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L-invariants for cohomological representations of PGL(2) over arbitrary number fields

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Abstract

Let π be a cuspidal, cohomological automorphic representation of an inner form G of PGL_2 over a number field F of arbitrary signature. Further, let $\mathfrak{p}$ be a prime of F such that G is split at $\mathfrak{p}$ and the local component $\pi_{\mathfrak{p}}$ of π at $\mathfrak{p}$ is the Steinberg representation. Assuming that the representation is noncritical at $\mathfrak{p}$, we construct automorphic $\mathcal{L}$-invariants for the representation π. If the number field F is totally real, we show that these automorphic $\mathcal{L}$-invariants agree with the Fontaine–Mazur $\mathcal{L}$-invariant of the associated p-adic Galois representation. This generalizes a recent result of Spieß respectively Rosso and the first named author from the case of parallel weight 2 to arbitrary cohomological weights.

Contents

Introduction

The purpose of this article is threefold: First, let π be a cuspidal, cohomological automorphic representation of an inner form G of PGL_2 over a number field F of arbitrary signature and $\mathfrak{p}$ be a prime of F such that G is split at $\mathfrak{p}$ and the local component $\pi_{\mathfrak{p}}$ of π at $\mathfrak{p}$ is the Steinberg representation. We want to give a general construction of automorphic $\mathcal{L}$-invariants (also known as Teitelbaum, Darmon or Orton $\mathcal{L}$-invariants) for π. For representations which are cohomological with respect to the trivial coefficient system, or in other words for forms of parallel weight 2, these $\mathcal{L}$-invariants have been defined in general (see, for example, [23]) but for higher weights they have been defined only in certain situations:

1. In case $F = \mathbb{Q}$ and G is compact at infinity by Teitelbaum in [41],
2. In case $F = \mathbb{Q}$ and G is split by Orton in [33],
3. In case $F = \mathbb{Q}$ and G is split at infinity by Rotger and Seveso in [34],
4. In case F is imaginary quadratic and G is split by Barrera-Salazar and Williams in [3] and
5. In case F is totally real and G is compact at infinity by Chida, Mok and Park in [14].

An obstacle that might have prevented the construction in general is the following: Whereas in case of the trivial coefficient system the representation is always ordinary and therefore noncritical at $\mathfrak{p}$, this is no longer true for higher weights. But note that for the cases 1–4 above the representation has still noncritical slope at $\mathfrak{p}$ and, thus, is noncritical at $\mathfrak{p}$ (see the end of Section 2.4 for a detailed discussion on noncritical slopes). It seems to the authors of this paper that in [14] it is implicitly assumed that the representation is noncritical at $\mathfrak{p}$ (see Remark 2.6 for more details). Our first main result (see Definition 3.2) is the construction of automorphic $\mathcal{L}$-invariants under the assumption that the representation π is noncritical at $\mathfrak{p}$. We point out that our construction of $\mathcal{L}$-invariants is novel as it does not involve the Bruhat–Tits tree at any stage.

Our second goal is to bridge the gap between works using overconvergent cohomology à la Ash–Stevens, for example, [3] and [2], and Spieß' more representation-theoretic approach (cf. [39]). In particular, we show that the noncriticality condition for classes in overconvergent cohomology that is discussed in [2] respectively [5] is equivalent to a more representation-theoretic one: Assume for the moment that $\pi_{\mathfrak{p}}$ is not necessarily Steinberg but merely has an Iwahori-fixed vector. We explain that choosing a $\mathfrak{p}$-stabilization of π, that is, an Iwahori-fixed vector of $\pi_{\mathfrak{p}}$, yields a cohomology class of a $\mathfrak{p}$-arithmetic subgroup of $G(F)$ with values in the dual of a locally algebraic principal series representation of $G(F_{\mathfrak{p}})$. Noncriticality is then equivalent to that this class can be lifted uniquely to a class in the cohomology with values in the continuous dual of the corresponding locally analytic principal series representation (see Proposition 2.13). The main tool to prove this equivalence is the resolution of locally analytic principal series representations by Kohlhaase and Schraen (see [31]).

Finally, we show that, if the number field F is totally real the automorphic $\mathcal{L}$-invariants attached to π agree with the derivatives of the $U_{\mathfrak{p}}$-eigenvalue of a p-adic family passing through π (cf. Theorem 4.3). This equality is known in case $F = \mathbb{Q}$ by the work of Bertolini–Darmon–Iovita (see [6]) and Seveso (see [38]). For Hilbert modular forms of parallel weight 2 the equality was recently proven by Rosso and the first named author (see [25]). As we do not work with general reductive groups as in *loc.cit* the arguments simplify substantially, making them more accessible to people who are only interested in Hilbert modular forms. Furthermore, it is known that the derivatives of the $U_{\mathfrak{p}}$-eigenvalue agree with the Fontaine–Mazur $\mathcal{L}$-invariant of the associated Galois representation, if that Galois representation is noncritical. Thus, we deduce the equality of automorphic and Fontaine–Mazur $\mathcal{L}$-invariants (see [40] for an independent proof of this equality in case of parallel weight 2). The equality of $\mathcal{L}$-invariants in parallel weight 2 is necessary for the construction of plectic Stark–Heegner points in recent work of Fornea and the first named author (cf. [20]). This article should be seen as a precursor for defining plectic Stark–Heegner cycles for arbitrary cohomological weights.

Notations

All rings will be commutative and unital. The units of a ring R will be denoted by $R^{\times}$. Given a prime ideal $\mathfrak{m}$ of a ring R and an R-module M, we write $M_{\mathfrak{m}}$ for the localization of M at $\mathfrak{m}$. If R is a ring and G is a group, we denote the group algebra of G over R by $R[G]$. The trivial character of any group will be denoted by $\mathbb{1}$. Given two sets X and Y, we will write $\mathcal{F}(X, Y)$ for the set of all maps from X to Y. If X and Y are topological spaces, we write $C(X, Y) \subseteq \mathcal{F}(X, Y)$ for the set of all continuous maps. If Y is a topological group, we denote by $C_c(X, Y) \subseteq C(X, Y)$ the subset of functions with compact support.

Setup

A number field. We fix an algebraic number field $F \subseteq \mathbb{C}$ with ring of integers $\mathcal{O}_F$. We write S_{∞} for the set of infinite places of F and Σ for the set of all embeddings from F into the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. The action of complex conjugation on Σ will be denoted by c. We write δ for the number of complex places of F. For any place $\mathfrak{q}$ of F, we will denote by $F_{\mathfrak{q}}$ the completion of F at $\mathfrak{q}$. If $\mathfrak{q}$ is a finite place, we let $\mathcal{O}_{\mathfrak{q}}$ denote the valuation ring of $F_{\mathfrak{q}}$ and $\mathrm{ord}_{\mathfrak{q}}$ the additive valuation such that $\mathrm{ord}_{\mathfrak{q}}(\varpi) = 1$ for any

local uniformizer $\varpi \in \mathcal{O}_{\mathfrak{q}}$. We write $\mathcal{N}(\mathfrak{q})$ for the cardinality of the residue field $\mathcal{O}/\mathfrak{q}$. We normalize the $\mathfrak{q}$-adic absolute valuation $| \cdot |_{\mathfrak{q}}$ by $|\varpi|_{\mathfrak{p}} = \mathcal{N}(\mathfrak{q})^{-1}$.

For a finite set S of places of F, we define the 'S-truncated adeles' $\mathbb{A}^S$ as the restricted product of all completions F_v with $v \notin S$. In case S is the empty set, we drop the superscript S from the notation. We will often write $\mathbb{A}^{S,\infty}$ instead of $\mathbb{A}^{S \cup S_\infty}$. If H is an algebraic group over F, we will put $H_{\mathfrak{q}} = H(F_{\mathfrak{q}})$ for any place $\mathfrak{q}$ of F. If S is a finite set of places of F, we will write $H_S = \prod_{\mathfrak{q} \in S} H_{\mathfrak{q}}$. Further, we abbreviate $H_\infty = H_{S_\infty}$.

A quaternion algebra. We fix a quaternion algebra D over F. We denote by $\mathrm{ram}(D)$ the set of places of F at which D is ramified and put

$$\mathrm{disc}(D) = \prod_{\mathfrak{q} \in \mathrm{ram}(D), \mathfrak{q} \nmid \infty} \mathfrak{q}.$$

Let $D^\times$ be the group of units of D considered as an algebraic group over F. The centre $Z \subseteq D^\times$ is naturally isomorphic to the multiplicative group $\mathbb{G}_m$. We put $G = D^\times/Z$. For any place $\mathfrak{q} \notin \mathrm{ram}(D)$, we fix an isomorphism $D_{\mathfrak{q}} \cong M_2(F_{\mathfrak{q}})$ that in turn induces an isomorphism $G_{\mathfrak{q}} \cong \mathrm{PGL}_2(F_{\mathfrak{q}})$. For any Archimedean place $\mathfrak{q} \in \mathrm{ram}(D)$, we fix an isomorphism of $D_{\mathfrak{q}}$ with the Hamilton quaternions, which yields an embedding $G_{\mathfrak{q}} \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$. In particular, we get an injection $j_\sigma \colon G(F) \subseteq G(F_{\mathfrak{q}}) \overset{\sigma_*}{\hookrightarrow} \mathrm{PGL}_2(\mathbb{C})$ for every embedding $\sigma \in \Sigma$ with underlying place $\mathfrak{q}$. We write

$$j \colon G(F) \hookrightarrow \prod_{\sigma \in \Sigma} \mathrm{PGL}_2(\mathbb{C})$$

for the diagonal embedding.

Let $S_\infty(D)$ be the set of all Archimedean places of F at which D is split. We put

$$q = \#S_\infty(D).$$

Let $S_{\mathbb{R}}(D) \subseteq S_\infty(D)$ be the subset of real places. We denote by G_∞^+ the connected component of the identity of G_∞. The group $\mathrm{PGL}_2(\mathbb{C})$ and the units of the Hamilton quaternions are connected, whereas $\mathrm{PGL}_2(\mathbb{R})$ has two connected components. Therefore, we can identify

$$\pi_0(G_\infty) = G_\infty/G_\infty^+ \cong \{\pm 1\}^{S_{\mathbb{R}}(D)}.$$

If $A \subseteq G_\infty$ is a subgroup, we put $A^+ = A \cap G_\infty^+$.

An automorphic representation. Let $\pi' = \otimes_{\mathfrak{q}} \pi'_{\mathfrak{q}}$ be a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ that is cohomological (see Section 1.1.2). If F is totally real, then such automorphic representations (up to twists by the norm character) are in one-to-one correspondence with cuspidal Hilbert modular newforms with even weights and trivial Nebentypus. We assume that the local component $\pi'_{\mathfrak{q}}$ is either a twist of the Steinberg representation or supercuspidal for all primes $\mathfrak{q}$ dividing $\mathrm{disc}(D)$. Thus, there exists a Jacquet–Langlands transfer π of π' to $G(\mathbb{A})$, that is, an automorphic representation of $G(\mathbb{A})$ such that $\pi'_{\mathfrak{q}} \cong \pi_{\mathfrak{q}}$ for all places $\mathfrak{q} \notin \mathrm{ram}(D)$. Moreover, the representation $\pi_{\mathfrak{q}}$ is one-dimensional for all $\mathfrak{q} \mid \mathrm{disc}(D)$ such that $\pi'_{\mathfrak{q}}$ is a twist of the Steinberg representation. We put

$$\pi_\infty := \bigotimes_{v \in S_\infty} \pi_v \quad \text{and} \quad \pi^\infty := \bigotimes_{v \notin S_\infty} \pi_v.$$

1. Cohomology of p-arithmetic groups

We recollect some basic facts about the cohomology of $\mathfrak{p}$-arithmetic groups with values in duals of smooth representations.

1.1. The Eichler–Shimura isomorphism

In this section, we recall how the representation π contributes to the cohomology of the locally symmetric space attached to G.

1.1.1. Weights and coefficient modules

Let $k \geq 0$ be an even integer. For any ring R, we let $V_k(R) \subseteq R[X, Y]$ be the space of homogeneous polynomials of degree k with $\mathrm{PGL}_2(R)$-action given by

$$(g.f)(X, Y) = \det(g)^{-k/2} f(bY + dX, aY + cX) \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(R).$$

We may attach to $f \in V_k(R)$ an R-valued function ψ_f on $\mathrm{PGL}_2(R)$ via

$$\psi_f(g) = (g.f)(1, 0).$$

In case $b = \begin{pmatrix} b_1 & u \\ 0 & b_2 \end{pmatrix}$ is an upper triangular matrix, we have

$$\begin{aligned}
\psi_f(bg) = (bg.f)(1, 0) &= (b.(g.f))(1, 0) \\
&= b_1^{-k/2} b_2^{k/2} \cdot (g.f)(1, 0) \\
&= b_1^{-k/2} b_2^{k/2} \cdot \psi_f(g)
\end{aligned} \tag{1.1}$$

for all $g \in \mathrm{PGL}_2(R)$.

