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OSCILLATING PROPERTIES OF THE SOLUTIONS OF A CLASS OF NEUTRAL TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

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The present paper deals with some oscillating and asymptotic properties of the functional differential equations of the form

$$x''(t) + \lambda x''(t-\tau) + F\big(t, x(t-\tau), x'(t-\tau)\big) = 0$$

where λ is an arbitrary positive constant and $\tau > 0$ is a constant delay.

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$$x''(t) + \lambda x''(t-\tau) + F\big(t, x(t-\tau), x'(t-\tau)\big) = 0 \tag{1}$$

where λ is an arbitrary positive constant, and $\tau > 0$ is a constant delay. We should point out that the oscillating properties of second order functional differential equations when $\lambda = 0$ have been studied in many papers. Shevelo's monograph [2] contains a detailed bibliography on that subject.

We introduce some definitions.

DEFINITION 1. We shall consider as a solution of equation (1) every function

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$$x(t) = \begin{cases} x^0(t), & t \in [t_0-\tau, t_0], \\ \tilde{x}(t), & t \in [t_0, +\infty), \end{cases}$$

for every $t \geq t_0$, $t_0 \in \mathbb{R}^1$, where $x^0(t) \in C^1\left([t_0-\tau, t_0], \mathbb{R}^1\right)$, $\tilde{x}(t) \in C^2\left([t_0, +\infty), \mathbb{R}^1\right)$ and by $x'(t_0)$ we shall denote a right derivative.

By W we shall denote the set of solutions of equation (1), satisfying the condition $x(t) \not\equiv 0$ in every interval $[\bar{t}, +\infty)$, $\bar{t} \geq t_0$, and we shall assume that $W \neq \emptyset$.

DEFINITION 2. The solution $x(t) \in W$ will be called oscillating if it changes its sign in every interval $[\bar{t}, +\infty)$, $\bar{t} \geq t_0$.

THEOREM 1. *Let the conditions* (A) *be satisfied:*

A1. *the function* $F(t, u, v) : \mathbb{D} \to \mathbb{R}^1$ $\left(\mathbb{D} = [t_0, +\infty) \times \mathbb{R}^2\right)$ *is continuous,* $F(t, 0, 0) \equiv 0$ *for* $t \geq t_0$ *and it satisfies the inequality*

$$F(t, u, v) \operatorname{sign} u \geq p(t)f(u)$$

for all points $(t, u, v) \in \mathbb{D}$;

A2. *the function* $f(u) : \mathbb{R}^1 \to \mathbb{R}^1$ *is continuous,* $uf(u) > 0$ *for* $u \neq 0$ *and* $\inf|f(u)| > 0$ *for* $|u| \geq \varepsilon > 0$;

A3. *the function* $p(t) : [t_0, +\infty) \to [0, +\infty)$ *is continuous and for every closed set* E *whose intersection with every segment of the form* $[t, t+2\tau]$ $(t_0 \leq t < +\infty)$ *has a measure not smaller that* τ, *the equality*

(2) $$\int_E p(t)dt = +\infty$$

holds.

Then every solution $x(t) \in W$ *will be an oscillating one.*

In order to prove Theorem 1 we need the following

LEMMA 1. *Let* $t_0 \in \mathbb{R}^1$ *be an arbitrary fixed point,* $\lambda > 0$ *be an arbitrary constant and* $\tau > 0$ *be a constant delay. We shall assume that the conditions* (B) *are satisfied:*

B1. *the function* $\varphi(t) : [t_0, +\infty) \to [0, +\infty)$ *is continuous in the interval* $[t_0, +\infty)$;

B2. *for every* $t \geq t_0$ *the function* $\varphi(t) + \lambda\varphi(t-\tau)$ *is monotone increasing and* $\varphi(t) + \lambda\varphi(t-\tau) \geq C$, $C > 0$.

Then for every $t_1 \in [t_0, +\infty)$ *there exists a set*

$$A = \{t \mid t_1 \leq t \leq t_1+2\tau, \lambda\varphi(t-\tau) \geq \beta\}$$

whose measure is not smaller than τ . *Here* $\beta = \min(C/2, \lambda C/2)$.

Proof of Lemma 1. Let $t_1 \in [t_0, +\infty)$ be an arbitrary fixed point and let us consider the set $P = \{t \mid t \in [t_1, t_1+\tau], \varphi(t) > C/2\}$. If we assume that $P = \emptyset$, then from B2 it follows that the inequality $\lambda\varphi(t-\tau) \geq C/2$ will hold for every $t \in [t_1, t_1+\tau]$ and therefore we could set $A = [t_1, t_1+\tau]$.

