a^b S_X(x)\mathrm{d}x \leq M \right\}. \tag{3.8}$$

Clearly, $\mathscr{D}_2 \neq \emptyset$ and $\emptyset \neq \mathscr{H}_2 \subseteq \mathscr{H}$.

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For any $a \in [0,\ \mathrm{VaR}_\alpha(X)]$, we define

$$\mathscr{B}(a) := \left\{ b \in [a,\ a+L] : (1+\rho)\int_a^b S_X(x)\mathrm{d}x \le M \right\}.$$

Obviously, $\mathscr{B}(a) \neq \emptyset$. Let

$$\beta(a) := \sup_{b \in \mathscr{B}(a)} b. \tag{3.9}$$

Clearly, $a < \beta(a) \le a+L$ and

$$(1+\rho)\int_a^{\beta(a)} S_X(x)\mathrm{d}x \le M. \tag{3.10}$$

Moreover, given $a \in [0,\ \mathrm{VaR}_\alpha(X)]$, if

$$(1+\rho)\int_a^{a+L} S_X(x)\mathrm{d}x \le M,$$

then $\beta(a) = a+L$. If

$$(1+\rho)\int_a^{a+L} S_X(x)\mathrm{d}x > M,$$

then $\beta(a)$ is the unique solution to the equation

$$(1+\rho)\int_a^b S_X(x)\mathrm{d}x = M$$

with respect to $b \in (a,\ a+L)$. In this latter case, the first-order derivative of $\beta(a)$ with respect to $a$ is given by

$$\beta'(a) = \frac{S_X(a)}{S_X(\beta(a))} > 0, \tag{3.11}$$

since $S_X(\beta(a)) > 0$.

For any $(a,\ b) \in \mathscr{D}_2$, from (3.9) we have that $0 \le a \le \mathrm{VaR}_\alpha(X) \le b \le \beta(a)$, and hence

$$(1+\rho)\int_a^b S_X(x)\mathrm{d}x \le (1+\rho)\int_a^{\beta(a)} S_X(x)\mathrm{d}x \le M,$$

where the last inequality holds because of (3.10).

By the above analysis, we know that the set $\mathscr{D}_2$ defined by (3.8) can be rewritten as

$$\mathscr{D}_2 = \{(a,\ b) : 0 \le a \le \mathrm{VaR}_\alpha(X) \le b \le \beta(a)\}. \tag{3.12}$$

We denote by $\mathscr{A}_2$ the projection of $\mathscr{D}_2$ to the first coordinate; that is,

$$\mathscr{A}_2 := \{a \in [0,\ \mathrm{VaR}_\alpha(X)] : \text{there exists } b \in [\mathrm{VaR}_\alpha(X),\ \beta(a)] \text{ such that } (a,\ b) \in \mathscr{D}_2\}. \tag{3.13}$$

Clearly, $\mathrm{VaR}_\alpha(X) \in \mathscr{A}_2 \subseteq [0,\ \mathrm{VaR}_\alpha(X)]$, since $(\mathrm{VaR}_\alpha(X),\ \mathrm{VaR}_\alpha(X)) \in \mathscr{D}_2$.

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Note that for any $a \in \mathscr{A}_2$, if $\beta(a)$ is the unique solution to the equation

$$(1+\rho)\int_a^b S_X(x)\mathrm{d}x = M$$

with respect to $b \in (a,\ \ a+L)$, then by (3.12) we know that $\mathrm{VaR}_\alpha(X) \le \beta(a) < a+L$. Furthermore, we denote by $\mathscr{S}$ the set of $a \in \mathscr{A}_2$ such that $\beta(a)$ is the unique solution to the equation

$$(1+\rho)\int_a^b S_X(x)\mathrm{d}x = M$$

with respect to $b \in [\mathrm{VaR}_\alpha(X),\ \ a+L)$. If $\mathscr{S} \neq \emptyset$, we define

$$\hat{a} := \sup_{a\in\mathscr{S}} a. \tag{3.14}$$

Clearly, $\hat{a} \in [0,\ \ \mathrm{VaR}_\alpha(X)]$.

The following four lemmas concern the properties of $\beta(\cdot)$, $\mathscr{A}_2$, and $\mathscr{S}$, which will play important roles in our discussion later. Their proofs are postponed to the appendix.

**Lemma 3.1.** *The function* $\beta(\cdot)$ *defined by* (3.9) *is strictly increasing on* $[0,\ \ \mathrm{VaR}_\alpha(X)]$.

**Lemma 3.2.** *We have* $\hat{a}$, $\mathrm{VaR}_\alpha(X) \in \mathscr{A}_2$. *Moreover,* $\mathscr{A}_2 = [\underline{a},\ \ \mathrm{VaR}_\alpha(X)]$, *where* $\underline{a}$ *is defined by*

$$\underline{a} := \inf_{a\in\mathscr{A}_2} a. \tag{3.15}$$

**Lemma 3.3.** *The set* $\mathscr{S} = \emptyset$ *if and only if* $\beta(a) = a+L$ *for all* $a \in \mathscr{A}_2$.

**Lemma 3.4.** *Assume that* $\mathscr{S} \neq \emptyset$*; then the following hold:*

*(i)* $\mathscr{S} = [\underline{a},\ \ \hat{a}]$ *if* $\beta(\hat{a}) < a+L$*;*

*(ii)* $\mathscr{S} = [\underline{a},\ \ \hat{a})$ *if* $\beta(\hat{a}) = a+L$.

Provided that $\mathscr{S} \neq \emptyset$, from Lemmas 3.2 and 3.4, we know that $\mathrm{VaR}_\alpha(X) \le \beta(a) < a+L$ if $a \in \mathscr{S}$, and that $\mathrm{VaR}_\alpha(X) \le \beta(a) = a+L$ if $a \in \mathscr{A}_2 \backslash \mathscr{S}$.

### 3.2. VaR-based optimal reinsurance with limits on coverage

In this subsection we discuss the optimal reinsurance treaty for the optimal reinsurance model (2.2).

The next lemma is crucial in deriving the optimal reinsurance treaty for the optimal reinsurance model (2.2). Its proof is postponed to the appendix.

**Lemma 3.5.** *Let the confidence level* $\alpha \in (0,\ 1-p_0)$. *For any* $f \in \mathscr{H}$*, there exists a function* $h_f \in \mathscr{H}_1$ *such that*

$$\mathrm{VaR}_\alpha[T_{h_f}(X)] \le \mathrm{VaR}_\alpha[T_f(X)]. \tag{3.16}$$

By Lemma 3.5, the optimization model (2.2) can be equivalently translated into

$$\mathrm{VaR}_\alpha[T_{f^*}(X)] = \min_{f\in\mathscr{H}_1} \mathrm{VaR}_\alpha[T_f(X)] \tag{3.17}$$

The following theorem is the main result of this subsection; it provides the optimal reinsurance treaties for the model (2.2). Its proof is postponed to the appendix.

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**Theorem 3.1.** *Let the confidence level* $\alpha \in (0, 1 - p_0)$*, and let* $a^*$ *be as in* (3.6).

(i) *If* $\alpha \geq \frac{1}{1+\rho}$*, then*

$$\min_{f \in \mathscr{H}} \mathrm{VaR}_\alpha[T_f(X)] = \mathrm{VaR}_\alpha(X),$$

*and the minimum of* $\mathrm{VaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = 0$.

(ii) *If* $\alpha < \frac{1}{1+\rho}$ *and* $a^* < \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$*, then*

$$\min_{f \in \mathscr{H}} \mathrm{VaR}_\alpha[T_f(X)] = \mathrm{VaR}_{\frac{1}{1+\rho}}(X) + (1+\rho) \int_{\mathrm{VaR}_{\frac{1}{1+\rho}}(X)}^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x,$$

*and the minimum of* $\mathrm{VaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = (x - \mathrm{VaR}_{\frac{1}{1+\rho}}(X))_+ - (x - \mathrm{VaR}_\alpha(X))_+$.

(iii) *If* $\alpha < \frac{1}{1+\rho}$ *and* $a^* \geq \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$*, then*

$$\min_{f \in \mathscr{H}} \mathrm{VaR}_\alpha[T_f(X)] = a^* + (1+\rho) \int_{a^*}^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x,$$

*and the minimum of* $\mathrm{VaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = (x - a^*)_+ - (x - \mathrm{VaR}_\alpha(X))_+$.