Given a *weight* $\boldsymbol{k} = (k_\sigma)_{\sigma \in \Sigma} \in 2\mathbb{Z}_{\geq 0}[\Sigma]$ we define the $\mathrm{PGL}_2(R)^\Sigma$-representation

$$V_{\boldsymbol{k}}(R) = \otimes_{\sigma \in \Sigma} V_{k_\sigma}(R)$$

and $V_{\boldsymbol{k}}(R)^\vee$ as the R-linear dual of $V_{\boldsymbol{k}}(R)$. One may view $V_{\boldsymbol{k}}(\mathbb{C})$ as a representation of $G(F)$ via the embedding j. In fact, there exists a number field $E \subseteq \mathbb{C}$ such that every embedding $\sigma \colon F \hookrightarrow \mathbb{C}$ factors over E and such that $G(F)$ acts on $V_{\boldsymbol{k}}(E) \subseteq V_{\boldsymbol{k}}(\mathbb{C})$. We fix E for the remainder of the article.

1.1.2. $(\mathfrak{g}, K_\infty^+)$-cohomology

Let $\mathfrak{g}$ denote the complexification of the Lie algebra of G_∞. We fix a maximal compact subgroup K_∞ of G_∞ with connected component K_∞^+. Let us recall that π is *cohomological* if and only if there exists a weight $\boldsymbol{k} = (k_\sigma)_{\sigma \in \Sigma}$ such that

$$\mathrm{H}^\bullet(\mathfrak{g}, K_\infty^+, \pi_\infty \otimes V_{\boldsymbol{k}}(\mathbb{C})^\vee) \neq 0.$$

See [8] for the notion of $(\mathfrak{g}, K_\infty^+)$-cohomology. The weight $\boldsymbol{k}$ is uniquely determined by π_∞, and we fix it from here on. By [28, 3.6.1], the equality

$$k_\sigma = k_{c\sigma} \tag{1.2}$$

holds for all $\sigma \in \Sigma$. The group $\pi_0(G_\infty) \cong \pi_0(K_\infty)$ acts on $(\mathfrak{g}, K_\infty^+)$-cohomology. For every character $\epsilon \colon \pi_0(G_\infty) \to \{\pm 1\}$ we have the following dimension formulas for the ϵ-isotypic component:

$$\dim_{\mathbb{C}} \mathrm{H}^i(\mathfrak{g}, K_\infty^+, \pi_\infty \otimes V_{\boldsymbol{k}}(\mathbb{C})^\vee)^\epsilon = \begin{pmatrix} \delta \\ i - q \end{pmatrix}. \tag{1.3}$$

Via the Künneth theorem one may reduce the computation to that of the cohomology for each place $\mathfrak{q} \mid \infty$ separately. In case $G_\mathfrak{q}$ is split, the computation is spelled out in [28, Section 3.6.2]. The nonsplit case is trivial.

1.1.3. Local systems

Let us fix an open compact subgroup $K \subseteq G(\mathbb{A}^\infty)$ and consider the locally symmetric space

$$\mathcal{X}_K = G(F) \backslash G(\mathbb{A}) / K K_\infty^+$$

together with the projection map

$$p_K \colon G(\mathbb{A}) / K K_\infty^+ \to \mathcal{X}_K.$$

Let $\{g_1, \ldots, g_h\}$ be a set of representatives of the finite double coset

$$G(F)^+ \backslash G(\mathbb{A}^\infty) / K$$

and put

$$\Gamma^{g_i} = G(F)^+ \cap g_i K g_i^{-1}.$$

These are discrete subgroups of the real Lie group $G(F_\infty)^+$. We may decompose

$$\mathcal{X}_K = \bigcup_{i=1}^h \Gamma^{g_i} \backslash G(F_\infty)^+ / K_\infty^+. \tag{1.4}$$

If K is neat, then Γ^{g_i} is torsion free for every $i = 1, \ldots, h$ and, thus, the topological space $\mathcal{X}_K$ carries the structure of a locally symmetric space.

We fix a field Ω of characteristic zero. Given an $\Omega[G(F)]$-module N define the sheaf $\underline{N}$ on $\mathcal{X}_K$ by

$$\underline{N}(U) = \{s \colon p_K^{-1}(U) \to N \text{ locally constant} \mid s(g.u) = gs(u) \; \forall g \in G(F)\}.$$

In case K is neat, $\underline{N}$ is a local system. In any case, the group of global sections of $\underline{N}$ is given by

$$\Gamma(\mathcal{X}_K, \underline{N}) = \mathcal{F}(G(\mathbb{A}) / G(F_\infty)^+ K, N)^{G(F)},$$

where $G(F)$ acts on $\mathcal{F}(G(\mathbb{A}) / G(F_\infty)^+ K, N)$ via $(\gamma.f)(g) = \gamma.f(\gamma^{-1}g)$. Right translation defines commuting actions of the component group $\pi_0(G_\infty)$ and the Hecke algebra $\mathbb{T}_K(\Omega) := \Omega[K \backslash G(\mathbb{A}^\infty) / K]$ of level K on $\underline{N}$-valued cohomology $\mathrm{H}^\bullet(\mathcal{X}_K, \underline{N})$.

We assume in the following that K is of the form $K = \prod_{\mathfrak{q}} K_{\mathfrak{q}}$ and that

$$(\pi^\infty)^K \neq 0.$$

Let S be a finite set of primes of F such that S contains every $\mathfrak{q}$ such that $K_{\mathfrak{q}}$ is not maximal and put $K^S = \prod_{\mathfrak{q} \notin S} K_{\mathfrak{q}}$. The Hecke algebra

$$\mathbb{T}_{K^S}^S(\Omega) := \Omega[K^S \backslash G(\mathbb{A}^{S,\infty}) / K^S]$$

away from S is central in $\mathbb{T}_K(\Omega)$. We will assume for the rest of this article that π^∞ has a model over E, which we denote π_E^∞. We may always assume this by enlarging E slightly (see [29, Theorem C]). If Ω is an extension of E, we put $\pi_\Omega^\infty = \pi_E^\infty \otimes_E \Omega$. The Hecke algebra $\mathbb{T}_{K^S}^S(\Omega)$ acts on π_Ω^∞. For every $\mathfrak{q} \notin S$, the subgroup $K_{\mathfrak{q}} \subset G_{\mathfrak{q}}$ is a maximal open compact subgroup and, thus, by Casselman's theorem on newforms (see [13]) we have

$$\dim_{\mathbb{C}}(\pi_{\mathfrak{q}})^{K_{\mathfrak{q}}} = 1.$$

It follows that $\mathbb{T}_{K^S}^S(\Omega)$ acts on $(\pi_\Omega^\infty)^K$ via a character

$$\lambda_\pi^S \colon \mathbb{T}_{K^S}^S(\Omega) \longrightarrow \Omega.$$

Let $\mathfrak{m}_\pi^S \subseteq \mathbb{T}_{K^S}^S(\Omega)$ be the kernel of λ_π^S. It is a maximal ideal. Let M be a $\mathbb{T}_K(\Omega)$-module. Since $\mathbb{T}_{K^S}^S(\Omega) \subseteq \mathbb{T}_K(\Omega)$ is a central subalgebra it follows that $\mathbb{T}_K(\Omega)$ acts on the localization $M_{\mathfrak{m}_\pi^S}$.

Theorem 1.1. *Let Ω be any extension of E. For every character $\epsilon \colon \pi_0(G_\infty) \to \{\pm 1\}$ we have*

$$\dim_\Omega \mathrm{Hom}_{\mathbb{T}_K(\Omega)}((\pi_\Omega^\infty)^K, \mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})^\epsilon) = \binom{\delta}{i-q}.$$

Moreover, the localization $\mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})_{\mathfrak{m}_\pi^S}$ is equal to the sum of the images of all homomorphisms from $(\pi_\Omega^\infty)^K$ to $\mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})$.

Proof. The Borel–Serre compactification $\bar{\mathcal{X}}_K$ of $\mathcal{X}_K$ (see [7]) is homotopic to $\mathcal{X}_K$ and every local system – such as $\underline{V_{\boldsymbol{k}}(\Omega)^\vee}$ – naturally extends to it. Moreover, it is homeomorphic to a compact manifold with boundary and, thus, has a finite triangulation. One deduces that the canonical map

$$\mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(E)^\vee})^\epsilon \otimes_E \Omega \longrightarrow \mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})^\epsilon$$

is an isomorphism for every extension Ω of E. Thus, we may reduce to the case $\Omega = \mathbb{C}$. In that case, the first claim follows from standard arguments about cohomological representations and Equation (1.3) (see, for example, [28, Section II] for details in the case G is split).

The second claim follows from strong multiplicity one. $\square$

1.1.4. Cohomology of arithmetic groups

We are going to recast the above cohomology groups in terms of group cohomology. Let $K \subseteq G(\mathbb{A}^\infty)$ be an open compact subgroup and N an $\Omega[G(F)]$-module. The group $G(F)$ acts on the space $\mathcal{F}(G(\mathbb{A}^\infty)/K, N)$ via $(\gamma.f)(g) = \gamma.f(\gamma^{-1}g)$ and the Hecke algebra $\mathbb{T}_K(\Omega)$ via right translation. Thus, we have commuting actions of the component group $\pi_0(G_\infty)$ and the Hecke algebra $\mathbb{T}_K(\Omega) = \Omega[K \backslash G(\mathbb{A}^\infty)/K]$ on the spaces

$$\mathrm{H}^i(X_K, N) := \mathrm{H}^i(G(F)^+, \mathcal{F}(G(\mathbb{A}^\infty)/K, N)).$$

Lemma 1.2. *There are canonical isomorphisms*

$$\mathrm{H}^\bullet(X_K, N) \xrightarrow{\cong} \mathrm{H}^\bullet(\mathcal{X}_K, \underline{N})$$

that are equivariant with respect to the actions of the component group and the Hecke algebra.

Proof. The proof is rather standard. For the sake of completeness, we give a sketch of it. Since the homomorphism $G(F) \to \pi_0(G(F_\infty))$ is surjective, Frobenius reciprocity implies that

$$\begin{aligned}
\mathrm{H}^0(X_K, N) &= \mathcal{F}(G(\mathbb{A}^\infty)/K, N)^{G(F)^+} \\
&= \mathcal{F}(G(\mathbb{A})/G(F_\infty)^+ K, N)^{G(F)} \\
&= \Gamma(\mathcal{X}_K, \underline{N}).
\end{aligned}$$

The assignment $N \mapsto \mathrm{H}^\bullet(X_K, N)$ defines an effaceable δ-functor. Thus, by [26, Proposition 2.2.1] it is enough to prove that the assignment $N \mapsto \mathrm{H}^\bullet(\mathcal{X}_K, \underline{N})$ defines an effaceable δ-functor as well. It suffices to show that the functor $N \mapsto \underline{N}$ from the category of $\Omega[G(F)]$-modules to the category of sheaves on $\mathcal{X}_K$ sends injective $G(F)$-modules to acyclic sheaves and is exact.

Firstly, as every injective $G(F)$-modules is a direct summand of a coinduced module, it is enough to check that sheaves associated to coinduced modules are acyclic. Let V be a Ω-vector space. Consider the coinduced $G(F)$-module $\operatorname{Coind}_1^{G(F)} V$ as well as the constant sheaf $\tilde{V}$ on $G(\mathbb{A})/KK_\infty^+$. One readily computes that

$$(p_K)_*(\tilde{V}) \cong \underline{\operatorname{Coind}_1^{G(F)} V}.$$

Since the fibres of p_K are discrete, the higher direct images $R^q(p_K)(\tilde{V})$ vanish and we get

$$\operatorname{H}^\bullet(G(\mathbb{A})/KK_\infty^+, \tilde{V}) \cong \operatorname{H}^\bullet(\mathcal{X}_K, (p_K)_*(\tilde{V})) \cong \operatorname{H}^\bullet(\mathcal{X}_K, \underline{\operatorname{Coind}_1^{G(F)} V}).$$

Since $G(\mathbb{A})/KK_\infty^+$ is a disjoint union of contractible spaces, the claim follows.