Let $P \neq \emptyset$ and let us denote by α , $0 < \alpha \leq \tau$, its measure. If we denote by $\overline{P}$ the closure of P , then from B1 it follows that the inequality $\varphi(t) \geq C/2$ will be satisfied for every $t \in \overline{P}$. Let us consider the set $\overline{P} + \tau = \{t \mid t-\tau \in \overline{P}\}$. From B2 it follows that the inequality $\lambda\varphi(t-\tau) \geq \lambda C/2$ will hold for every $t \in \overline{P} + \tau$. Let us set $A = ([t_1, t_1+\tau]\backslash P) \cup (\overline{P}+\tau)$. Since the measure of $[t_1, t_1+\tau]\backslash P$ is $\tau - \alpha$, and the measure of the set $\overline{P} + \tau$ is equal to that of P , then the set A will have a measure τ . From the definition of the sets P and $\overline{P} + \tau$ it follows that the inequality $\lambda\varphi(t-\tau) \geq \beta$ holds for every $t \in A$. Thus Lemma 1 has been proved.

Proof of Theorem 1. Let us assume that there exists a non-oscillating solution $x(t)$ of equation (1) belonging to the set W . Without loss of generality we consider that there exists a point $\bar{t} \geq t_0$ such that

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$x(t) > 0$ and $x(t-\tau) \geq 0$ for every $t \geq \bar{t}$. (The case when there exists a point $\bar{t}$ such that $x(\bar{t}) < 0$ and $x(t-\tau) < 0$ for $t \geq \bar{t}$ follows similarly.)

Let us rewrite equation (1) in the form

$$[x'(t)+\lambda x'(t-\tau)]' = -F(t, x(t-\tau), x'(t-\tau)) . \tag{3}$$

From (3) it follows that the function $x'(t) + \lambda x'(t-\tau)$ is a monotone decreasing one for $t \geq \bar{t}$. If we assume that there exists a point $t_2 \geq \bar{t}$ such that

$$x'(t_2) + \lambda x'(t_2-\tau) = -C_1 < 0$$

then for every point $t \geq t_2$ the inequality

$$x'(t) + \lambda x'(t-\tau) \leq -C_1$$

is satisfied.

Integrating the latter inequality from t_2 to $t > t_2$, we obtain

$$x(t) + \lambda x(t-\tau) \leq x(t_2) + \lambda x(t_2-\tau) - C_1(t-t_2) . \tag{4}$$

After a limit transition $t \to +\infty$ in inequality (4) we come to a contradiction with the assumption that the function $x(t)$ is non-negative for $t \geq \bar{t}$. Therefore for every point $t \geq \bar{t}$ the following inequality

$$x'(t) + \lambda x'(t-\tau) \geq 0 \tag{5}$$

holds. From (3) and (5) it follows that

$$\lim_{t\to+\infty} \int_{\bar{t}}^{t} F(s, x(s-\tau), x'(x-\tau))ds$$
$$= [x'(\bar{t})+\lambda x'(\bar{t}-\tau)]' - \lim_{t\to+\infty} [x'(t)+\lambda x'(t-\tau)] < +\infty . \tag{6}$$

Furthermore from (5) it follows that $x(t) + \lambda x(t-\tau) \geq C_2 > 0$ for $t \geq \bar{t}$. Then according to Lemma 1 the intersection of the set $E = \{t \mid \bar{t} \leq t < +\infty,\ \lambda x(t-\tau) \geq \beta_1\}$, $\beta_1 = \min(C_2/2, \lambda C_2/2)$ with the segment $[s, s+2\tau]$, $s \in [\bar{t}, +\infty)$, will have a measure not smaller than τ and therefore from conditions (A) we shall have

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$$\int_E F\big(t,\ x(t-\tau),\ x'(t-\tau)\big)dt \geq \inf_{u \geq \beta, \lambda^{-1}} f(u) \int_E p(t)dt = +\infty ,$$

which contradicts inequality (6). Thus Theorem 1 has been proved.

DEFINITION 3. The solution $x(t) \in W$ will be called k_x-oscillating [1] if there exists a number $k_x \in \mathbb{R}^1$ such that the function $x(t) - k_x$ changes its sign in every interval $[\bar{t},\ +\infty)$, $\bar{t} \geq t_0$.

THEOREM 2. *Let the conditions* A1 *and* A2 *of Theorem* 1 *be satisfied and let the function* $p(t) : [t_0,\ +\infty) \to [0,\ +\infty)$ *satisfy*

$$\int_{t_0}^{+\infty} p(t)dt = +\infty . \tag{7}$$

Then every solution $x(t) \in W$ *will be* k_x*-oscillating.*

In order to prove Theorem 2 we need the following

LEMMA 2. *Let the conditions of Theorem* 2 *be satisfied. Then all non-oscillating solutions of equation* (1) *belonging to* W *will have the property* $\liminf_{t \to +\infty} |x(t)| = 0$.