**Remark 3.1.**

(1) The insurance economics of Theorem 3.1(iii) can be interpreted as follows. The limit $M$ may influence the deductible. Indeed, $a^*$ represents the deductible. The larger $M$ is, the less $a^*$ is. In other words, the more reinsurance premium the insurer would like to pay, the more reinsurance the insurer would like to purchase.

(2) When $M$ goes to infinity, $a^*$ decreases to $[\mathrm{VaR}_\alpha(X) - L]_+$. Hence Theorem 3.1 recovers Theorem 3.2 of Lu *et al.* [34].

### 3.3. CVaR-based optimal reinsurance with limits on coverage

In this subsection, we discuss the optimal reinsurance treaty for the optimal reinsurance model (2.3).

Note that for any $Z \in \mathscr{X}$ and any $\alpha \in (0, 1)$,

$$\begin{aligned} \mathrm{CVaR}_\alpha(Z) &= \mathrm{VaR}_\alpha(Z) + \frac{1}{\alpha} \cdot \mathbb{E}\left[(Z - \mathrm{VaR}_\alpha(Z))_+\right] \\ &= \mathrm{VaR}_\alpha(Z) + \frac{1}{\alpha} \int_{\mathrm{VaR}_\alpha(Z)}^{+\infty} S_Z(x)\mathrm{d}x. \end{aligned} \tag{3.18}$$

For any $f \in \mathscr{H}$, by Lemma 2.1(f), the increasing property, and the Lipschitz continuity of $f(x)$, it is not hard to verify that

$$\begin{aligned} [R_f(X) - \mathrm{VaR}_\alpha(R_f(X)]_+ &= [X - \mathrm{VaR}_\alpha(X) - (f(X) - f(\mathrm{VaR}_\alpha(X)))]_+ \\ &= [X - \mathrm{VaR}_\alpha(X)]_+ - [f(X) - f(\mathrm{VaR}_\alpha(X))]_+. \end{aligned} \tag{3.19}$$

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By Parts (d) and (f) of Lemma 2.1 and (3.18), we know that for any $f \in \mathscr{H}$,

$$\begin{aligned} \mathrm{CVaR}_\alpha[T_f(X)] = \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X)) + (1+\rho)\mathbb{E}[f(X)] \\ + \frac{1}{\alpha} \cdot \mathbb{E}[[R_f(X) - \mathrm{VaR}_\alpha(R_f(X)]_+]. \end{aligned} \tag{3.20}$$

By (3.19) we have that

$$\begin{aligned} \mathbb{E}[[R_f(X) - \mathrm{VaR}_\alpha(R_f(X)]_+] &= \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] - \mathbb{E}[[f(X) - f(\mathrm{VaR}_\alpha(X))]_+] \\ &= \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] - \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} f(x)\mathrm{d}F_X(x) \\ &\quad + \alpha f(\mathrm{VaR}_\alpha(X)). \end{aligned} \tag{3.21}$$

Hence, from (3.20) and (3.21) it follows that

$$\begin{aligned} \mathrm{CVaR}_\alpha[T_f(X)] = \mathrm{VaR}_\alpha(X) + \frac{1}{\alpha} \cdot \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] + (1+\rho)\mathbb{E}[f(X)] \\ - \frac{1}{\alpha} \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} f(x)\mathrm{d}F_X(x), \end{aligned} \tag{3.22}$$

or equivalently,

$$\begin{aligned} \mathrm{CVaR}_\alpha[T_f(X)] = \mathrm{VaR}_\alpha(X) + \frac{1}{\alpha} \cdot \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] + (1+\rho) \int_0^{\mathrm{VaR}_\alpha(X)} f(x)\mathrm{d}F_X(x) \\ + \delta \cdot \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} f(x)\mathrm{d}F_X(x), \end{aligned} \tag{3.23}$$

where $\delta := 1 + \rho - \dfrac{1}{\alpha}$.

The next lemma is crucial in deriving the optimal reinsurance treaty for the optimal reinsurance model (2.3). Its proof is postponed to the appendix.

**Lemma 3.6.** *Let the confidence level $\alpha \in (0, 1 - p_0)$. For any $f \in \mathscr{H}$, there exists a function $h_f \in \mathscr{H}_2$ such that*

$$\mathrm{CVaR}_\alpha[T_{h_f}(X)] \le \mathrm{CVaR}_\alpha[T_f(X)]. \tag{3.24}$$

By Lemma 3.6, the optimization model (2.3) can be equivalently translated into

$$\mathrm{CVaR}_\alpha[T_{f^*}(X)] = \min_{f \in \mathscr{H}_2} \mathrm{CVaR}_\alpha[T_f(X)]. \tag{3.25}$$

For any $g(x) := g(x;\ a,\ b) \in \mathscr{H}_2$ with $(a,\ b) \in \mathscr{D}_2$, from (3.23) and an elementary calculation it follows that

$$\begin{aligned} \mathrm{CVaR}_\alpha[T_g(X)] = \mathrm{VaR}_\alpha(X) + \frac{1}{\alpha} \cdot \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] + (1+\rho) \int_0^{\mathrm{VaR}_\alpha(X)} g(x;\ a,\ b)\mathrm{d}F_X(x) \\ + \delta \cdot \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} g(x;\ a,\ b)\mathrm{d}F_X(x) \end{aligned} \tag{3.26}$$

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$$= \frac{1}{\alpha} \cdot \mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] + a + (1+\rho) \int_a^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x$$
$$+ \delta \cdot \int_{\mathrm{VaR}_\alpha(X)}^{b} S_X(x)\mathrm{d}x \tag{3.27}$$

$$:= \phi(a,\ b). \tag{3.28}$$

For any $a \in \mathscr{A}_2$, taking the partial derivative of $\phi(a,\ b)$ with respect to $b \in [\mathrm{VaR}_\alpha\ (X),\ \beta(a)]$ yields

$$\frac{\partial \phi(a,\ b)}{\partial b} = \delta \cdot S_X(b) < 0. \tag{3.29}$$

The following theorem is the main result of this subsection; it provides the optimal reinsurance treaties for the model (2.3). Its proof is postponed to the appendix. Recall that $\rho^* := \frac{1}{1+\rho}$.

**Theorem 3.2.** *Let the confidence level* $\alpha \in (0,\ 1 - p_0)$*, and let* $\beta(a)$ *and* $\hat{a}$ *be defined as in (3.9) and (3.14), respectively.*

*(i) If* $\alpha > \rho^*$*, then*

$$\min_{f \in \mathscr{H}} \mathrm{CVaR}_\alpha[T_f(X)] = \mathrm{VaR}_\alpha(X) + \frac{1}{\alpha}\mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] = \mathrm{CVaR}_\alpha(X),$$

*and the minimum of* $\mathrm{CVaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = 0$.

*(ii) If* $\alpha = \rho^*$*, then*

$$\min_{f \in \mathscr{H}} \mathrm{CVaR}_\alpha[T_f(X)] = \mathrm{VaR}_\alpha(X) + \frac{1}{\alpha}\mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] = \mathrm{CVaR}_\alpha(X),$$

*and the minimum of* $\mathrm{CVaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = (x - \mathrm{VaR}_\alpha(X))_+ - (x - b)_+$*, where* $b$ *is any real number satisfying* $\mathrm{VaR}_\alpha(X) \le b \le \beta(\mathrm{VaR}_\alpha(X))$.

*(iii) If* $\alpha < \rho^*$ *and* $\mathscr{S} = \emptyset$*, then*

$$\min_{f \in \mathscr{H}} \mathrm{CVaR}_\alpha[T_f(X)] = \frac{1}{\alpha}\mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] + a_0$$
$$+ (1+\rho) \int_{a_0}^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x + \delta \int_{\mathrm{VaR}_\alpha(X)}^{a_0+L} S_X(x)\mathrm{d}x,$$

*and the minimum of* $\mathrm{CVaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = (x - a_0)_+ - (x - a_0 - L)_+$*, where* $a_0$ *is the unique solution to the equation*

$$1 - (1+\rho)S_X(a) + \delta S_X(a + L) = 0$$

*with respect to* $a \in [0,\ \mathrm{VaR}_\alpha(X)]$*, satisfying* $\max\{\mathrm{VaR}_{\rho^*}(X),\ \mathrm{VaR}_\alpha(X) - L\} < a_0 < \mathrm{VaR}_\alpha(X)$.