Secondly, to prove exactness it is enough to check exactness at the level of stalks. Let x be a point of $\mathcal{X}_K$ and $g_i \in G(F)^+$ such that x lives in the connected component corresponding to g_i with respect to the decomposition (1.4). Consider a preimage $\tilde{x} \in G(F_\infty)^+/K_\infty^+$ of x and denote by $\Gamma_{\tilde{x}}^{g_i} \subseteq \Gamma^{g_i}$ the stabilizer of $\tilde{x}$ in Γ^{g_i}. The stalk of $\underline{N}$ at x is given by

$$\underline{N}_x \cong N^{\Gamma_{\tilde{x}}^{g_i}}.$$

Exactness follows since the group $\Gamma_{\tilde{x}}^{g_i}$ is finite and, hence, taking invariance in characteristic zero is exact. $\qquad\square$

1.2. Cohomology of $\mathfrak{p}$-arithmetic groups

Let $\mathfrak{p}$ be a prime of F and Ω a field of characteristic zero. Given an open compact subgroup $K^\mathfrak{p} \subseteq G(\mathbb{A}^{\mathfrak{p},\infty})$, an $\Omega[G_\mathfrak{p}]$-module M and an $\Omega[G(F)]$-module N, we let $G(F)$ act on the Ω-vector space $\mathcal{F}(G(\mathbb{A}^{\mathfrak{p},\infty})/K^\mathfrak{p}, \operatorname{Hom}_\Omega(M, N))$ via $(\gamma.f)(g)(m) = \gamma.f(\gamma^{-1}g)(\gamma^{-1}(m))$ and put

$$\operatorname{H}_\Omega^i(X_{K^\mathfrak{p}}^\mathfrak{p}, M, N) := \operatorname{H}^i(G(F)^+, \mathcal{F}(G(\mathbb{A}^{\mathfrak{p},\infty})/K^\mathfrak{p}, \operatorname{Hom}_\Omega(M, N))).$$

These cohomology groups carry commuting actions of the component group $\pi_0(G_\infty)$ and the Hecke algebra

$$\mathbb{T}_{K^\mathfrak{p}}^\mathfrak{p}(\Omega) := \Omega[K^\mathfrak{p}\backslash G(\mathbb{A}^{\mathfrak{p},\infty})/K^\mathfrak{p}].$$

In case $M = \Omega$ with the trivial $G_\mathfrak{p}$-action, we put

$$\operatorname{H}^i(X_{K^\mathfrak{p}}^\mathfrak{p}, N) := \operatorname{H}_\Omega^i(X_{K^\mathfrak{p}}^\mathfrak{p}, \Omega, N).$$

Suppose that $M = M_1 \otimes_\Omega M_2$ with both M_1 and M_2 being $\Omega[G_\mathfrak{p}]$-modules. Then by definition, we have

$$\operatorname{H}_\Omega^i(X_{K^\mathfrak{p}}^\mathfrak{p}, M_1 \otimes M_2, N) = \operatorname{H}_\Omega^i(X_{K^\mathfrak{p}}^\mathfrak{p}, M_1, \operatorname{Hom}_\Omega(M_2, N)), \tag{1.5}$$

where $G(F)$ acts on M_2 via the embedding $G(F) \hookrightarrow G_\mathfrak{p}$. For later purposes, we also define the following continuous variant: Let A be an affinoid $\mathbb{Q}_p$-algebra. Given a continuous A-module M with a continuous $G_\mathfrak{p}$-action and an $A[G(F)]$-module N that is finitely generated and free over A, we put

$$\operatorname{H}_{A,\operatorname{ct}}^i(X_{K^\mathfrak{p}}^\mathfrak{p}, M, N) := \operatorname{H}^i(G(F)^+, \mathcal{F}(G(\mathbb{A}^{\mathfrak{p},\infty})/K^\mathfrak{p}, \operatorname{Hom}_{A,\operatorname{ct}}(M, N))).$$

It also carries actions by the Hecke algebra $\mathbb{T}_{K^\mathfrak{p}}^\mathfrak{p}(A) := A[K^\mathfrak{p}\backslash G(\mathbb{A}^{\mathfrak{p},\infty})/K^\mathfrak{p}]$ and the component group $\pi_0(G_\infty)$.

Let us discuss an example of the above construction. First, we are going to recall the notion of compact induction: Let $K_\mathfrak{p} \subset G_\mathfrak{p}$ be an open compact subgroup and L a $\Omega[K_\mathfrak{p}]$-module. The *compact induction*

c-ind$_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L$ of L to $G_{\mathfrak{p}}$ is given by the set of functions $f \colon G_{\mathfrak{p}} \to L$ that satisfy $f(gk) = k^{-1}.f(g)$ for all $g \in G_{\mathfrak{p}}$, $k \in K_{\mathfrak{p}}$ and have finite support modulo $K_{\mathfrak{p}}$. The group $G_{\mathfrak{p}}$ acts on c-ind$_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L$ via left translation. If L is a $\Omega[G_{\mathfrak{p}}]$-module, the map

$$L \otimes_{\Omega} \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega \longrightarrow \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L, \ l \otimes f \longmapsto [g \mapsto f(g) \cdot g^{-1}.l] \tag{1.6}$$

is a $G_{\mathfrak{p}}$-equivariant isomorphism. For any $\Omega[G(F)]$-module N, the bilinear pairing

$$\mathcal{F}(G_{\mathfrak{p}}/K_{\mathfrak{p}}, N) \times \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega \longrightarrow N, \quad (f_1, f_2) \longmapsto \sum_{k \in G_{\mathfrak{p}}/K_{\mathfrak{p}}} f_1(g) \cdot f_2(g)$$

induces an isomorphism

$$\mathcal{F}(G_{\mathfrak{p}}/K_{\mathfrak{p}}, N) \xrightarrow{\cong} \text{Hom}_{\Omega}(\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega, N)$$

of $G(F)$-modules. This in turn induces a canonical $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-equivariant isomorphism

$$\text{H}_{\Omega}^i(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega, N) \cong \text{H}^i(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, N)$$

for every $i \geq 0$. More generally, by using Equation (1.5) we get canonical $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-equivariant isomorphisms

$$\text{H}_{\Omega}^i(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L, N) \cong \text{H}^i(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, \text{Hom}_{\Omega}(L, N)). \tag{1.7}$$

Consider the projective limit

$$\tilde{\text{H}}^i(X_{K^{\mathfrak{p}}}, N) = \varinjlim_{K_{\mathfrak{p}}} \text{H}^i(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, N)$$

taken over all open compact subgroups $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$. This space carries commuting actions of $\pi_0(G_{\infty})$, $G_{\mathfrak{p}}$ and $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$. Since Ω has characteristic 0, one deduces that the canonical map

$$\text{H}^i(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, N) \longrightarrow \tilde{\text{H}}^i(X_{K^{\mathfrak{p}}}, N)^{K_{\mathfrak{p}}}$$

is an isomorphism for all open compact subgroups $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$. Thus, $\tilde{\text{H}}^i(X_{K^{\mathfrak{p}}}, N)$ is a smooth representation of $G_{\mathfrak{p}}$. Moreover, it is admissible in case N is finite-dimensional. The goal of this section is to compare $\text{Hom}_{\Omega[G_{\mathfrak{p}}]}(M, \tilde{\text{H}}^i(X_{K^{\mathfrak{p}}}, N))$ and $\text{H}_{\Omega}^i(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, M, N)$ in certain situations.

Lemma 1.3. *Let M be a smooth $\Omega[G_{\mathfrak{p}}]$-representation of finite length. Then, there exists a resolution*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where P_i, $i = 0, 1$ are finitely generated projective smooth representations.

Proof. In case $\mathfrak{p} \in \text{ram}(D)$, the group $G_{\mathfrak{p}}$ is compact. Thus, the category of smooth representations of $G_{\mathfrak{p}}$ with Ω-coefficients is semisimple and M is projective itself. In case D is split at $\mathfrak{p}$ and therefore $G_{\mathfrak{p}} \cong \text{PGL}_2(F_{\mathfrak{p}})$, this is a consequence of the main theorem of [36]. $\square$

Let $\pi^{\mathfrak{p}, \infty}$ be the restricted tensor product of all local components of π away from $\mathfrak{p}$ and ∞. This is a smooth irreducible representation of $G(\mathbb{A}^{\mathfrak{p}, \infty})$. We fix models $\pi_E^{\mathfrak{p}, \infty}$, respectively $\pi_{\mathfrak{p}, E}$ of $\pi^{\mathfrak{p}, \infty}$, respectively $\pi_{\mathfrak{p}}$ over E and a $G(\mathbb{A}^{\infty})$-equivariant isomorphism

$$\pi_E^{\mathfrak{p}, \infty} \otimes_E \pi_{\mathfrak{p}, E} \cong \pi_E^{\infty}.$$

From now on, Ω will always be an extension of E and we put $\pi_{\Omega}^{\mathfrak{p},\infty} = \pi_{E}^{\mathfrak{p},\infty} \otimes_E \Omega$ as well as $\pi_{\mathfrak{p},\Omega} = \pi_{\mathfrak{p},E} \otimes_E \Omega$. Further, we assume that $K^{\mathfrak{p}}$ is of the form $K^{\mathfrak{p}} = \prod_{\mathfrak{q} \neq \mathfrak{p}} K_{\mathfrak{q}}$ and that $(\pi_{\Omega}^{\mathfrak{p},\infty})^{K^{\mathfrak{p}}} \neq 0$. We fix a finite set S of primes of F as before such that $\mathfrak{p} \in S$. In particular, the Hecke algebra $T_{K^S}^S(\Omega)$ is a central subalgebra of $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$.

Proposition 1.4. *Let M be a smooth $\Omega[G_{\mathfrak{p}}]$-representation of finite length. There are isomorphisms*

$$\mathrm{H}_{\Omega}^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, M, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} \xrightarrow{\cong} \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(M, \tilde{\mathrm{H}}^q(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S})$$

that are functorial in M and equivariant under the actions of $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$ and $\pi_0(G_{\infty})$. Furthermore, we have

$$\dim_{\Omega} \mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, M, V_{\boldsymbol{k}}(\Omega)^{\vee}) < \infty$$

for all $d \geq 0$.

Proof. If $M = P$ is projective, there are functorial isomorphisms

$$\mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, P, V_{\boldsymbol{k}}(\Omega)^{\vee}) \xrightarrow{\cong} \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(P, \tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})) \qquad (1.8)$$

for all $d \geq 0$ by [24, Lemma 3.5(b)]. In particular, we have

$$\mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, P, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} = 0$$

for all $d < q$ by Theorem 1.1. Thus, the short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of Lemma 1.3 induces the exact sequence

$$0 \to \mathrm{H}_{\Omega}^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, M, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} \to \mathrm{H}_{\Omega}^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, P_0, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} \to \mathrm{H}_{\Omega}^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, P_1, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S},$$

and the first claim follows from the isomorphism (1.8) for $P = P_0, P_1$.

Since $\tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})$ is admissible, the isomorphism (1.8) implies that

$$\dim_{\Omega} \mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, P, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} < \infty \qquad \forall d \geq 0$$

in case P is finitely generated and projective. For M of finite length, consider again the long exact sequence induces by the short exact sequence of Lemma 1.3. $\qquad \square$

Remark 1.5. A typical projective smooth representation is the compact induction $\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega$ of the trivial representation from an open compact subgroup $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$. In that case, the result of [24] used above simply follows from the composition of isomorphisms

$$\mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega, N) \cong \mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, N)$$

$$\cong \tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, N)^{K_{\mathfrak{p}}}$$

$$\cong \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega, \tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, N)),$$

where the first isomorphism is a special case of Equation (1.7) and the third follows from Frobenius reciprocity.

Proposition 1.4 together with Theorem 1.1 implies the following:

Corollary 1.6. *Let $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ be a character and M an irreducible smooth $\Omega[G_{\mathfrak{p}}]$-representation. Then*

$$\mathrm{H}_\Omega^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, M, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}_\pi^S}^\epsilon = 0$$

unless $M \cong \pi_{\mathfrak{p},\Omega}$. Furthermore, there is an isomorphism

$$\mathrm{H}_\Omega^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \pi_{\mathfrak{p},\Omega}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}_\pi^S}^\epsilon \cong (\pi_\Omega^{\mathfrak{p},\infty})^{K^{\mathfrak{p}}}$$

of $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-modules. In particular, it is an absolutely irreducible $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-module.