Proof of Lemma 2. Let us assume that there exists a point $\bar{t} \geq t_0$ such that $x(t) \geq 0$ for $t \geq \bar{t}$. The case when $x(t) \leq 0$ for $t \geq \bar{t}$ is similar. In the proof of Theorem 1 it has been established that if there exists a point $\bar{t} \geq t_0$ such that $x(t) \geq 0$ for $t \geq \bar{t}$, the following inequality will hold:

$$\int_{\bar{t}}^{+\infty} F\big(t,\ x(t-\tau),\ x'(t-\tau)\big)dt < +\infty . \tag{8}$$

If we assume that $\liminf_{t \to +\infty} x(t) \geq C_3 > 0$ then there exists a point $t_4 \geq \bar{t}$ such that $x(t-\tau) \geq C_3/2$ for $t \geq t_4$. From (8) it follows that

$$\inf_{u \geq C_3/2} f(u) \int_{t_4}^{+\infty} p(t)dt \leq \int_{t_4}^{+\infty} F\big(t,\ x(t-\tau),\ x'(t-\tau)\big)dt < +\infty ,$$

which contradicts equality (7). Thus Lemma 2 has been proved.

Proof of Theorem 2. In the proof of Theorem 1 it has been established that if there exists a point $\bar{t} \geq t_0$ such that for $t \geq \bar{t}$, $x(\bar{t}) > 0$ and $x(t-\tau) \geq 0$, then

(9) $$x'(t) + \lambda x'(t-\tau) \geq 0$$

holds for every $t \geq \bar{t}$.

From (9) it follows that the function $x(t) + \lambda x(t-\tau)$ is a monotone increasing one and that is why two cases are possible:

(a) $\lim\limits_{t\to+\infty} [x(t)+\lambda x(t-\tau)] = +\infty$;

(b) $\lim\limits_{t\to+\infty} [x(t)+\lambda x(t-\tau)] = C_4 < +\infty$;

(the constant C_4 can not be zero because $x(t) \in W$).

First let us consider (a). Then from Lemma 2 it follows that for every number $k \in (0, +\infty)$ there exists a point $t_k \geq \bar{t}$ such that the function $x(t) - k$ changes its sign in every interval $[s, +\infty)$, $s \geq t_k$, and therefore $x(t)$ is k-oscillating for every $k \in (0, +\infty)$.

For (b) we set

$$k = C_3/2 = \tfrac{1}{2} \lim_{t\to+\infty} [x(t)+\lambda x(t-\tau)] ,$$

and from Lemma 2 we obtain that $x(t)$ is k-oscillating for $k = C_3/2$. Thus Theorem 2 has been proved.

Finally we give two examples.

EXAMPLE 1. We consider the equation

(10) $$x''(t) + x''(t-2\pi) + 2x(t-2\pi) = 0 .$$

In this case $F(t, u, v) = 2f(u)$, $f(u) = u$, and $p(t) = 2$. One can immediately verify that the functions $F(t, u, v)$, $f(u)$ and $p(t)$ satisfy the conditions A1-A3 of Theorem 1 and, therefore, all solutions of (10) will be oscillating. (For example, the functions $x(t) = C_1 \cos t + C_2 \sin t$ where C_1 and C_2 are arbitrary constants,

$|C_1| + |C_2| > 0$, define oscillating solutions of (10).

EXAMPLE 2. In the equation

$$x''(t) + x''(t-\tau) + p(t)x^3(t-\tau) = 0 \tag{11}$$

we have $F(t, u, v) = p(t)f(u)$, $f(u) = u^3$, and

$$p(t) = \begin{cases} e^{-t}, & t \in [t_0+2k\tau,\ t_0+(2k+1)\tau], \\ \left(2\left(\tau-e^{-[t_0+(2k+2)\tau]}\right)/\tau\right)[t-(t_0+(2k+1)\tau)] + e^{-[t_0+(2k+1)\tau]}, & \\ \qquad t \in [t_0+(2k+1)\tau,\ t_0+(2k+(3/2))\tau], & \\ \left(2\left(e^{-[t_0+(2k+2)\tau]}-\tau\right)/\tau\right)[t-[t_0+(2k+(3/2))\tau]] + \tau, & \\ \qquad t \in [t_0+(2k+(3/2))\tau,\ t_0+(2k+2)\tau], & \end{cases}$$

$k = 0, 1, 2, \ldots$.

The functions $F(t, u, v)$ and $f(u)$ satisfy the conditions A1 and A2 of Theorem 1, and the function $p(t)$ $(p(t) > 0$ for $t \geq t_0)$ satisfies (7). Then, from Theorem 2, it follows that there exists numbers k, $k \in (-\infty, +\infty)$ such that all the solutions of (11) are k-oscillating.

Note. The function $p(t)$ from (11) is an example of a function satisfying the conditions of Theorem 2 but not satisfying the condition A3 of Theorem 1.

Actually, let us denote by E the set

$$E = \bigcup_{k=0}^{\infty} [t_0+2k\tau,\ t_0+(2k+1)\tau]$$

whose intersection with every interval of the form $[t, t+2\tau]$ has a measure τ; then

$$\int_E p(t)dt = \int_E e^{-t}dt \leq \int_{t_0}^{+\infty} e^{-t} < +\infty .$$

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