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*(iv) If* $\alpha < \rho^*$, $\mathscr{S} \neq \emptyset$ *and* $\hat{a} = \mathrm{VaR}_\alpha(X)$, *then*

$$\begin{aligned}\min_{f\in\mathscr{H}} \mathrm{CVaR}_\alpha[T_f(X)] &= \frac{1}{\alpha}\mathbb{E}[(X-\mathrm{VaR}_\alpha(X))_+] + \mathrm{VaR}_\alpha(X) + \delta\int_{\mathrm{VaR}_\alpha(X)}^{\beta(\mathrm{VaR}_\alpha(X))} S_X(x)\mathrm{d}x \\ &= \mathrm{CVaR}_\alpha(X) + \delta\int_{\mathrm{VaR}_\alpha(X)}^{\beta(\mathrm{VaR}_\alpha(X))} S_X(x)\mathrm{d}x,\end{aligned}$$

*and the minimum of* $\mathrm{CVaR}_\alpha[T_f(X)]$ *is attained at* $f^*(x) = (x - \mathrm{VaR}_\alpha(X))_+ - (x - \beta(\mathrm{VaR}_\alpha(X)))_+$.

**Remark 3.2.**

(1) Since $X$ is of finite expectation, if the limit $M$ is large enough so that $(1+\rho)\mathbb{E}(X) \leq M$, then $\beta(a) = a + L$ for all $a \in [0, \mathrm{VaR}_\alpha(X)]$. Hence by Lemma 3.3, $\mathscr{S} = \emptyset$; thus Theorem 3.2 recovers Theorem 4.2 of Lu *et al.* [34]. In general, Theorem 3.2(iv) suggests that in the presence of premium budget constraints, one more case needs to be discussed than in Theorem 4.2 of Lu *et al.* [34]. Example 3.1 below will show that this case, corresponding to Theorem 3.2(iv), does exist.

(2) Parts (ii) and (iv) of Theorem 3.2 provide an interesting interpretation of the limit $M$ in terms of insurance economics. For any $x \geq 0$ and any $b$ satisfying $\mathrm{VaR}_\alpha(X) \leq b \leq \beta(\mathrm{VaR}_\alpha(X))$, we have that

$$(x - \mathrm{VaR}_\alpha(X))_+ - (x-b)_+ \leq (x - \mathrm{VaR}_\alpha(X))_+ - (x - \beta(\mathrm{VaR}_\alpha(X)))_+. \tag{3.30}$$

Recall that the formula on the left-hand side of (3.30) is the optimal reinsurance treaty for the case (ii) of Theorem 3.2, while the one on the right-hand side of (3.30) is the optimal reinsurance treaty for the case (iv) of Theorem 3.2. Therefore, the limit $M$ influences the maximum of the optimal ceded loss function. Note that the less the limit $M$ is, the less $\beta(\mathrm{VaR}_\alpha(X))$ is. In other words, when the insurer would like to pay less reinsurance premium, the insurer would also be willing to accept a lower maximum for the ceded loss, which is fair for the reinsurer.

(3) Theorem 3.2(iii) provides an interesting interpretation of the limit $L$ in terms of insurance economics. From Theorem 3.2(iii), we know that when $L \leq \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_{\rho^*}(X)$, we have $\mathrm{VaR}_{\rho^*}(X) \leq \mathrm{VaR}_\alpha(X) - L$, and thus

$$\mathrm{VaR}_\alpha(X) - L < a_0 < \mathrm{VaR}_\alpha(X),$$

where $a_0$ is the deductible of the optimal reinsurance treaty with lower bound $\mathrm{VaR}_\alpha(X) - L$. Note that $\mathrm{VaR}_\alpha(X) - L$ is decreasing with respect to $L$. Therefore, for given fixed $\alpha$, $\rho$ with $\alpha < \rho^* := \frac{1}{1+\rho}$ and $M$ large enough so that $\mathscr{S} = \emptyset$, the less the tolerance level $L$ is, the less reinsurance the insurer will purchase. This implication coincides with the intuition in practice. Furthermore, when the tolerance level $L$ decreases to zero, the deductible $a_0$ converges to $\mathrm{VaR}_\alpha(X)$, which implies that the optimal reinsurance treaty $f^*(x)$ converges to zero. This observation also coincides with the intuition in practice.

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TABLE 1. Goodness of fit for automobile insurance loss distribution function

| Distribution | *AIC* | *BIC* |
|---|---|---|
| Weibull | 2699.53 | −1397.91 |
| Log-normal | 2614.33 | −1426.25 |
| Pareto | 2903.28 | −1372.70 |

We end this subsection with an example which shows that the case (iv) of Theorem 3.2 does occur.

**Example 3.1.** *Let the random loss X be exponentially distributed with density function $f_X(x)$; that is,*

$$f_X(x) = \begin{cases} 0, & x \le 0, \\ \lambda e^{-\lambda x}, & x > 0, \end{cases}$$

*where $\lambda > 0$ is a constant. Clearly, for any given $\alpha \in (0,\ 1)$,* $\mathrm{VaR}_\alpha(X) = -\frac{\log \alpha}{\lambda}$.

For any given $L > 0$ and $\rho > 0$ satisfying $\alpha < \frac{1}{1+\rho}$,

$$(1+\rho)\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)+L} S_X(x)\mathrm{d}x = (1+\rho)\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)+L} e^{-\lambda x}\mathrm{d}x = \frac{\alpha(1+\rho)}{\lambda}\left[1-e^{-\lambda L}\right].$$

Hence, if the limit $M > 0$ is chosen so that

$$\frac{\alpha(1+\rho)}{\lambda}\left[1-e^{-\lambda L}\right] > M,$$

then $\beta(\mathrm{VaR}_\alpha(X)) < \mathrm{VaR}_\alpha(X) + L$.

Note that for any $a \in [0,\ \mathrm{VaR}_\alpha(X)]$,

$$\begin{aligned}(1+\rho)\int_a^{a+L} S_X(x)\mathrm{d}x &= (1+\rho)\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)+L} S_X(x-\mathrm{VaR}_\alpha(X)+a)\mathrm{d}x \\ &\ge (1+\rho)\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)+L} S_X(x)\mathrm{d}x \\ &> M;\end{aligned}$$

thus, for any $a \in [0,\ \mathrm{VaR}_\alpha(X)]$, $\beta(a) < a + L$. Consequently, $\mathscr{S} \neq \emptyset$ and $\hat{a} = \mathrm{VaR}_\alpha(X)$.

## 4. Numerical study

We devote this section to a numerical analysis based on real data. The data are taken from the public China Insurance Yearbook [23], which records amounts of automobile insurance claims from 87 Chinese insurance companies. The sample mean is about 6668.68 million Chinese yuan. Table 1 shows the goodness of fit of the loss distribution, for three of the most likely possible distributions.

According to the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), from Table 1 we can see that the distribution of claims approximately obeys the log-normal distribution with parameters $\mu$ and $\sigma^2$; that is, the probability density function $p_X$ of

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TABLE 2. $M = 17000$

| $L$ | $a^*$ | $\mathrm{VaR}_{\frac{1}{1+\rho}}(X)$ | $f^*(x)$ | $\min_{f\in\mathcal{H}} \mathrm{VaR}_\alpha\left[T_f(X)\right]$ |
|---|---|---|---|---|
| 13890 | 52.15 | 42.27 | $(x-52.15)_+ - (x-13942.15)_+$ | 3312.07 |
| 13910 | 32.15 | 42.27 | $(x-32.15)_+ - (x-13942.15)_+$ | 3311.93 |

TABLE 3. $L = 13000$

| $M$ | $a^*$ | $\mathrm{VaR}_{\frac{1}{1+\rho}}(X)$ | $f^*(x)$ | $\min_{f\in\mathcal{H}} \mathrm{VaR}_\alpha\left[T_f(X)\right]$ |
|---|---|---|---|---|
| 15000 | 942.15 | 42.27 | $(x-942.15)_+ - (x-13942.15)_+$ | 3638.75 |
| 16000 | 942.15 | 42.27 | $(x-942.15)_+ - (x-13942.15)_+$ | 3638.75 |

TABLE 4. $M = 17000$

| $L$ | $a_0$ | $b$ | $f^*(x)$ | $\min_{f\in\mathcal{H}} \mathrm{CVaR}_\alpha\left[T_f(X)\right]$ |
|---|---|---|---|---|
| 12000 | 4800 | 16800 | $(x-4800)_+ - (x-16800)_+$ | 122505.73 |
| 15000 | 3300 | 18300 | $(x-3300)_+ - (x-18300)_+$ | 120257.88 |

the claim $X$ is given by

$$p_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}, \qquad x > 0,$$

where the estimators $\hat{\mu}$ and $\hat{\sigma}$ for the parameters $\mu$ and $\sigma$ are $\hat{\mu} = 8.5578$ and $\hat{\sigma} = 2.6053$, respectively.