Remark 1.7. The corollary above was implicitly proven in [39] under the assumption that the local representation $\pi_{\mathfrak{p}}$ has an Iwahori-fixed vector by using an explicit resolution of $\pi_{\mathfrak{p}}$ constructed via the Bruhat–Tits tree.

1.3. The Steinberg case

We now assume that D is split at $\mathfrak{p}$. We define the Ω-*valued smooth Steinberg representation* $\mathrm{St}_{\mathfrak{p}}^\infty(\Omega)$ of $G_{\mathfrak{p}}$ as the space of locally constant Ω-valued functions on $\mathbb{P}^1(F_{\mathfrak{p}})$ modulo constant functions. The group $G_{\mathfrak{p}} \cong \mathrm{PGL}_2(F_{\mathfrak{p}})$ naturally acts on $\mathbb{P}^1(F_{\mathfrak{p}})$ and thus also on $\mathrm{St}_{\mathfrak{p}}^\infty(\Omega)$. We assume for the moment that $\pi_{\mathfrak{p}} \cong \mathrm{St}_{\mathfrak{p}}^\infty(\mathbb{C})$. Then, Corollary 1.6 implies that the $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-module $\mathrm{H}_\Omega^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p}}^\infty(\Omega), V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}_\pi^S}^\epsilon$ is irreducible.

Given smooth Ω-representations V and W of $G_{\mathfrak{p}}$, we denote by $\mathrm{Ext}_\infty^i(V, W)$ the Ext-groups in the category of smooth representations. It is well known that

$$\dim_\Omega \mathrm{Ext}_\infty^i(\Omega, \mathrm{St}_{\mathfrak{p}}^\infty(\Omega)) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

and

$$\dim_\Omega \mathrm{Ext}_\infty^i(\mathrm{St}_{\mathfrak{p}}^\infty(\Omega), \mathrm{St}_{\mathfrak{p}}^\infty(\Omega)) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

The above calculations follow directly from the existence of the following two projective resolutions:

$$0 \longrightarrow \mathrm{c\text{-}ind}_{J_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \chi_- \longrightarrow \mathrm{c\text{-}ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega \longrightarrow \Omega \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{c\text{-}ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega \longrightarrow \mathrm{c\text{-}ind}_{J_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \chi_- \longrightarrow \mathrm{St}_{\mathfrak{p}}^\infty(\Omega) \longrightarrow 0.$$

Here $K_{\mathfrak{p}} = \mathrm{PGL}_2(\mathcal{O}_{\mathfrak{p}})$ is a maximal compact subgroup, $J_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ is an open compact subgroup that contains an Iwahori subgroup as a normal subgroup and $\chi_-\colon J_{\mathfrak{p}} \to \{\pm 1\}$ is a nontrivial character that is trivial on the Iwahori subgroup. The exactness of the first sequence is just a reformulation of the contractibility of the Bruhat–Tits tree. The second follows from the fact that the cohomology with compact support of the Bruhat–Tits tree is the Steinberg representation (see, for example, [39, equation (18)]). To keep with our promise not to use the Bruhat–Tits tree, let us mention that alternative calculations of these Ext-groups that do not invoke the Bruhat–Tits tree – and work for more general reductive groups – can be found in [32] and [15].

Let $\mathcal{E}^{\infty}$ be the unique (up to multiplication by a scalar) nontrivial extension of Ω by $\mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega)$, that is, there exists a nonsplit exact sequence

$$0 \longrightarrow \mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega) \longrightarrow \mathcal{E}^{\infty} \longrightarrow \Omega \longrightarrow 0 \tag{1.11}$$

of $G_{\mathfrak{p}}$-modules. The short exact sequence (1.11) induces the short exact sequence

$$0 \longrightarrow V_{\boldsymbol{k}}(\Omega)^{\vee} \longrightarrow \mathrm{Hom}_{\Omega}(\mathcal{E}^{\infty}, V_{\boldsymbol{k}}(\Omega)^{\vee}) \longrightarrow \mathrm{Hom}_{\Omega}(\mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee}) \longrightarrow 0,$$

which in turn induces the boundary map

$$\mathrm{H}_{\Omega}^{d}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee}) \longrightarrow \mathrm{H}_{\Omega}^{d+1}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^{\vee})$$

in cohomology. Given a character $\epsilon \colon \pi_0(G_{\infty}) \to \{\pm 1\}$, we write

$$c_{\infty}^{d,\epsilon} \colon \mathrm{H}_{\Omega}^{d}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon} \longrightarrow \mathrm{H}_{\Omega}^{d+1}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon} \tag{1.12}$$

for the induced map on the $\mathfrak{m}_{\pi}^{S}$-localization of the ϵ-isotypic part. It is a homomorphism of $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-modules.

The following generalization of [39, Lemma 6.2 (b)] holds. Its proof is an adaption of that of [24, Lemma 3.7].

Lemma 1.8. *The map*

$$c_{\infty}^{d,\epsilon} \colon \mathrm{H}_{\Omega}^{d}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon} \longrightarrow \mathrm{H}_{\Omega}^{d+1}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon}$$

is an isomorphism for every sign character $\epsilon \colon \pi_0(G_{\infty}) \to \{\pm 1\}$ and every $d \geq 0$.

Proof. We have to show that

$$\mathrm{H}_{\Omega}^{d}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathcal{E}^{\infty}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon} = 0$$

for all $d \geq 0$. Let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathcal{E}^{\infty} \longrightarrow 0 \tag{1.13}$$

be a projective resolution of $\mathcal{E}^{\infty}$ as in Lemma 1.3. Again, by [24, Lemma 3.5(b)] we have

$$\mathrm{H}_{\Omega}^{d}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, P_i, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon} \xrightarrow{\cong} \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(P_i, \tilde{\mathrm{H}}^{d}(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon})$$

for all $d \geq 0$ and $i = 0, 1$. As a $G_{\mathfrak{p}}$-module $\tilde{\mathrm{H}}^{d}(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^{S}}^{\epsilon}$ is isomorphic to some copies of $\mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega)$ by our assumption and Theorem 1.1. Thus, analyzing the long exact sequence induced from Equation (1.13) it is enough to show that

$$\mathrm{Ext}_{\infty}^{d}(\mathcal{E}^{\infty}, \mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega)) = 0$$

for all $d \geq 0$. But this follows directly from applying $\mathrm{Ext}_{G_{\mathfrak{p}}}(\cdot, \mathrm{St}_{\mathfrak{p}}^{\infty}(\Omega))$ to the short exact sequence (1.11) and the computations (1.9) and (1.10) of dimensions of smooth Ext-groups. □

2. Stabilizations

We explain the connection between overconvergent cohomology and the cohomology of $\mathfrak{p}$-arithmetic subgroups with values in duals of locally analytic principal series representations. We fix a prime $\mathfrak{p}$ of F, at which the quaternion algebra D is split, and an embedding

$$\iota_P \colon \overline{\mathbb{Q}} \lhook\joinrel\longrightarrow \overline{\mathbb{Q}_p},$$

where p denotes the rational prime underlying $\mathfrak{p}$. We define E_p to be the completion of E with respect to the topology induced by ι_p.

2.1. Smooth principal series

We will give a representation-theoretic description of $\mathfrak{p}$-stabilizations as in [21, Section 2.2]. Most of the basic results on smooth representations of PGL_2 over local fields that are used in this section can be found in [12, Section 4.5].

As mentioned before, we may identify $G_\mathfrak{p}$ with $\mathrm{PGL}_2(F_\mathfrak{p})$. Let B be the standard Borel subgroup of $G_\mathfrak{p}$ of upper triangular matrices. Given a smooth Ω-representation τ of B its *smooth parabolic induction* is the space

$$i_B(\tau) = \{ f \colon G_\mathfrak{p} \to \tau \text{ locally constant} \mid f(bg) = b.f(g) \; \forall b \in B, \; g \in G_\mathfrak{p} \}.$$

The group $G_\mathfrak{p}$ acts on $i_B(\tau)$ via right translation. We identify the spaces of locally constant characters on B with those on $F_\mathfrak{p}^\times$ by mapping a locally constant character $\chi \colon F_\mathfrak{p}^\times \longrightarrow \Omega^\times$ to the character

$$B \longrightarrow \Omega^\times, \quad \begin{pmatrix} b_1 & u \\ 0 & b_2 \end{pmatrix} \longmapsto \chi(b_1/b_2). \tag{2.1}$$

Definition 2.1. A $\mathfrak{p}$-stabilization (χ, ϑ) of π_Ω consists of a locally constant character $\chi \colon F_\mathfrak{p}^\times \longrightarrow \Omega^\times$ together with a nonzero $G_\mathfrak{p}$-equivariant homomorphism

$$\vartheta \colon i_B(\chi) \longrightarrow \pi_{\mathfrak{p}, \Omega}.$$

Let (χ, ϑ) be a $\mathfrak{p}$-stabilization of π_Ω. Since $\pi_{\mathfrak{p}, \Omega}$ is irreducible, ϑ is automatically surjective. Moreover, (χ, ϑ) induces a $\mathfrak{p}$-stabilization of $\pi_{\Omega'}$ for every extension Ω' of Ω. By the classification of smooth irreducible representations of $G_\mathfrak{p}$, we know that $\pi_\mathfrak{p}$ is either supercuspidal or a quotient of a smooth parabolic induction as above (with $\Omega = \mathbb{C}$). In the first case, π_Ω admits no $\mathfrak{p}$-stabilization. In the later case, there always exists a finite extension Ω' of Ω such that $\pi_{\Omega'}$ admits a $\mathfrak{p}$-stabilization (χ, ϑ). Furthermore, the map ϑ is unique up to multiplication with a scalar. The character χ is in general not unique. Suppose $\pi_{\Omega'}$ admits a $\mathfrak{p}$-stabilization (χ, ϑ). If ϑ is an isomorphism, then $\chi^2 \neq \mathbb{1}$ and $\chi^2 \neq |\cdot|_\mathfrak{p}^2$. Moreover, $i_B(\chi)$ is isomorphic to $i_B(\chi^{-1} |\cdot|_\mathfrak{p})$ but to no other principal series. Thus, as long as $\chi^2 \neq |\cdot|_\mathfrak{p}$ there are two essentially different $\mathfrak{p}$-stabilizations.

If ϑ is not an isomorphism, then $\chi^2 = \mathbb{1}$ and the kernel of ϑ is one-dimensional generated by the function $g \mapsto \chi(\det(g))$. Moreover, the sequence

$$0 \longrightarrow \ker(\vartheta) \longrightarrow i_B(\chi) \longrightarrow \pi_{\mathfrak{p}, \Omega} \longrightarrow 0$$

is nonsplit. We have $\pi_{\mathfrak{p}, \Omega} \cong \mathrm{St}_\mathfrak{p}^\infty(\Omega) \otimes \chi$ in this case.

From the discussion above and Proposition 1.4, we get the following:

Corollary 2.2. *Suppose π_Ω admits a $\mathfrak{p}$-stabilization (χ, ϑ). Then for every character $\epsilon \colon \pi_0(G_\infty) \to \{\pm 1\}$, there is an isomorphism*

$$\mathrm{H}_\Omega^q(X_{K^\mathfrak{p}}^\mathfrak{p}, i_B(\chi), V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}_\pi^S}^\epsilon \cong (\pi_\Omega^{\mathfrak{p}, \infty})^{K^\mathfrak{p}}$$

of $\mathbb{T}_{K^\mathfrak{p}}(\Omega)$-modules. In particular, it is an absolutely irreducible $\mathbb{T}_{K^\mathfrak{p}}(\Omega)$-module.

Remark 2.3. We will be mostly interested in the case that the representation $\pi_{\mathfrak{p}}$ admits an invariant vector under the Iwahori group

$$I_{\mathfrak{p}} = \{g \in \mathrm{PGL}_2(\mathcal{O}_{\mathfrak{p}}) \mid g \text{ is upper triangular mod } \mathfrak{p}\}.$$

It is well known that having an Iwahori-fixed vector is equivalent to having a $\mathfrak{p}$-stabilization (χ, ϑ) with respect to an unramified character χ, that is, $\chi(\mathcal{O}_{\mathfrak{p}}^{\times}) = 1$.

2.2. Noncritical $\mathfrak{p}$-stabilizations

Composing an embedding $\sigma \colon F \hookrightarrow E \subseteq \overline{\mathbb{Q}}$ with ι_p induces a p-adic topology on F. We define $\Sigma_{\mathfrak{p}} \subseteq \Sigma$ to be the set of all embeddings inducing the topology coming from our chosen prime $\mathfrak{p}$ and put $\Sigma^{\mathfrak{p}} = \Sigma \setminus \Sigma_{\mathfrak{p}}$. We identify $\Sigma_{\mathfrak{p}} \subseteq \Sigma$ with the set of embeddings $F_{\mathfrak{p}} \hookrightarrow E_p$.

Suppose Ω is a finite extension of E_p, which we will do from now on. We may decompose

$$V_{\boldsymbol{k}}(\Omega) = V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \otimes_{\Omega} V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)$$

with

$$V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) = \bigotimes_{\sigma \in \Sigma_{\mathfrak{p}}} V_{k_{\sigma}}(\Omega) \qquad \text{and} \qquad V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega) = \bigotimes_{\sigma \in \Sigma^{\mathfrak{p}}} V_{k_{\sigma}}(\Omega).$$

For each $\sigma \in \Sigma_{\mathfrak{p}}$, the group $G_{\mathfrak{p}}$ acts on $V_{k_{\sigma}}$ via the embedding $\sigma \colon F_{\mathfrak{p}} \hookrightarrow E_p \subseteq \Omega$. Hence, the representation $V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)$ of $G(F)$ extends to an algebraic representation of the group $G_{\mathfrak{p}}$.

We will assume for the reminder of this section that π_{Ω} admits a $\mathfrak{p}$-stabilization (χ, ϑ). We define the locally algebraic $G_{\mathfrak{p}}$-representation

$$i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}) = i_B(\chi) \otimes_{\Omega} V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega).$$

Then, by Equation (1.5), we have a canonical $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-equivariant isomorphism

$$\mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, i_B(\chi), V_{\boldsymbol{k}}(\Omega)^{\vee}) \xrightarrow{\cong} \mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee}).$$

Let A be an affinoid $\mathbb{Q}_p$-algebra. As in §2.1, we identify locally $\mathbb{Q}_p$-analytic characters from $F_{\mathfrak{p}}^{\times}$ to $A^{\times}$ with those from B to $A^{\times}$. Given a locally analytic representation τ of B, its *locally analytic parabolic induction* is given by

$$\mathbb{I}_B(\tau) = \{f \colon G_{\mathfrak{p}} \to \tau \text{ locally analytic} \mid f(bg) = b.f(g) \;\forall b \in B, \; g \in G_{\mathfrak{p}}\}.$$

The group $G_{\mathfrak{p}}$ acts on it via right translation. Suppose that $A = \Omega$ and τ is finite-dimensional. In that case, the locally analytic parabolic induction is a strongly admissible locally analytic representation of $G_{\mathfrak{p}}$. The case of one-dimensional representations is [19, Proposition 1.21]. The proof works verbatim for every finite-dimensional representation.

To any locally constant character $\chi \colon F_{\mathfrak{p}}^{\times} \to \Omega^{\times}$, we associate the locally analytic character $\chi_{\boldsymbol{k}_{\mathfrak{p}}}$ by

$$\chi_{\boldsymbol{k}_{\mathfrak{p}}}(x) = \chi(x) \prod_{\sigma \in \Sigma_{\mathfrak{p}}} \sigma(x)^{-k_{\sigma}/2}.$$

We may identify $i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}})$ with a subspace of $\mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}})$ as follows: Given an embedding $\sigma \in \Sigma_{\mathfrak{p}}$ and $f_{\sigma} \in V_{k_{\sigma}}(\Omega)$, we consider the Ω-valued function $\psi_{f_{\sigma}}$ on $\mathrm{PGL}_2(\Omega)$ constructed in Section 1.1.1. The embedding $\sigma \colon F_{\mathfrak{p}} \hookrightarrow E_p$ induces an embedding of $G_{\mathfrak{p}}$ into $\mathrm{PGL}_2(\Omega)$. We denote the restriction of $\psi_{f_{\sigma}}$

to $G_{\mathfrak{p}}$ also by ψ_{f_σ}. It is clearly a $\mathbb{Q}_p$-analytic function. By Equation (1.1), the map

$$\beta \colon i_B(\chi) \otimes \bigotimes_{\sigma \in \Sigma_{\mathfrak{p}}} V_{k_\sigma}(\Omega) \longrightarrow \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}})$$

$$(f_\infty, (f_\sigma)_{\sigma \in \Sigma_{\mathfrak{p}}}) \longmapsto f_\infty \cdot \prod_{\sigma \in \Sigma_{\mathfrak{p}}} \psi_{f_\sigma}$$

is well defined.

Definition 2.4. A $\mathfrak{p}$-stabilization (χ, ϑ) of π_Ω is called noncritical if the canonical map

$$\beta^* \colon \mathrm{H}^d_{\Omega,\mathrm{ct}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \longrightarrow \mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \qquad (2.2)$$

is an isomorphism for all $d \geq 0$.

Note that the notion of noncriticality depends on the level $K^{\mathfrak{p}}$ away from $\mathfrak{p}$ and the set S: If β^* is an isomorphism for $K^{\mathfrak{p}}$, it is clearly an isomorphism for every open compact subgroup containing $K^{\mathfrak{p}}$. But on the other hand, while one can completely describe the right-hand side of Equation (2.2) when making $K^{\mathfrak{p}}$ smaller, one a priori does not have any control over the left-hand side. Similarly, strong multiplicity one implies that the right-hand side does not change when enlarging S, while the left-hand side might get larger.

2.2.1. Locally algebraic and locally analytic Steinberg representation

Assume for the moment that $\pi_{\mathfrak{p}}$ is the Steinberg representation. As mentioned above there is a unique $\mathfrak{p}$-stabilization $\vartheta \colon i_B(\mathbb{1}) \to \mathrm{St}^\infty_{\mathfrak{p}}(\Omega)$ which has a one-dimensional kernel. We say π is noncritical at $\mathfrak{p}$ if this unique $\mathfrak{p}$-stabilization is noncritical.

We define the *locally algebraic Steinberg representation of weight $\boldsymbol{k}_{\mathfrak{p}}$* via

$$\mathrm{St}^\infty_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) = \mathrm{St}^\infty_{\mathfrak{p}}(\Omega) \otimes_\Omega V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega).$$

Sending $(f_\sigma)_{\sigma \in \Sigma_{\mathfrak{p}}} \in V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)$ to $\prod_{\sigma \in \Sigma_{\mathfrak{p}}} \psi_{f_\sigma}$, we can view $V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)$ as a subspace of $\mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}})$. We define the *locally analytic Steinberg representation of weight $\boldsymbol{k}_{\mathfrak{p}}$* as the quotient

$$\mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) = \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}})/V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega).$$

Thus, we have a natural embedding

$$\kappa \colon \mathrm{St}^\infty_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \hookrightarrow \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega).$$

Proposition 2.5. *Suppose that $\pi_{\mathfrak{p}} \cong \mathrm{St}^\infty_{\mathfrak{p}}(\mathbb{C})$ and π is noncritical at $\mathfrak{p}$. Then the canonical map*

$$\kappa^* \colon \mathrm{H}^d_{\Omega,\mathrm{ct}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \longrightarrow \mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^\infty_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi}$$

is an isomorphism for all $d \geq 0$.

Proof. We have the short exact sequence

$$0 \longrightarrow V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \longrightarrow i_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}) \longrightarrow \mathrm{St}^\infty_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \longrightarrow 0$$

and the commutative diagram

$$
\begin{array}{ccc}
i_B(\mathbb{1}_{k_{\mathfrak{p}}}) & \hookrightarrow & \mathbb{I}_B(\mathbb{1}_{k_{\mathfrak{p}}}) \\
\downarrow & & \downarrow \\
\mathrm{St}_{k_{\mathfrak{p}}}^{\infty}(\Omega) & \hookrightarrow & \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega),
\end{array}
$$

where the vertical arrows on both sides are given by quotient out by $V_{k_{\mathfrak{p}}}(\Omega)$. For all $d \geq 0$, it induces the following commutative diagram in cohomology

$$
\begin{array}{ccc}
\mathrm{H}_{\Omega,\mathrm{ct}}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} & \longrightarrow & \mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{k_{\mathfrak{p}}}^{\infty}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} \\
\downarrow & & \downarrow \\
\mathrm{H}_{\Omega,\mathrm{ct}}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_B(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S} & \longrightarrow & \mathrm{H}_{\Omega}^d(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, i_B(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S}.
\end{array}
$$

Since π is noncritical at $\mathfrak{p}$, the lower horizontal map is an isomorphism and hence also the upper one. □

Remark 2.6. On page 653 of [14], it is claimed that a property closely related to noncriticality always holds if the quaternion algebra D is totally definite. It is alluded to an Amice–Vélu and Vishik-type argument. But to the knowledge of the authors of this article the most general results of that type are in [11, Section 7], which essentially only cover the case of noncritical slope.

In the following, we are going to show that if the representation $\pi_{\mathfrak{p}}$ has an Iwahori-fixed vector the above definition of noncriticality is equivalent to the one given in terms of overconvergent cohomology that is used, for example, in [2] or [5]. The classicality theorem for overconvergent cohomology will give a numerical criterion for the noncriticality of a $\mathfrak{p}$-stabilization. In order to state this criterion, later we will need the following definition.

Definition 2.7. Let (χ, ϑ) be a $\mathfrak{p}$-stabilization of π_{Ω}. The p-adic valuation of $\prod_{\sigma \in \Sigma_{\mathfrak{p}}} \sigma(\varpi_{\mathfrak{p}})^{\frac{k_{\sigma}}{2}} \chi(\varpi_{\mathfrak{p}})$ is called the slope of (χ, ϑ). We say that (χ, ϑ) has noncritical slope if its slope is less than $\frac{1}{e_{\mathfrak{p}}} \min_{\sigma \in \Sigma_{\mathfrak{p}}}(k_{\sigma} + 1)$, where $e_{\mathfrak{p}}$ denotes the ramification index of $\mathfrak{p}$.

2.3. Overconvergent cohomology

We give a quick overview over the basics of overconvergent cohomology.

2.3.1. Locally analytic inductions
For $n \geq 1$, let $I_{\mathfrak{p}}^n$ be the subgroup

$$
I_{\mathfrak{p}}^n = \left\{ g \in \mathrm{PGL}_2(\mathcal{O}_{\mathfrak{p}}) \mid g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{p}^n \right\},
$$

and set $I_{\mathfrak{p}}^0 = I_{\mathfrak{p}}$. Then, $(I_{\mathfrak{p}}^n)_{n \geq 0}$ is a family of open normal subgroups of $I_{\mathfrak{p}}$ and it is a fundamental system of neighbourhoods of the identity. Let A be an affinoid $\mathbb{Q}_p$-algebra and $\chi : B \cap I_{\mathfrak{p}} \to A^{\times}$ a locally analytic character. This means that there exists a minimal integer $n_{\chi} \geq 1$ such that χ restricted to $B \cap I_{\mathfrak{p}}^{n_{\chi}}$ is analytic. For any integer $n \geq n_{\chi}$, define the $A[I_{\mathfrak{p}}]$-module $\mathcal{A}_{\chi}^n$ of functions $f : I_{\mathfrak{p}} \longrightarrow A$ such that

- f is analytic on any coset of $I_{\mathfrak{p}}/I_{\mathfrak{p}}^n$,
- $f(bk) = \chi(b)f(k) \quad \forall b \in B \cap I_{\mathfrak{p}}, \ k \in I_{\mathfrak{p}}$

with the action of $I_\mathfrak{p}$ given by $(h \cdot f)(k) = f(kh)$, and put

$$
\mathcal{A}_\chi = \bigcup_{n \geq n_\chi} \mathcal{A}_\chi^n.
$$

The $A[I_\mathfrak{p}]$-module $\mathcal{A}_\chi$ is the locally analytic induction of χ to $I_\mathfrak{p}$. In the special case that $A = \Omega$ is a finite extension of E_p and $\chi = \mathbb{1}_{k_\mathfrak{p}}$, we put

$$
\mathcal{A}_{k_\mathfrak{p}} = \mathcal{A}_{\mathbb{1}_{k_\mathfrak{p}}}.
$$

The Iwahori decomposition gives an isomorphism $I_\mathfrak{p} \cong (I_\mathfrak{p} \cap \overline{\mathbf{N}}) \times (I_\mathfrak{p} \cap B)$, where $\overline{\mathbf{N}}$ denotes the group of unipotent lower triangular matrices. Thus, restricting a function $f \in \mathcal{A}_\chi$ to $I_\mathfrak{p} \cap \overline{\mathbf{N}}$ induces an isomorphism between $\mathcal{A}_\chi$ and the space $\mathcal{A}(I_\mathfrak{p} \cap \overline{\mathbf{N}}, A)$ of locally analytic A-valued functions on $I_\mathfrak{p} \cap \overline{\mathbf{N}}$. An analogous bijection holds between $\mathcal{A}_\chi^n$ and $\mathcal{A}_n(I_\mathfrak{p} \cap \overline{\mathbf{N}}, A)$, the space of n-locally analytic functions on $I_\mathfrak{p} \cap \overline{\mathbf{N}}$.