Next, we discuss the impact of the premium budget constraint $M$ and the reinsurer's risk limit $L$ on the optimal reinsurance treaties. For this purpose, we set $\alpha = 0.1$, $\rho = 0.2$. We then know that $\mathrm{VaR}_\alpha(X) = 13942.15$.

By Theorem 3.1, we can obtain the optimal reinsurance treaties for the model (2.2), which are displayed in Tables 2 and 3. From Table 2 we can see that given the premium budget constraint $M$, the reinsurer's risk limit $L$ influences the deductible of the optimal reinsurance treaty. More precisely, the larger $L$ is, the smaller the deductible is. In contrast to this phenomenon, Table 3 suggests that given the reinsurer's risk limit $L$, the premium budget constraint $M$ has almost no influence on the optimal reinsurance treaty.

Using Theorem 3.2, we can obtain the optimal reinsurance treaties for the model (2.3), which are displayed in Tables 4 and 5. From Table 4 we can see that given the premium budget constraint $M$, the reinsurer's risk limit $L$ strongly influences the optimal reinsurance treaty. More precisely, the larger the $L$ is, the smaller the deductible is, and the larger the right endpoint of the layer is as well. In contrast to this phenomenon, Table 5 suggests that given the reinsurer's risk limit $L$, the premium budget constraint $M$ has almost no influence on the optimal reinsurance treaty.

TABLE 5. $L = 13000$

| $M$ | $a_0$ | $b$ | $f^*(x)$ | $\min_{f\in\mathscr{H}} \mathrm{CVaR}_\alpha\big[T_f(X)\big]$ |
|---|---|---|---|---|
| 15000 | 4300 | 17300 | $(x-4300)_+ - (x-17300)_+$ | 121740.86 |
| 16000 | 4300 | 17300 | $(x-4300)_+ - (x-17300)_+$ | 121740.86 |

## 5. Concluding remarks

We propose two new optimal reinsurance models with premium budget constraints and Cummins–Mahul-type reinsurer's risk limits. Under the optimality criteria of minimizing the VaR and CVaR of the insurer's total risk exposure, we derive explicit optimal reinsurance treaties, which are layer reinsurance treaties. A new approach is developed to construct the optimal reinsurance treaties.

We would like to mention that Assa [2] and Zhuang *et al.* [44] developed a powerful tool called the marginal indemnification function formulation to study optimal reinsurance treaties. This paper does not use this powerful tool. Nevertheless, we are not certain whether the marginal indemnification function formulation, together with Lagrangian and convex programming methods, could be applied to the present optimal reinsurance models; it would be interesting to see this worked out in the future.

We would also like to mention that there is an interesting strand of literature on reciprocal reinsurance that takes care of both the insurer and the reinsurer; for instance, see Cai *et al.* [6, 7], Lo and Tang [33], Asimit *et al.* [1], and the references therein, to name just a few. Unlike the above reciprocal reinsurance models, the model in the present paper takes care of both parties by considering the optimal problem from the insurer's point of view and imposing a reinsurer's risk limit from the reinsurer's point of view. It would be interesting to explore whether there is a possibility of merging the two kinds of optimal reinsurance models described above.

## Appendix

In this appendix we provide the proofs of all the main results, as well as the lemmas, in this paper.

*Proof of Lemma* 3.1. Given any $0 \le a_1 < a_2 \le \mathrm{VaR}_\alpha(X)$, we will show that $\beta(a_1) < \beta(a_2)$ by considering two cases.

Case I: Assume that $\beta(a_1) = a_1 + L$. Then by the definition of $\beta(\cdot)$, we know that

$$(1+\rho)\int_{a_1}^{a_1+L} S_X(x)\mathrm{d}x \le M.$$

Hence

$$\begin{aligned}(1+\rho)\int_{a_2}^{a_2+L} S_X(x)\mathrm{d}x &= (1+\rho)\int_{a_1}^{a_1+L} S_X(x+a_2-a_1)\mathrm{d}x \\ &\le (1+\rho)\int_{a_1}^{a_1+L} S_X(x)\mathrm{d}x \\ &\le M,\end{aligned}$$

which results in $\beta(a_2) = a_2 + L > a_1 + L = \beta(a_1)$.

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Case II: Assume that $\beta(a_1) < a_1 + L$. Then by the definition of $\beta(\cdot)$, $\beta(a_1)$ is the unique solution to the equation

$$(1+\rho)\int_{a_1}^{b} S_X(x)\mathrm{d}x = M$$

with respect to $b \in (a_1,\ a_1+L)$; that is,

$$(1+\rho)\int_{a_1}^{\beta(a_1)} S_X(x)\mathrm{d}x = M.$$

We proceed by considering two subcases.

Subcase 1: $\beta(a_1) \le a_2$. Obviously, by the definition of $\beta(\cdot)$, $\beta(a_1) \le a_2 < \beta(a_2)$.

Subcase 2: $\beta(a_1) > a_2$. It is not hard to see that $a_2 < \beta(a_1) < a_1 + L < a_2 + L$, and

$$(1+\rho)\int_{a_2}^{\beta(a_1)} S_X(x)\mathrm{d}x < (1+\rho)\int_{a_1}^{\beta(a_1)} S_X(x)\mathrm{d}x = M.$$

Hence, $\beta(a_1) \in \mathscr{B}(a_2)$, and thus $\beta(a_1) < \beta(a_2)$. Lemma 3.1 is proved. □

*Proof of Lemma* 3.2. Clearly, $\mathrm{VaR}_\alpha(X) \in \mathscr{A}_2$ since $(\mathrm{VaR}_\alpha(X),\ \mathrm{VaR}_\alpha(X)) \in \mathscr{D}_2$.

For all $0 < \varepsilon < \mathrm{VaR}_\alpha(X)$ satisfying $(1+\rho)\varepsilon \le M$,

$$(1+\rho)\int_{\mathrm{VaR}_\alpha(X)-\varepsilon}^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x \le (1+\rho)\varepsilon \le M,$$

which yields that $\mathrm{VaR}_\alpha(X) \in \mathscr{B}(\mathrm{VaR}_\alpha(X)-\varepsilon)$. Hence $\mathrm{VaR}_\alpha(X) \le \beta(\mathrm{VaR}_\alpha(X)-\varepsilon)$. Thus $(\mathrm{VaR}_\alpha(X)-\varepsilon,\ \beta(\mathrm{VaR}_\alpha(X)-\varepsilon)) \in \mathscr{D}_2$, and therefore $\mathrm{VaR}_\alpha(X)-\varepsilon \in \mathscr{A}_2$. We further conclude that $\mathscr{A}_2$ must be an interval with right endpoint $\mathrm{VaR}_\alpha(X)$. For this purpose, it suffices to show that given $a_1 \in \mathscr{A}_2$ with $a_1 < \mathrm{VaR}_\alpha(X)$, we have $a \in \mathscr{A}_2$ for all $a \in (a_1,\ \mathrm{VaR}_\alpha(X))$. Indeed, by Lemma 3.1, $\mathrm{VaR}_\alpha(X) \le \beta(a_1) < \beta(a)$. From (3.12) it follows that

$$(1+\rho)\int_{a}^{\beta(a)} S_X(x)\mathrm{d}x \le M.$$

Therefore, $(a,\ \beta(a)) \in \mathscr{D}_2$. Consequently, $a \in \mathscr{A}_2$.