2.3.2. The $U_\mathfrak{p}$ operator

Consider the compact induction c-ind$_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n)$. By Frobenius reciprocity, the ring $\mathrm{End}_{A[G_\mathfrak{p}]}(\mathrm{c\text{-}ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n))$ can be identified with the space of all functions $\Psi \colon G_\mathfrak{p} \longrightarrow \mathrm{End}_A(\mathcal{A}_\chi^n)$ such that

- Ψ is $I_\mathfrak{p}$-biequivariant, that is, $\Psi(k_1 g k_2) = k_1 \Psi(g) k_2$ in $\mathrm{End}_A(\mathcal{A}_\chi^n)$, for all $k_1, k_2 \in I_\mathfrak{p}$, $g \in G_\mathfrak{p}$, and
- for any element $f \in \mathcal{A}_\chi^n$, the function $G_\mathfrak{p} \to \mathcal{A}_\chi^n$, $g \mapsto \Psi(g)(f)$ is compactly supported.

Let $u_\mathfrak{p} := \begin{pmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}$. Consider the element $\varphi_{u_\mathfrak{p}} \in \mathrm{End}_A(\mathcal{A}_\chi^n)$ defined by

$$
\varphi_{u_\mathfrak{p}}(f)(\overline{n}) = f(u_\mathfrak{p} \overline{n} u_\mathfrak{p}^{-1}) \qquad \text{for all } f \in \mathcal{A}_\chi^n, \ \overline{n} \in I_\mathfrak{p} \cap \overline{\mathbf{N}}.
$$

By [31, Lemma 2.2], there exists a unique $I_\mathfrak{p}$-biequivariant function $\Psi_{u_\mathfrak{p}} \colon G_\mathfrak{p} \to \mathrm{End}_A(\mathcal{A}_\chi^n)$ such that $\mathrm{supp}(\Psi_{u_\mathfrak{p}}) = I_\mathfrak{p} u_\mathfrak{p}^{-1} I_\mathfrak{p}$ and $\Psi_{u_\mathfrak{p}}(u_\mathfrak{p}^{-1}) = \varphi_{u_\mathfrak{p}}$. Abusing notation, we will simply denote by $u_\mathfrak{p}$ the $G_\mathfrak{p}$-equivariant endomorphism of c-ind$_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n)$ corresponding to $\Psi_{u_\mathfrak{p}}$.

We define the module of distributions $\mathcal{D}_{\chi,k^\mathfrak{p}}^n := \mathrm{Hom}_{A,\mathrm{ct}}(\mathcal{A}_\chi^n, A) \otimes_\Omega V_{k^\mathfrak{p}}(\Omega)^\vee$ with its natural $I_\mathfrak{p}$-action. For $d \geq 0$, we set

$$
\mathrm{H}^d(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, \mathcal{D}_{\chi,k^\mathfrak{p}}^n) := \mathrm{H}_{A,\mathrm{ct}}^d(X_{K^\mathfrak{p}}^\mathfrak{p}, \mathrm{c\text{-}ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n), A \otimes_\Omega V_{k^\mathfrak{p}}(\Omega)^\vee).
$$

This notation is justified as follows: To any $I_\mathfrak{p}$-module M, one can attach a sheaf on the locally symmetric space $\mathcal{X}_{K^\mathfrak{p} \times I_\mathfrak{p}}$ (see, for example, [4, Definition 3.2 (ii)]). With similar arguments as in the proof of Lemma 1.2, one can show that the cohomology of the sheaf associated to $\mathcal{D}_{\chi,k^\mathfrak{p}}^n$ is computed by the groups above.

For now, we are mostly interested in the following special case: $A = \Omega$ is a finite extension of E_p and $\chi = \mathbb{1}_{k_\mathfrak{p}}$. In this situation, we abbreviate

$$
\mathrm{H}^d(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, \mathcal{D}_k^n) := \mathrm{H}^d(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, \mathcal{D}_{\mathbb{1}_{k_\mathfrak{p}}, k^\mathfrak{p}}^n).
$$

Later, we will also need the case that A is the coordinate ring of an affinoid subspace of the weight space (see Section 4.1).

Remark 2.8. In [2], overconvergent cohomology groups are introduced depending on a subset of the set of all primes of F lying above p. The spaces defined above correspond to the subset consisting only of the prime $\mathfrak{p}$.

The endomorphism $u_{\mathfrak{p}}$ induces an operator, that we denote by $U_{\mathfrak{p}}^{\circ}$, on cohomology:

$$U_{\mathfrak{p}}^{\circ} \colon H^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\chi, \boldsymbol{k}^{\mathfrak{p}}}) \longrightarrow H^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\chi, \boldsymbol{k}^{\mathfrak{p}}}).$$

Similarly as before, we may identify $V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)$ with the space of (globally) algebraic vectors in $\mathcal{A}^n_{\boldsymbol{k}_{\mathfrak{p}}}$. It induces an embedding

$$\text{c-ind}_{I_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)) \longrightarrow \text{c-ind}_{I_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(\mathcal{A}^n_{\boldsymbol{k}_{\mathfrak{p}}})$$

and the subspace $\text{c-ind}_{I_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega))$ is clearly invariant under the action of $u_{\mathfrak{p}}$. Thus, by invoking Equation (1.7) we get a $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-equivariant map

$$\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}}) \longrightarrow \mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee}) \tag{2.3}$$

in cohomology. We denote the natural operator on the right-hand side induced by $u_{\mathfrak{p}}$ by $U_{\mathfrak{p}}$. The map (2.3) intertwines the action of the Hecke operator $U_{\mathfrak{p}}^{\circ}$ on $H^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})$ with the action of $\prod_{\sigma \in \Sigma_{\mathfrak{p}}} \sigma(\varpi_{\mathfrak{p}})^{\frac{k_{\sigma}}{2}} U_{\mathfrak{p}}$ on $H^d_c(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})$. This follows from a simple analysis of the change of action of $u_{\mathfrak{p}}$ under the isomorphism

$$\text{c-ind}_{I_{\mathfrak{p}}}^{G_{\mathfrak{p}}} V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \cong V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \otimes_{\Omega} \text{c-ind}_{I_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega$$

given by Equation (1.6). We define the Hecke operator $U_{\mathfrak{p}}^{\circ}$ on the right-hand side of Equation (2.3) by

$$U_{\mathfrak{p}}^{\circ} = \prod_{\sigma \in \Sigma_{\mathfrak{p}}} \sigma(\varpi_{\mathfrak{p}})^{\frac{k_{\sigma}}{2}} U_{\mathfrak{p}},$$

and similarly we define an action of $U_{\mathfrak{p}}$ on the left-hand side of Equation (2.3).

2.3.3. Slope decompositions and classicality

We give a reminder on slope decompositions. As before, A denotes a affinoid $\mathbb{Q}_p$-algebra. Let M be an A-module equipped with an A-linear endomorphism $u \colon M \to M$. Fix a rational number $h \geq 0$. A polynomial $Q \in A[x]$ is multiplicative of slope $\leq h$ if

- the leading coefficient of Q is a unit in A and
- every edge of the Newton polygon of Q has slope $\leq h$.

We put $Q^*(x) = x^{\deg Q} Q(1/x)$. An element $m \in M$ is said to be of slope $\leq h$ if there is a multiplicative polynomial $Q \in A[x]$ of slope $\leq h$ such that $Q^*(u)m = 0$. Let $M^{\leq h} \subseteq M$ be the submodule of elements of M of slope $\leq h$.

Definition 2.9. A slope $\leq h$ decomposition of M is an $A[u]$-module isomorphism

$$M \cong M^{\leq h} \oplus M^{>h}$$

such that

- $M^{\leq h}$ is a finitely generated A-module and
- $Q^*(u)$ acts invertibly on $M^{>h}$ for every multiplicative polynomial $Q \in A[x]$ of slope $\leq h$.

Note that if $A = \Omega$ is a finite extension of E_P and M is finite-dimensional, then a slope $\leq h$ decomposition always exists. If M admits a slope $\leq h$ decomposition for all $h \geq 0$, we put

$$M^{<\infty} = \bigcup_{h \geq 0} M^{\leq h}.$$

The most remarkable result about slope decomposition is the following theorem which was first proved by Ash and Stevens [1] in special cases over $\mathbb{Q}$ and then generalized by Urban [42] and Hansen [27] to more general settings. See also [4] for a detailed treatment of the case of PGL_2 over arbitrary number fields. In these results, one always considers all primes above $\mathfrak{p}$ simultaneously. The modifications necessary to allow subsets of all primes above $\mathfrak{p}$ are explained in the proof of [2, Theorem 2.7].

Theorem 2.10. *For every $d \geq 0$ and every $h \geq 0$ the cohomology groups*

$$H^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}}),$$

admit a slope $\leq h$ decomposition with respect to the Hecke operator $U^\circ_{\mathfrak{p}}$.

If $h < \frac{1}{e_{\mathfrak{p}}} \min_{\sigma \in \Sigma_{\mathfrak{p}}}(k_\sigma + 1)$, where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$, then for all $d \geq 0$ the map (2.3) induces the following $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}$-equivariant isomorphism

$$H^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})^{\epsilon, \leq h} \xrightarrow{\sim} H^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)^{\epsilon, \leq h}.$$

Here, the slope decomposition is taken with respect to $U^\circ_{\mathfrak{p}}$ on both sides.

2.4. Overconvergent cohomology and noncritical $\mathfrak{p}$-stabilization

Let A be an affinoid algebra and $\chi \colon F^\times_{\mathfrak{p}} \longrightarrow A^\times$ be a locally analytic character. Denote by χ_0 its restriction to $\mathfrak{O}^\times_{\mathfrak{p}}$. An element $f \in \mathcal{A}^n_{\chi_0}$ can uniquely extended to a function on $BI_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ by putting $f(bk) = \chi(b)f(k)$. Since $BI_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ is open, extension by zero yields an $I_{\mathfrak{p}}$-equivariant A-linear injection

$$\mathcal{A}^n_{\chi_0} \hookrightarrow \mathbb{I}_B(\chi)|_{I_{\mathfrak{p}}}.$$

By Frobenius reciprocity, it induces a unique $G_{\mathfrak{p}}$-equivariant A-linear morphism

$$\mathrm{aug}_\chi \colon \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \longrightarrow \mathbb{I}_B(\chi). \tag{2.4}$$

The following theorem is due to Kohlhaase and Schraen.

Theorem 2.11. *For every $n \geq n_{\chi_0}$, the sequence*

$$0 \longrightarrow \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \xrightarrow{u_{\mathfrak{p}} - \chi(\varpi_{\mathfrak{p}})} \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \xrightarrow{\mathrm{aug}_\chi} \mathbb{I}_B(\chi) \longrightarrow 0 \tag{2.5}$$

is exact.

Proof. See [31, Proposition 2.4 and Theorem 2.5]. $\qquad\square$

Let $\chi \colon F^\times_{\mathfrak{p}} \to \Omega^\times$ be an unramified character. Similarly as above, the sequence

$$0 \longrightarrow \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\Omega) \xrightarrow{u_{\mathfrak{p}} - \chi(\varpi_{\mathfrak{p}})} \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\Omega) \xrightarrow{\mathrm{aug}_\chi} i_B(\chi) \longrightarrow 0$$

is exact. This can be deduced from Borel's theorem that $\mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\Omega)$ is a flat module over the Iwahori–Hecke algebra (see the end of [25, Section 3.1]). Tensoring the above exact sequence with $V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)$ and using Equation (1.6), we get a short exact sequence

$$0 \longrightarrow \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)) \xrightarrow{u_{\mathfrak{p}} - \chi_{\boldsymbol{k}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})} \mathrm{c\text{-}ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)) \xrightarrow{\mathrm{aug}_{\chi_{\boldsymbol{k}_{\mathfrak{p}}}}} i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}) \longrightarrow 0. \tag{2.6}$$

Suppose that $\pi_{\mathfrak{p}}$ has an Iwahori-fixed vector. Given a $\mathfrak{p}$-stabilization (χ, ϑ) of π_{Ω} (with χ not necessarily unramified), we define

$$\mathfrak{m}^S_{\pi,(\chi,\vartheta)} \subseteq \mathbb{T}^S_K(\Omega)[U_{\mathfrak{p}}]$$

to be the maximal ideal generated by $\mathfrak{m}^S_\pi$ and $U_{\mathfrak{p}} - \chi(\varpi_{\mathfrak{p}})$. In accordance with [2, Definition 2.12] and [5, Definition 1.5.1], we make the following definition:

Definition 2.12. The maximal ideal $\mathfrak{m}^S_{\pi,(\chi,\vartheta)} \subseteq \mathbb{T}^S_K(\Omega)[U_{\mathfrak{p}}]$ is noncritical if the map

$$\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}} \longrightarrow \mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}} \tag{2.7}$$

induced by Equation (2.3) is an isomorphism for all $d \geq 0$.

Proposition 2.13. *Suppose that $\pi_{\mathfrak{p}}$ has an Iwahori-fixed vector, and let (χ, ϑ) be a $\mathfrak{p}$-stabilization of π_{Ω}. Then the following are equivalent:*

(i) *(χ, ϑ) is noncritical*
(ii) *$\mathfrak{m}^S_{\pi,(\chi,\vartheta)}$ is noncritical.*

Proof. Note that χ is an unramified character. Thus, the long exact sequences induced by Equations (2.5) and (2.6) yield the following diagram with exact columns:

$$
\begin{array}{ccc}
\mathrm{H}^d_{\Omega,\mathrm{ct}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} & \xrightarrow{\ \beta^* \ } & \mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \\
{\scriptstyle\mathrm{aug}^*_{\chi_{\boldsymbol{k}_{\mathfrak{p}}}}} \Big\downarrow & & {\scriptstyle\mathrm{aug}^*_{\chi_{\boldsymbol{k}_{\mathfrak{p}}}}} \Big\downarrow \\
\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_\pi} & \xrightarrow{\ (2.3) \ } & \mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \\
{\scriptstyle U^\circ_{\mathfrak{p}} - \chi_{\boldsymbol{k}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})} \Big\downarrow & & \Big\downarrow {\scriptstyle U^\circ_{\mathfrak{p}} - \chi_{\boldsymbol{k}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})} \\
\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_\pi} & \xrightarrow{\ (2.3) \ } & \mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \\
{\scriptstyle\partial} \Big\downarrow & & \Big\downarrow {\scriptstyle\partial} \\
\mathrm{H}^{d+1}_{\Omega,\mathrm{ct}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} & \xrightarrow{\ \beta^* \ } & \mathrm{H}^{d+1}_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi}
\end{array}
$$

The existence of slope decompositions (see Theorem 2.10) implies that we may replace $\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_\pi}$ by $\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}}$ in the diagram above (and similarly for cohomology with coefficients in $V_{\boldsymbol{k}}(\Omega)$): Indeed, the operator $U^\circ_{\mathfrak{p}} - \chi_{\boldsymbol{k}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})$ is an isomorphism on the part of cohomology where the slope is bigger than h as long as h is large enough. On the other hand, the slope $\leq h$-part is finite-dimensional and localizing at the ideal generated by $U^\circ_{\mathfrak{p}} - \chi_{\boldsymbol{k}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})$ is the same as taking the corresponding generalized eigenspace.

The five lemma immediately gives the implication (ii)$\Rightarrow$(i). The other direction is proven by induction on d: Let us assume that (χ, ϑ) is noncritical and that

$$\mathrm{H}^{d-1}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}} \xrightarrow{\ (2.7)\ } \mathrm{H}^{d-1}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}}$$

is an isomorphism. Choose a positive integer N that is large enough. The assumptions above imply that the map

$$\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}} \xrightarrow{\ (2.7)\ } \mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}}$$

viewed as a homomorphism of finitely generated $\Omega[T]/T^N$-modules by letting T act via $U_{\mathfrak{p}}^\circ - \chi_{k_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})$ fulfils the assumption of Lemma 2.14 below. In particular, it is an isomorphism. $\qquad\square$

Given a $\Omega[T]/T^N$-module M put

$$M[T] = \ker(M \xrightarrow{\cdot T} M).$$

Lemma 2.14. *Let $f \colon M_1 \to M_2$ be a homomorphism of finitely generated $\Omega[T]/T^N$-modules. Assume that*

○ *f induces an isomorphism from $M_1[T]$ to $M_2[T]$ and*
○ *f induces an injection from M_1/TM_1 to M_2/TM_2.*

Then f is an isomorphism.

Proof. One immediately checks that f also induces an isomorphism from M_1/TM_1 to M_2/TM_2 since $\dim_\Omega M_i[T] = \dim_\Omega M_i/TM_i$ for $i = 1, 2$. Surjectivity of f then follows from Nakayama's lemma. It remains to prove injectivity: Let a be in the kernel of f. Assume that $a \neq 0$. There exists a maximal integer $0 \leq n \leq N$ such that $T^n \cdot a \neq 0$. By construction, $T^n \cdot a$ is an element of $M_1[T] \cap \ker(f) = \{0\}$, which is a contradiction. $\qquad\square$

Applying the second part of Theorem 2.10 one deduces the following:

Corollary 2.15. *Suppose that $\pi_{\mathfrak{p}}$ has an Iwahori-fixed vector. If a $\mathfrak{p}$-stabilization (χ, ϑ) of π_Ω has noncritical slope, then it is noncritical.*

Suppose $\pi_{\mathfrak{p}} = \mathrm{St}_{\mathfrak{p}}^\infty(\mathbb{C})$. Then the corollary above shows that π is noncritical at $\mathfrak{p}$ if

(i) $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$ or
(ii) $F_{\mathfrak{p}} = \mathbb{Q}_p$ or
(iii) $[F_{\mathfrak{p}} : \mathbb{Q}_p] = 2$ and $k_{\sigma_1} = k_{\sigma_2}$, where $\Sigma_{\mathfrak{p}} = \{\sigma_1, \sigma_2\}$.

This always holds in case $F = \mathbb{Q}$ or F is imaginary quadratic by Equation (1.2).

3. Automorphic L-invariants

The main aim of this section is to define automorphic $\mathcal{L}$-invariants for the representation π under the assumption that the local component of π at a prime $\mathfrak{p}$ is Steinberg.

3.1. Extensions of locally analytic Steinberg representations

Let Ω be a finite extension of E_p. The following construction of extensions is due to Breuil (see [9, Section 2.1]). Let $\lambda \colon F_{\mathfrak{p}}^\times \to \Omega$ be a continuous homomorphism. Note that λ is automatically locally $\mathbb{Q}_p$-analytic. We define τ_λ to be the two-dimensional Ω-representation of B given by

$$\begin{pmatrix} a & u \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \lambda(a/d) \\ 0 & 1 \end{pmatrix}$$

and put $\tau_{\lambda,k_{\mathfrak{p}}} = \tau_\lambda \otimes \mathbb{1}_{k_{\mathfrak{p}}}$. As the short exact sequence

$$0 \longrightarrow \mathbb{1}_{k_{\mathfrak{p}}} \longrightarrow \tau_{\lambda,k_{\mathfrak{p}}} \longrightarrow \mathbb{1}_{k_{\mathfrak{p}}} \longrightarrow 0$$

is split in the category of topological vector spaces, the induced sequence of locally analytic representations

$$0 \longrightarrow \mathbb{I}_B(\mathbb{1}_{k_{\mathfrak{p}}}) \longrightarrow \mathbb{I}_B(\tau_{\lambda,k_{\mathfrak{p}}}) \longrightarrow \mathbb{I}_B(\mathbb{1}_{k_{\mathfrak{p}}}) \longrightarrow 0$$

is exact (see, for example, Proposition 5.1 and Remark 5.4 of [30]). Pullback via $V_{k_{\mathfrak{p}}}(\Omega) \hookrightarrow \mathbb{I}_B(\mathbb{1}_{k_{\mathfrak{p}}})$ and pushforward along $\mathbb{I}_B(\mathbb{1}_{k_{\mathfrak{p}}}) \twoheadrightarrow \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega)$ yields the exact sequence

$$0 \longrightarrow \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega) \longrightarrow \mathcal{E}_{\lambda, k_{\mathfrak{p}}} \longrightarrow V_{k_{\mathfrak{p}}}(\Omega) \longrightarrow 0. \tag{3.1}$$

Let W_1 and W_2 be $\mathbb{Q}_p$-analytic Ω-representations of $G_{\mathfrak{p}}$. We write $\mathrm{Ext}_{\mathrm{an}}^1(W_1, W_2)$ for the space of locally $\mathbb{Q}_p$-analytic extensions of W_2 by W_1. The map

$$\mathrm{Hom}_{\mathrm{ct}}(F_{\mathfrak{p}}^{\times}, \Omega) \longrightarrow \mathrm{Ext}_{\mathrm{an}}^1(V_{k_{\mathfrak{p}}}(\Omega), \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega)), \ \lambda \longmapsto \mathcal{E}_{\lambda, k_{\mathfrak{p}}}$$

is an isomorphism. In the case $F_{\mathfrak{p}} = \mathbb{Q}_p$, this is due to Breuil. In fact, an analogous statement is true for more general split reductive groups (see [18, Theorem 1] and [24, Theorem 2.15]).

Tensoring with $V_{k_{\mathfrak{p}}}(\Omega)$ yields a canonical homomorphism

$$\mathrm{Ext}_{\mathrm{an}}^1(\Omega, \mathrm{St}_{\mathfrak{p}}^{\infty}) \longrightarrow \mathrm{Ext}_{\mathrm{an}}^1(V_{k_{\mathfrak{p}}}(\Omega), \mathrm{St}_{k_{\mathfrak{p}}}^{\infty}(\Omega)),$$

which is an isomorphism by [37, Proposition 4.14]. Moreover, by [37, Proposition 4.15] the canonical injection

$$\mathrm{Ext}_{\infty}^1(\Omega, \mathrm{St}_{\mathfrak{p}}^{\infty}) \longrightarrow \mathrm{Ext}_{\mathrm{an}}^1(\Omega, \mathrm{St}_{\mathfrak{p}}^{\infty})$$

is an isomorphism. Let $\mathcal{E}^{\infty}$ be the smooth extension of Equation (1.11). It follows that

$$\mathcal{E}_{k_{\mathfrak{p}}}^{\infty} := \mathcal{E}^{\infty} \otimes_{\Omega} V_{k_{\mathfrak{p}}}(\Omega)$$

is a generator of $\mathrm{Ext}_{\mathrm{an}}^1(V_{k_{\mathfrak{p}}}(\Omega), \mathrm{St}_{k_{\mathfrak{p}}}^{\infty}(\Omega))$. The following lemma compares this generator with Breuil's extension associated to the *p*-adic valuation.

Lemma 3.1. *The natural map* $\mathrm{Ext}_{\mathrm{an}}^1(V_{k_{\mathfrak{p}}}(\Omega), \mathrm{St}_{k_{\mathfrak{p}}}^{\infty}(\Omega)) \to \mathrm{Ext}_{\mathrm{an}}^1(V_{k_{\mathfrak{p}}}(\Omega), \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega))$ *is injective. It sends the generator* $\mathcal{E}_{k_{\mathfrak{p}}}^{\infty}$ *to a multiple of* $\mathcal{E}_{\mathrm{ord}_{\mathfrak{p}}, k_{\mathfrak{p}}}$.

Proof. This is [37, Corollary 4.16]. □

3.2. Definition of the L-invariant

We assume for the rest of this article that $\pi_{\mathfrak{p}}$ is the Steinberg representation.

Let $\lambda \colon F_{\mathfrak{p}}^{\times} \to \Omega$ be a continuous character. The short exact sequence (3.1) induces the short exact sequence

$$0 \longrightarrow V_k(\Omega)^{\vee} \longrightarrow \mathrm{Hom}_{\Omega, \mathrm{ct}}(\mathcal{E}_{\lambda, k_{\mathfrak{p}}}, V_{k^{\mathfrak{p}}}(\Omega)^{\vee}) \longrightarrow \mathrm{Hom}_{\Omega, \mathrm{ct}}(\mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee}) \longrightarrow 0,$$

which in turn induces the boundary map

$$\mathrm{H}_{\Omega, \mathrm{ct}}^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee}) \longrightarrow \mathrm{H}_{\Omega}^{q+1}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_k(\Omega)^{\vee})$$

in cohomology. Given a character $\epsilon \colon \pi_0(G_{\infty}) \to \{\pm 1\}$, we write

$$c_{\lambda}^{\epsilon} \colon \mathrm{H}_{\Omega, \mathrm{ct}}^q(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S}^{\epsilon} \longrightarrow \mathrm{H}_{\Omega}^{q+1}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_k(\Omega)^{\vee})_{\mathfrak{m}_{\pi}^S}^{\epsilon}$$

for the induced map, which is clearly a $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-module homomorphism.