Next we show that $\underline{a} \in \mathscr{A}_2$. Choose a sequence of $a_n \in \mathscr{A}_2$ with $a_n \le \mathrm{VaR}_\alpha(X)$ such that $a_n$ decreases to $\underline{a}$; that is, for all $n \ge 1$,

$$\mathrm{VaR}_\alpha(X) \le \beta(a_n) \le a_n + L \le \mathrm{VaR}_\alpha(X) + L$$

and

$$(1+\rho)\int_{a_n}^{\beta(a_n)} S_X(x)\mathrm{d}x \le M.$$

Then, by Lemma 3.1, we know that $\Delta := \lim_{n\to+\infty} \beta(a_n)$ exists,

$$\max\{\mathrm{VaR}_\alpha(X),\ \beta(\underline{a})\} \le \Delta \le \underline{a} + L,$$

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and

$$(1+\rho)\int_{\underline{a}}^{\Delta} S_X(x)\mathrm{d}x \le M.$$

Together with the definition of $\beta(\cdot)$, this implies that $\Delta = \beta(\underline{a}) \ge \mathrm{VaR}_\alpha(X)$. Hence $(\underline{a},\ \beta(\underline{a})) \in \mathscr{D}_2$, and therefore $\underline{a} \in \mathscr{A}_2$. Consequently, $\mathscr{A}_2 = [\underline{a},\ \mathrm{VaR}_\alpha(X)]$. Since $\underline{a} \le \hat{a} \le \mathrm{VaR}_\alpha(X)$, $\hat{a} \in \mathscr{A}_2$. Lemma 3.2 is proved. □

*Proof of Lemma* 3.3. The sufficiency is obvious. We now show the necessity. Assume that $\mathscr{S} = \emptyset$. We will prove the lemma by contradiction. If there exists an $a \in \mathscr{A}_2$ such that $\beta(a) < a + L$, then by the definition of $\beta(a)$, we know that $\mathrm{VaR}_\alpha(X) \le \beta(a)$,

$$(1+\rho)\int_{a}^{\beta(a)} S_X(x)\mathrm{d}x \le M,$$

and for all $b \in (\beta(a),\ a+L]$,

$$(1+\rho)\int_{a}^{b} S_X(x)\mathrm{d}x > M,$$

which also yields that

$$(1+\rho)\int_{a}^{\beta(a)} S_X(x)\mathrm{d}x \ge M,$$

since $b \in (\beta(a),\ a+L]$ was arbitrary.

Therefore,

$$(1+\rho)\int_{a}^{\beta(a)} S_X(x)\mathrm{d}x = M.$$

Consequently, $a \in \mathscr{S}$, which is a contradiction. Lemma 3.3 is proved. □

*Proof of Lemma* 3.4. We will prove the lemma in three steps.

**Step 1:** We claim that given $a \in \mathscr{S}$, for all $a_1 \in \mathscr{A}_2$ satisfying $a_1 \le a$, we have $a_1 \in \mathscr{S}$. Note that

$$\mathrm{VaR}_\alpha(X) \le \beta(a) < a + L, \qquad \mathrm{VaR}_\alpha(X) \le \beta(a_1) \le a_1 + L,$$

and

$$(1+\rho)\int_{a}^{\beta(a)} S_X(x)\mathrm{d}x = M.$$

We have that

$$\begin{aligned}(1+\rho)\int_{a_1}^{a_1+L} S_X(x)\mathrm{d}x &= (1+\rho)\int_{a}^{a+L} S_X(x+a_1-a)\mathrm{d}x \\ &\ge (1+\rho)\int_{a}^{a+L} S_X(x)\mathrm{d}x \\ &> (1+\rho)\int_{a}^{\beta(a)} S_X(x)\mathrm{d}x \\ &= M,\end{aligned}$$

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which implies that $\beta(a_1)$ is the unique solution of the equation

$$(1+\rho)\int_{a_1}^{b} S_X(x)\mathrm{d}x = M$$

with respect to $b \in [\mathrm{VaR}_\alpha(X),\ a_1 + L)$. Therefore, $a_1 \in \mathscr{S}$.

**Step 2:** We further claim that $\mathscr{S}$ must be an interval with left endpoint $\underline{a}$.

Indeed, from Step 1 we know that $\underline{a} \in \mathscr{S}$; that is,

$$\mathrm{VaR}_\alpha(X) \le \beta(\underline{a}) < \underline{a} + L$$

and

$$(1+\rho)\int_{\underline{a}}^{\beta(\underline{a})} S_X(x)\mathrm{d}x = M.$$

Note that for any $\varepsilon \in (0,\ \beta(\underline{a}) - \underline{a})$ and any $\tau$ satisfying $\beta(\underline{a}) < \underline{a} + L - \tau < \underline{a} + L$,

$$\begin{aligned}
&(1+\rho)\int_{\underline{a}+\varepsilon}^{\underline{a}+L-\tau} S_X(x)\mathrm{d}x \\
&= (1+\rho)\int_{\underline{a}+\varepsilon}^{\beta(\underline{a}} S_X(x)\mathrm{d}x + (1+\rho)\int_{\beta(\underline{a})}^{\underline{a}+L-\tau} S_X(x)\mathrm{d}x \\
&= (1+\rho)\int_{\underline{a}}^{\beta(\underline{a})} S_X(x)\mathrm{d}x - (1+\rho)\int_{\underline{a}}^{\underline{a}+\varepsilon} S_X(x)\mathrm{d}x + (1+\rho)\int_{\beta(\underline{a})}^{\underline{a}+L-\tau} S_X(x)\mathrm{d}x \\
&\ge M + (1+\rho)\int_{\beta(\underline{a})}^{\underline{a}+L-\tau} S_X(x)\mathrm{d}x - (1+\rho)\varepsilon.
\end{aligned}$$

We can choose $\tau$ and $\varepsilon$ such that

$$(1+\rho)\int_{\beta(\underline{a})}^{\underline{a}+L-\tau} S_X(x)\mathrm{d}x > (1+\rho)\varepsilon.$$

Therefore,

$$(1+\rho)\int_{\underline{a}+\varepsilon}^{\underline{a}+\varepsilon+L} S_X(x)\mathrm{d}x > (1+\rho)\int_{\underline{a}+\varepsilon}^{\underline{a}+L-\tau} S_X(x)\mathrm{d}x > M,$$

which yields that $\beta(\underline{a}+\varepsilon)$ is the unique solution to the equation

$$(1+\rho)\int_{\underline{a}+\varepsilon}^{b} S_X(x)\mathrm{d}x = M$$

with respect to $b \in (\underline{a}+\varepsilon,\ \underline{a}+\varepsilon+L)$. By Lemmas 3.1 and 3.3, we also know that

$$\mathrm{VaR}_\alpha(X) \le \beta(\underline{a}) < \beta(\underline{a}+\varepsilon) < \underline{a}+\varepsilon+L.$$

Therefore, $\underline{a}+\varepsilon \in \mathscr{S}$. Consequently, taking into account the claim shown in Step 1, we know that $\mathscr{S}$ must be an interval with left endpoint $\underline{a}$.

**Step 3:** From the previous two steps, we know that $[\underline{a},\ \hat{a}) \subseteq \mathscr{S} \subseteq [\underline{a},\ \hat{a}]$. Moreover, by the definition of $\mathscr{S}$, we have that $\mathscr{S} = [\underline{a},\ \hat{a})$ if $\beta(\hat{a}) = \hat{a} + L$, and $\mathscr{S} = [\underline{a},\ \hat{a}]$ if $\beta(\hat{a}) < \hat{a} + L$. Lemma 3.4 is proved. □

*Proof of Lemma* 3.5. Note that by Parts (d) and (f) of Lemma 2.1, for any $h \in \mathscr{H}$,

$$\mathrm{VaR}_\alpha[T_h(X)] = \mathrm{VaR}_\alpha(X) - h(\mathrm{VaR}_\alpha(X)) + (1+\rho)\mathbb{E}[h(X)]. \tag{A.1}$$

Given $f \in \mathscr{H}$, we define a function $g(x)$ by

$$g(x) := g(x;\kappa) := (x-\kappa)_+ - (x - \mathrm{VaR}_\alpha(X))_+, \quad x \geq 0,$$

where $\kappa := \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X))$. Obviously, $g(\mathrm{VaR}_\alpha(X)) = f(\mathrm{VaR}_\alpha(X))$.

We further claim that

$$g(x) \leq f(x), \qquad x \geq 0.$$

To prove this, we consider three possibilities for $x \geq 0$. First, when $x \in [0, \kappa)$, $g(x) = 0 \leq f(x)$. Second, when $x \in [\kappa, \mathrm{VaR}_\alpha(X))$, (2.1) implies that $f(\mathrm{VaR}_\alpha(X)) - f(x) \leq \mathrm{VaR}_\alpha(X) - x$. Hence,

$$g(x) = x - \mathrm{VaR}_\alpha(X) + f(\mathrm{VaR}_\alpha(X)) \leq f(x).$$

Third, when $x \in [\mathrm{VaR}_\alpha(X), +\infty)$, $g(x) = f(\mathrm{VaR}_\alpha(X))$. The fact that $f(x)$ is increasing implies that

$$g(x) = f(\mathrm{VaR}_\alpha(X)) \leq f(x).$$

In summary, for any $x \in [0, +\infty)$, $g(x) \leq f(x)$. Therefore,

$$(1+\rho)\int_\kappa^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x = (1+\rho)\mathbb{E}[g(X)] \leq (1+\rho)\mathbb{E}[f(X)] \leq M, \tag{A.2}$$

since $f \in \mathscr{H}$. Consequently, $\kappa \in \mathscr{D}_1$ and $g \in \mathscr{H}_1$. Let $h_f := g$; then (3.16) follows from (A.1) and (A.2). Lemma 3.5 is proved. □

*Proof of Theorem* 3.1. For any given $g(x;a) \in \mathscr{H}_1$, by (A.1), we know that

$$\begin{aligned}\mathrm{VaR}_\alpha[T_g(X)] &= \mathrm{VaR}_\alpha(X) - g(\mathrm{VaR}_\alpha(X);a) + (1+\rho)\mathbb{E}[g(X;a)] \\ &= a + (1+\rho)\int_a^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x \\ &:= \varphi(a).\end{aligned} \tag{A.3}$$

For any $a \in \mathscr{D}_1$, taking the first-order derivative of $\varphi(a)$ yields that

$$\varphi'(a) = 1 - (1+\rho)S_X(a). \tag{A.4}$$

Hence

$$\varphi'(a) \lesseqqgtr 0 \Leftrightarrow a \lesseqqgtr \mathrm{VaR}_{\frac{1}{1+\rho}}(X). \tag{A.5}$$

(i) Assume that $\alpha \geq \frac{1}{1+\rho}$; then $\mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$. Hence, by (A.3) and (A.5), the minimum of $\mathrm{VaR}_\alpha[T_g(X)]$ is attained at $a = \mathrm{VaR}_\alpha(X)$, which implies that $f^*(x) := g(x;\mathrm{VaR}_\alpha(X)) = 0$.

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(ii) Assume that $\alpha < \frac{1}{1+\rho}$ and $a^* < \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$; then $\mathrm{VaR}_\alpha(X) > \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$. Hence, by (A.3) and (A.5), the minimum of $\mathrm{VaR}_\alpha[T_g(X)]$ is attained at $a = \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$, which implies that

$$f^*(x) := g\left(x;\, \mathrm{VaR}_{\frac{1}{1+\rho}}(X)\right) = \left(x - \mathrm{VaR}_{\frac{1}{1+\rho}}(X)\right)_+ - (x - \mathrm{VaR}_\alpha(X))_+.$$

(iii) Assume that $\alpha < \frac{1}{1+\rho}$ and $a^* \geq \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$; then $\mathrm{VaR}_\alpha(X) > \mathrm{VaR}_{\frac{1}{1+\rho}}(X)$. Hence, by (A.3) and (A.5), the minimum of $\mathrm{VaR}_\alpha[T_g(X)]$ is attained at $a = a^*$, which implies that $f^*(x) := g(x;\, a^*) = (x - a^*)_+ - (x - \mathrm{VaR}_\alpha(X))_+$. Theorem 3.1 is proved. □

*Proof of Lemma* 3.6. If $\delta \geq 0$, given any $f \in \mathscr{H}$, let $h_f(x) := g(x;\ a,\ a)$ with $a \in [0,\ \mathrm{VaR}_\alpha(X)]$. Then $h_f \in \mathscr{H}_2$, $h_f(x) = 0$ for all $x \geq 0$, and thus (3.23) implies (3.24).

If $\delta < 0$, given any $f \in \mathscr{H}$, let $\kappa := \mathrm{VaR}_\alpha(X) - f(\mathrm{VaR}_\alpha(X))$. Then $\kappa \leq \mathrm{VaR}_\alpha(X) \leq \kappa + L$. Let $\beta(\kappa)$ be defined by (3.9). We will prove Lemma 3.6 by considering the following two exclusive cases separately.

Case one: Assume that $\beta(\kappa) = \kappa + L$. Clearly, $(\kappa,\ \kappa + L) \in \mathscr{D}_2$. Let $h_f(x) := g(x;\ \kappa,\ \kappa + L) := (x - \kappa)_+ - (x - \kappa - L)_+$; then $h_f \in \mathscr{H}_2$. We further claim that

$$h_f(x) \leq f(x) \quad \text{for} \quad x \leq \mathrm{VaR}_\alpha(X) \tag{A.6}$$

and

$$h_f(x) \geq f(x) \quad \text{for} \quad x \geq \mathrm{VaR}_\alpha(X). \tag{A.7}$$

To prove the claim, we consider four possibilities for $x \geq 0$. First, when $x \in [0, \kappa)$, $h_f(x) = 0 \leq f(x)$. Second, when $x \in [\kappa, \mathrm{VaR}_\alpha(X))$, (2.1) implies that $f(\mathrm{VaR}_\alpha(X)) - f(x) \leq \mathrm{VaR}_\alpha(X) - x$. Thus,

$$h_f(x) = x - \mathrm{VaR}_\alpha(X) + f(\mathrm{VaR}_\alpha(X)) \leq f(x).$$

Third, when $x \in [\mathrm{VaR}_\alpha(X), \kappa + L)$, (2.1) implies that $f(x) - f(\mathrm{VaR}_\alpha(X)) \leq x - \mathrm{VaR}_\alpha(X)$. Thus,

$$h_f(x) = x - \mathrm{VaR}_\alpha(X) + f(\mathrm{VaR}_\alpha(X)) \geq f(x).$$

Fourth, when $x \in [\kappa + L, +\infty)$, $h_f(x) = L \geq f(x)$. Consequently, from (3.23), (A.6), and (A.7), it follows that (3.24) holds for the function $h_f$.

Case two: Assume that $\beta(\kappa)$ is the unique solution to the equation $(1 + \rho)\int_\kappa^b S_X(x)\mathrm{d}x = M$ with respect to $b \in (\kappa,\ \kappa + L)$. Then $\beta(\kappa) < \kappa + L$ and

$$(1 + \rho)\int_\kappa^{\beta(\kappa)} S_X(x)\mathrm{d}x = M,$$

which yields that

$$(1 + \rho)\int_\kappa^{\kappa + L} S_X(x)\mathrm{d}x > (1 + \rho)\int_\kappa^{\beta(\kappa)} S_X(x)\mathrm{d}x = M \geq (1 + \rho)\mathbb{E}[f(X)], \tag{A.8}$$

since $f \in \mathscr{H}$ and $S_X(x)$ is strictly decreasing on $(0,\ +\infty)$.

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For any $a \in [\kappa,\ \mathrm{VaR}_\alpha(X)]$, $b \in [\mathrm{VaR}_\alpha(X),\ \kappa + L]$, we define a function $h(a,\ b)$ by

$$h(a,\ b) := (1+\rho)\int_a^b S_X(x)\mathrm{d}x.$$

Keeping (A.8) in mind, it is easy to see that

$$\lim_{\substack{a\downarrow\kappa \\ b\uparrow\kappa+L}} h(a,\ b) = (1+\rho)\int_\kappa^{\kappa+L} S_X(x)\mathrm{d}x > (1+\rho)\mathbb{E}[f(X)]$$

and

$$\lim_{\substack{a\uparrow\mathrm{VaR}_\alpha(X) \\ b\downarrow\mathrm{VaR}_\alpha(X)}} h(a,\ b) = (1+\rho)\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x = 0 \le (1+\rho)\mathbb{E}[f(X)].$$

Hence, by the intermediate value theorem for the continuous function $h(a,\ b)$ with respect to $a$ and $b$, there exist $\kappa \le \tilde{a} \le \mathrm{VaR}_\alpha(X)$, $\mathrm{VaR}_\alpha(X) \le \tilde{b} \le \kappa + L$ such that

$$(1+\rho)\int_{\tilde{a}}^{\tilde{b}} S_X(x)\mathrm{d}x = h(\tilde{a},\ \tilde{b}) = (1+\rho)\mathbb{E}[f(X)] \le M, \tag{A.9}$$

which also implies that $\tilde{a} \le \tilde{b} \le \beta(\tilde{a})$, and thus that $(\tilde{a},\ \tilde{b}) \in \mathscr{D}_2$. Let $h_f(x) := g(x;\ \tilde{a},\ \tilde{b}) := (x-\tilde{a})_+ - (x-\tilde{b})_+$; then $h_f \in \mathscr{H}_2$. Moreover, from (A.9) it follows that

$$\mathbb{E}[h_f(X)] = \mathbb{E}[g(X;\ \tilde{a},\ \tilde{b})] = \int_{\tilde{a}}^{\tilde{b}} S_X(x)\mathrm{d}x = \mathbb{E}[f(X)]. \tag{A.10}$$

We further conclude that for $0 \le x \le \mathrm{VaR}_\alpha(X)$,

$$h_f(x) = g(x;\ \tilde{a},\ \tilde{b}) \le f(x). \tag{A.11}$$

In fact, when $x \in [0,\ \tilde{a}]$, $h_f(x) = 0 \le f(x)$. When $x \in [\tilde{a},\ \mathrm{VaR}_\alpha(X)]$, (2.1) results in

$$h_f(x) = x - \tilde{a} \le x - \kappa = x - \mathrm{VaR}_\alpha(X) + f(\mathrm{VaR}_\alpha(X)) \le f(x).$$

By (A.11),

$$\int_0^{\mathrm{VaR}_\alpha(X)} h_f(x)\mathrm{d}F_X(x) \le \int_0^{\mathrm{VaR}_\alpha(X)} f(x)\mathrm{d}F_X(x). \tag{A.12}$$

Taking into account the facts that

$$\mathbb{E}[h_f(X)] = \int_0^{\mathrm{VaR}_\alpha(X)} h_f(x)\mathrm{d}F_X(x) + \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} h_f(x)\mathrm{d}F_X(x)$$

and

$$\mathbb{E}[f(X))] = \int_0^{\mathrm{VaR}_\alpha(X)} f(x)\mathrm{d}F_X(x) + \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} f(x)\mathrm{d}F_X(x),$$

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from (A.10) and (A.12) it follows that

$$\int_{\mathrm{VaR}_\alpha(X)}^{+\infty} h_f(x)\mathrm{d}F_X(x) \geq \int_{\mathrm{VaR}_\alpha(X)}^{+\infty} f(x)\mathrm{d}F_X(x),$$

which, together with (3.23) and (A.12), implies that (3.24) holds for $h_f(x) := g(x;\ \tilde{a},\ \tilde{b})$. Lemma 3.6 is proved. □

*Proof of Theorem* 3.2.

(i) If $\alpha > \rho^*$, then $\delta > 0$. By (3.26) we know that $\mathrm{CVaR}_\alpha[T_f(X)]$ attains its minimum at $f^*(x) := g(x;\ a,\ a)$, for $a \in [0,\ \mathrm{VaR}_\alpha(X)]$, if and only if $f^*(x) = 0$, which implies the desired result.

(ii) If $\alpha = \rho^*$, then $\delta = 0$. By (3.26) we know that $\mathrm{CVaR}_\alpha[T_f(X)]$ attains its minimum at $f^*(x) := g(x;\ a,\ b)$, for $(a,\ b) \in \mathscr{D}_2$, if and only if $f^*(x) = 0$ for any $x \in [0,\ \mathrm{VaR}_\alpha(X)]$. Therefore, $f^*(x) = (x - \mathrm{VaR}_\alpha(X))_+ - (x - b)_+$, where $b$ is any real number satisfying $\mathrm{VaR}_\alpha(X) \leq b \leq \beta(\mathrm{VaR}_\alpha(X))$.

(iii) If $\alpha < \rho^*$ and $\mathscr{S} = \emptyset$, then $\delta < 0$ and $\mathrm{VaR}_\alpha(X) \leq \beta(a) = a + L$ for all $a \in \mathscr{A}_2$ by Lemma 3.3. Hence, for any $(a,\ b) \in \mathscr{D}_2$, by (3.12) and (3.29) we know that $(a,\ a + L) = (a,\ \beta(a)) \in \mathscr{D}_2$ and $\phi(a,\ b) \geq \phi(a,\ a + L)$. Thus, taking (3.25) and (3.28) into account, the minimum of $\mathrm{CVaR}_\alpha[T_f(X)]$ over $\mathscr{H}$ must be attained at $(a,\ b) \in \mathscr{D}_2$ with $b = a + L$. Therefore, it is sufficient for us to solve the following optimization problem:

$$\min_{a \in \mathscr{A}_2} \psi(a), \tag{A.13}$$

where $\psi(a)$ is defined by

$$\begin{aligned}\psi(a) &:= \phi(a,\ a + L)\\ &= \frac{1}{\alpha}\mathbb{E}[(X - \mathrm{VaR}_\alpha(X))_+] + a + (1 + \rho)\int_a^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x + \delta\int_{\mathrm{VaR}_\alpha(X)}^{a+L} S_X(x)\mathrm{d}x.\end{aligned} \tag{A.14}$$

The first- and second-order derivatives of $\psi(a)$ are given respectively by

$$\psi'(a) = 1 - (1 + \rho)S_X(a) + \delta S_X(a + L)$$

and

$$\psi''(a) = (1 + \rho)f_X(a) - \delta f_X(a + L) > 0,$$

which implies that $\psi(a)$ is strictly convex on $[0,\ \mathrm{VaR}_\alpha(X)]$. Moreover,

$$\begin{aligned}\psi'(\mathrm{VaR}_\alpha(X)) &= 1 - (1 + \rho)S_X(\mathrm{VaR}_\alpha(X)) + \delta S_X(\mathrm{VaR}_\alpha(X) + L)\\ &> 1 - (1 + \rho)\alpha + \delta S_X(\mathrm{VaR}_\alpha(X))\\ &= 0,\end{aligned} \tag{A.15}$$

$$\begin{aligned}\psi'(\mathrm{VaR}_{\rho^*}(X)) &= 1 - (1 + \rho)S_X(\mathrm{VaR}_{\rho^*}(X)) + \delta S_X(\mathrm{VaR}_{\rho^*}(X) + L)\\ &= 1 - (1 + \rho)\rho^* + \delta S_X(\mathrm{VaR}_{\rho^*}(X) + L)\\ &= \delta S_X(\mathrm{VaR}_{\rho^*}(X) + L)\\ &< 0,\end{aligned} \tag{A.16}$$

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and if $\mathrm{VaR}_\alpha(X) \geq L$, then

$$\begin{aligned}\psi'(\mathrm{VaR}_\alpha(X)-L) &= 1-(1+\rho)S_X(\mathrm{VaR}_\alpha(X)-L)+\delta S_X(\mathrm{VaR}_\alpha(X))\\ &= 1-(1+\rho)S_X(\mathrm{VaR}_\alpha(X)-L)+\delta\alpha\\ &< 1-(1+\rho)S_X(\mathrm{VaR}_\alpha(X))+\delta\alpha\\ &= 0. \end{aligned} \tag{A.17}$$

Therefore, there exists a unique solution, denoted by $a_0$, to the equation $\psi'(a)=0$ with respect to $a \in [0,\ \mathrm{VaR}_\alpha(X)]$. We further conclude that

$$\max\{\mathrm{VaR}_{\rho^*}(X),\ \mathrm{VaR}_\alpha(X)-L\} < a_0 < \mathrm{VaR}_\alpha(X), \tag{A.18}$$

which also implies that $(a_0,\ a_0+L)=(a_0,\ \beta(a_0)) \in \mathscr{D}_2$. In fact, if $\mathrm{VaR}_{\rho^*}(X) > [\mathrm{VaR}_\alpha(X)-L]_+$, then $\mathrm{VaR}_{\rho^*}(X) > \mathrm{VaR}_\alpha(X)-L$, and thus $\mathrm{VaR}_{\rho^*}(X) < a_0 < \mathrm{VaR}_\alpha(X)$. If $\mathrm{VaR}_{\rho^*}(X) \leq [\mathrm{VaR}_\alpha(X)-L]_+$, then $\mathrm{VaR}_{\rho^*}(X) \leq \mathrm{VaR}_\alpha(X)-L$, and thus $\mathrm{VaR}_\alpha(X)-L < a_0 < \mathrm{VaR}_\alpha(X)$.

By (A.18) and the strict convexity of $\psi(a)$, we know that $a_0$ is the unique optimal solution to the optimization problem (A.13). Consequently, $\mathrm{CVaR}_\alpha[T_f(X)]$ attains its minimum at $f^*(x) := g(x;\, a_0,\ a_0+L) = (x-a_0)_+ - (x-a_0-L)_+$.

(iv) If $\alpha < \rho^*$, $\mathscr{S} \neq \emptyset$, and $\hat{a} = \mathrm{VaR}_\alpha(X)$, then $\delta < 0$. For any $(a,\ b) \in \mathscr{D}_2$, by (3.12), (3.28), and (3.29) we know that $(a,\ \beta(a)) \in \mathscr{D}_2$ and $\phi(a,\ b) \geq \phi(a,\ \beta(a))$. Hence, taking (3.25) and (3.28) into account, the minimum of $\mathrm{CVaR}_\alpha[T_f(X)]$ over $\mathscr{H}$ must be attained at $(a,\ b) \in \mathscr{D}_2$ with $b=\beta(a)$. Therefore, it suffices for us to solve the following optimization problem:

$$\min_{a \in \mathscr{A}_2} \phi(a,\ \beta(a)), \tag{A.19}$$

where $\phi(a,\ \beta(a))$ is given by

$$\phi(a,\ \beta(a)) = \frac{1}{\alpha}\mathbb{E}[(X-\mathrm{VaR}_\alpha(X))_+] + a + (1+\rho)\int_a^{\mathrm{VaR}_\alpha(X)} S_X(x)\mathrm{d}x + \delta\int_{\mathrm{VaR}_\alpha(X)}^{\beta(a)} S_X(x)\mathrm{d}x. \tag{A.20}$$

Next, using Lemma 3.4, we will prove the desired result by considering two exclusive cases.

Case I: Assume that $\mathscr{S} = [\underline{a},\ \hat{a}]$. That is, $\mathscr{S} = [\underline{a},\ \mathrm{VaR}_\alpha(X)] = \mathscr{A}_2$. Then $\mathrm{VaR}_\alpha(X) \leq \beta(a) < a+L$ for all $a \in \mathscr{A}_2$.

Keeping (3.11) in mind, the first-order derivative of $\phi(a,\ \beta(a))$ with respect to $a \in \mathscr{A}_2$ is given by

$$\begin{aligned}\frac{\mathrm{d}\phi(a,\ \beta(a))}{\mathrm{d}a} &= 1-(1+\rho)S_X(a)+\delta\cdot S_X(\beta(a))\cdot\beta'(a)\\ &= 1-(1+\rho)S_X(a)+\delta\cdot S_X(a)\\ &= 1-\frac{1}{\alpha}S_X(a).\end{aligned}$$

Hence

$$\frac{\mathrm{d}\phi(a,\ \beta(a))}{\mathrm{d}a} \leqq 0 \Leftrightarrow a \leqq \mathrm{VaR}_\alpha(X). \tag{A.21}$$

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Note that $\mathscr{A}_2 = [\underline{a},\ \mathrm{VaR}_\alpha(X)]$, (A.21) implies that $a = \mathrm{VaR}_\alpha(X)$ is the unique optimal solution to the optimization problem (A.19). Consequently, $(\mathrm{VaR}_\alpha(X),\ \beta(\mathrm{VaR}_\alpha(X))) \in \mathscr{D}_2$, and $\mathrm{CVaR}_\alpha[T_f(X)]$ attains its minimum at

$$f^*(x) := g(x;\, \mathrm{VaR}_\alpha(X),\ \beta(\mathrm{VaR}_\alpha(X))) = (x - \mathrm{VaR}_\alpha(X))_+ - (x - \beta(\mathrm{VaR}_\alpha(X)))_+.$$

Case II: Assume that $\mathscr{S} = [\underline{a},\ \hat{a})$. That is, $\mathscr{S} = [\underline{a},\ \mathrm{VaR}_\alpha(X)) \subset \mathscr{A}_2$. Then $\mathrm{VaR}_\alpha(X) < \beta(\mathrm{VaR}_\alpha(X)) = \mathrm{VaR}_\alpha(X) + L$ and $\mathrm{VaR}_\alpha(X) \le \beta(a) < a + L$ for all $a \in [\underline{a},\ \mathrm{VaR}_\alpha(X))$. Note that $\mathscr{A}_2 = [\underline{a},\ \mathrm{VaR}_\alpha(X)]$; therefore,

$$\begin{aligned}\min_{a\in\mathscr{A}_2} \phi(a,\ \beta(a)) &= \min\left\{\min_{a\in[\underline{a},\ \mathrm{VaR}_\alpha(X))} \phi(a,\ \beta(a)),\ \phi(\mathrm{VaR}_\alpha(X),\ \mathrm{VaR}_\alpha(X) + L)\right\}\\ &= \min\left\{\min_{a\in[\underline{a},\ \mathrm{VaR}_\alpha(X))} \phi(a,\ \beta(a)),\ \psi(\mathrm{VaR}_\alpha(X))\right\},\end{aligned} \tag{A.22}$$

where $\psi(a)$ is given by (A.14) and $\phi(a,\ \beta(a))$ is given by (A.20).

By Lemma 3.1, we know that $\beta(\mathrm{VaR}_\alpha(X)-) := \lim_{a\uparrow \mathrm{VaR}_\alpha(X)} \beta(a)$ exists, and that $\mathrm{VaR}_\alpha(X) \le \beta(\mathrm{VaR}_\alpha(X)-) \le \mathrm{VaR}_\alpha(X) + L$. Hence, from (A.21), it follows that

$$\min_{a\in[\underline{a},\ \mathrm{VaR}_\alpha(X))} \phi(a,\ \beta(a)) = \phi(\mathrm{VaR}_\alpha(X),\ \beta(\mathrm{VaR}_\alpha(X)-)),$$

which, together with (A.22) and the fact that $\delta < 0$, implies that

$$\begin{aligned}\min_{a\in\mathscr{A}_2} \phi(a,\ \beta(a)) &= \min\left\{\min_{a\in[\underline{a},\ \mathrm{VaR}_\alpha(X))} \phi(a,\ \beta(a)),\ \psi(\mathrm{VaR}_\alpha(X))\right\}\\ &= \min\{\phi(\mathrm{VaR}_\alpha(X),\ \beta(\mathrm{VaR}_\alpha(X)-)),\ \psi(\mathrm{VaR}_\alpha(X))\}\\ &= \psi(\mathrm{VaR}_\alpha(X)).\end{aligned}$$

Consequently, $(\mathrm{VaR}_\alpha(X),\ \mathrm{VaR}_\alpha(X) + L) = (\mathrm{VaR}_\alpha(X),\ \beta(\mathrm{VaR}_\alpha(X))) \in \mathscr{D}_2$ and $\mathrm{CVaR}_\alpha\ [T_f(X)]$ attains its minimum at $f^*(x) := g(x;\, \mathrm{VaR}_\alpha(X),\ \mathrm{VaR}_\alpha(X) + L) = g(x;\ \mathrm{VaR}_\alpha(X),\ \beta(\mathrm{VaR}_\alpha(X))) = (x - \mathrm{VaR}_\alpha(X))_+ - (x - \beta(\mathrm{VaR}_\alpha(X)))_+$. Theorem 3.2 is proved. □

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## Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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## Data

The data related to the numerical study in Section 4 can be found in the public China Insurance Yearbook [23].

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