Definition 3.2. The automorphic $\mathcal{L}$-invariant of π at $\mathfrak{p}$ with respect to the sign character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ is the subspace

$$\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon := \ker(\lambda \longmapsto c_\lambda^\epsilon) \subseteq \mathrm{Hom}_{\mathrm{ct}}(F_{\mathfrak{p}}^\times, \Omega).$$

Proposition 3.3. *Assume that π is noncritical at $\mathfrak{p}$. Then for every sign character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ the codimension of the $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon \subseteq \mathrm{Hom}_{\mathrm{ct}}(F_{\mathfrak{p}}^\times, \Omega)$ is equal to one. Moreover, it does not contain the space of locally constant homomorphisms.*

Proof. By Equation (1.10), the Ω-vector space $\mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(\mathrm{St}_{\mathfrak{p}}^\infty, \mathrm{St}_{\mathfrak{p}}^\infty)$ is one-dimensional. Thus, combining Proposition 2.5 and Lemma 1.8 one deduces that the space of $\mathbb{T}_{K^{\mathfrak{p}}}^{\mathfrak{p}}(\Omega)$-linear homomorphisms between the two modules $\mathrm{H}_{\Omega,\mathrm{ct}}^q(X_{K^{\mathfrak{p}}}, \mathrm{St}_{\boldsymbol{k}_{\mathfrak{p}}}^{\mathrm{an}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)_{\mathfrak{m}_\pi^S}^\epsilon$ and $\mathrm{H}_\Omega^{q+1}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}_\pi^S}^\epsilon$ is one-dimensional as well. One concludes that the codimension of the $\mathcal{L}$-invariant is at most one.

We now show that the codimension is exactly one by showing that there exists a nontrivial element in $\mathrm{Hom}_{\mathrm{ct}}(F_{\mathfrak{p}}^\times, \Omega)$ which is not contained in $\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon$. Consider the homomorphism $c_{\mathrm{ord}_{\mathfrak{p}}}^\epsilon$. By Lemma 3.1, $c_{\mathrm{ord}_{\mathfrak{p}}}^\epsilon = c_\infty^{q,\epsilon} \circ \kappa^*$ up to a nonzero constant, where $c_\infty^{q,\epsilon}$ denotes the homomorphism constructed in Equation (1.12). By Lemma 1.8, the homomorphism $c_\infty^{q,\epsilon}$ is an isomorphism, while κ^* is an isomorphism by Proposition 2.5. Therefore, $\mathrm{ord}_{\mathfrak{p}} \notin \mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon$. The second statement follows observing that locally constant homomorphisms in $\mathrm{Hom}_{\mathrm{ct}}(F_{\mathfrak{p}}^\times, \Omega)$ are multiples of $\mathrm{ord}_{\mathfrak{p}}$. $\square$

Remark 3.4. As in [22], one could also define automorphic $\mathcal{L}$-invariants for higher degree cohomology groups, for which its π-isotypic component does not vanish. As these $\mathcal{L}$-invariants neither seem to show up in exceptional zero formulas nor are they used to define (plectic) Darmon cycles, we will not consider them here.

4. P-adic families

For this section, we assume that F is totally real, $\pi_{\mathfrak{p}}$ is the Steinberg representation and π is noncritical at $\mathfrak{p}$. We give a formula for the automorphic $\mathcal{L}$-invariant in terms of derivatives of $U_{\mathfrak{p}}$-eigenvalues of p-adic families passing through π. Comparing with the corresponding formula for the Fontaine–Mazur $\mathcal{L}$-invariant of the corresponding Galois representation we deduce that automorphic and Fontaine–Mazur $\mathcal{L}$-invariants agree.

4.1. The weight space

Let Ω be a finite extension of E_p. Define the (partial) weight space $\mathcal{W}_{\mathfrak{p}}$ to be the rigid analytic space over Ω associated to the completed group algebra $\mathcal{O}_\Omega[\![\mathcal{O}_{\mathfrak{p}}^\times]\!]$. There is a universal character

$$\kappa^{\mathrm{un}}\colon \mathcal{O}_{\mathfrak{p}}^\times \longrightarrow (\mathcal{O}_\Omega[\![\mathcal{O}_{\mathfrak{p}}^\times]\!])^\times.$$

Let $\mathcal{U} \subseteq \mathcal{W}_{\mathfrak{p}}$ be an affinoid and $\mathcal{O}(\mathcal{U})$ be the ring of its rigid analytic functions. We will denote by $\kappa_{\mathcal{U}}^{\mathrm{un}}\colon \mathcal{O}_{\mathfrak{p}}^\times \to \mathcal{O}(\mathcal{U})^\times$ the restriction of the universal character to $\mathcal{U}$. For an affinoid $\mathcal{U} \subseteq \mathcal{W}_{\mathfrak{p}}$ and a locally analytic character $\chi\colon B \cap I_{\mathfrak{p}} \to \mathcal{O}(\mathcal{U})^\times$, recall from section 2.3 the $\mathcal{O}(\mathcal{U})[I_{\mathfrak{p}}]$-module $\mathcal{A}_\chi^n$ defined as the locally n-analytic induction of χ to $I_{\mathfrak{p}}$, and the cohomology groups $\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}_{\chi,\boldsymbol{k}^{\mathfrak{p}}}^n)$. If χ is the universal character $\kappa_{\mathcal{U}}^{\mathrm{un}}$, we simply write $\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}}^n)$ in place of $\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}_{\kappa_{\mathcal{U}}^{\mathrm{un}},\boldsymbol{k}^{\mathfrak{p}}}^n)$. If $\mathcal{U}$ contains $\boldsymbol{k}_{\mathfrak{p}}$, the evaluation $\mathcal{O}(\mathcal{U}) \to \Omega$ at $\boldsymbol{k}_{\mathfrak{p}}$ induces the map

$$\mathrm{H}^d(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}}^n) \longrightarrow \mathrm{H}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}_{\boldsymbol{k}}^n). \tag{4.1}$$

4.2. Étaleness at $\mathfrak{m}_\pi^S$

Let $\mathcal{U}$ be an admissible open affinoid in $\mathcal{W}_\mathfrak{p}$ containing $\boldsymbol{k}_\mathfrak{p}$, and let $\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}}$ be the rigid localization of $\mathcal{O}(\mathcal{U})$ at $\boldsymbol{k}_\mathfrak{p} \in \mathcal{U}$. It is the local ring defined as

$$\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}} = \varinjlim_{\boldsymbol{k}_\mathfrak{p} \in \mathcal{U}' \subset \mathcal{U}} \mathcal{O}(\mathcal{U}'),$$

where the limit is taken over all admissible open subaffinoids $\mathcal{U}'$ in $\mathcal{U}$ containing $\boldsymbol{k}_\mathfrak{p}$. Thus, it contains the algebraic localization of $\mathcal{O}(\mathcal{U})$ at the maximal ideal $\mathfrak{m}_{\boldsymbol{k}_\mathfrak{p}}$ attached to $\boldsymbol{k}_\mathfrak{p}$. By a slight abuse of notation, we write $\mathfrak{m}_\pi^S \subseteq \mathbb{T}_{K^S}^S(\mathcal{O}(\mathcal{U})) = \mathbb{T}_{K^S}^S(\Omega) \otimes_\Omega \mathcal{O}(\mathcal{U})$ for the ideal generated by $\mathfrak{m}_\pi^S \subseteq \mathbb{T}_{K^S}^S(\Omega)$ and $\mathfrak{m}_{\boldsymbol{k}_\mathfrak{p}} \subseteq \mathcal{O}(\mathcal{U})$.

The assumption that π is noncritical at $\mathfrak{p}$ has strong implications on the existence of p-adic families interpolating the system of Hecke-eigenvalues attached to π.

Theorem 4.1. *Let $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ be a character. Up to shrinking $\mathcal{U}$ to a small enough open affinoid containing $\boldsymbol{k}_\mathfrak{p}$ the following holds:*

$$\mathrm{H}^d(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, \mathcal{D}_{\mathcal{U},\boldsymbol{k}^\mathfrak{p}}^n)_{(\mathfrak{m}_\pi^S, U_\mathfrak{p}-1)}^\epsilon = 0 \quad \textit{for every } d \neq q,$$

for $d = q$ it is a free $\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}}$-module of finite rank and the map of $\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}}$-modules obtained by localizing the composition of Equation (2.3) with the map (4.1) induces an isomorphism

$$\mathrm{H}^q(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, \mathcal{D}_{\mathcal{U},\boldsymbol{k}^\mathfrak{p}}^n)_{(\mathfrak{m}_\pi^S, U_\mathfrak{p}-1)}^\epsilon \otimes_{\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}}} \mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}} / \mathfrak{m}_{\boldsymbol{k}_\mathfrak{p}} \xrightarrow{\cong} \mathrm{H}^q(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{(\mathfrak{m}_\pi^S, U_\mathfrak{p}-1)}^\epsilon.$$

Moreover, the operator $U_\mathfrak{p}^\circ$ acts on it via a scalar $\alpha_\mathfrak{p}^\epsilon \in \mathcal{O}(\mathcal{U})_{\boldsymbol{k}_\mathfrak{p}}^\times$.

Proof. Since F is totally real, the constant δ is equal to 0. Thus, Theorem 1.1 implies that

$$\mathrm{H}^d(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{(\mathfrak{m}_\pi^S, U_\mathfrak{p}-1)} = 0 \quad \text{for all } d \neq q.$$

The first claims follow using the same arguments as in the proof of [2, Theorem 2.14]. The statement about the operator $U_\mathfrak{p}^\circ$ can be deduced from the fact that

$$\mathrm{H}^q(X_{K^\mathfrak{p} \times I_\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{(\mathfrak{m}_\pi^S, U_\mathfrak{p}-1)}^\epsilon$$

is an absolutely irreducible $\mathbb{T}_{K^\mathfrak{p}}(\Omega)$-module. $\qquad\square$

Theorem 4.1 can be rephrased in more geometric terms: It implies that the map from a certain eigenvariety to weight space is étale at the point corresponding to π.

4.3. Infinitesimal deformations and $\mathcal{L}$-invariants

Let $\Omega[\varepsilon] := \Omega[X]/(X^2)$ be the Ω-algebra of dual numbers over Ω and $\pi\colon \Omega[\varepsilon] \to \Omega$ be the natural surjection sending ε to 0. If $X = \mathrm{Spec}\,(A)$ is an affine Ω-scheme and $x\colon A \to A/\mathfrak{m}_x = \Omega$ is a Ω-valued point, then the space of morphisms $\boldsymbol{v}_x\colon A \to \Omega[\varepsilon]$ such that $\pi \circ \boldsymbol{v}_x = x$ is identified with the tangent space of X at x.

Let $\mathcal{U}$ be an admissible open affinoid containing $\boldsymbol{k}_\mathfrak{p}$ and $\chi\colon B \to \mathcal{O}(\mathcal{U})^\times$ a locally analytic character, that we identify with an element of $\mathrm{Hom}(F_\mathfrak{p}^\times, \mathcal{O}(\mathcal{U})^\times)$ as in section 2.1. Let $\boldsymbol{v}\colon \mathcal{O}(\mathcal{U}) \to \Omega[\varepsilon]$ be an element of the tangent space of $\mathcal{U}$ at $\boldsymbol{k}_\mathfrak{p}$. Then the pullback $\chi_{\boldsymbol{v}} = \boldsymbol{v} \circ \chi \in \mathrm{Hom}(F_\mathfrak{p}^\times, \Omega[\varepsilon])$ of χ along $\boldsymbol{v}$ can be written in a unique way as

$$\chi_{\boldsymbol{v}} = \overline{\chi}(1 + \partial_{\boldsymbol{v}}(\chi)\varepsilon), \tag{4.2}$$

where $\overline{\chi}\colon F_{\mathfrak{p}}^{\times} \to (\mathcal{O}(\mathcal{U})/\mathfrak{m}_{k_{\mathfrak{p}}})^{\times} = \Omega^{\times}$ denotes the reduction of χ modulo $\mathfrak{m}_{k_{\mathfrak{p}}}$ and $\partial_{\nu}(\chi)$ is a homomorphism from $F_{\mathfrak{p}}^{\times}$ to Ω.

Now, assume that $\chi\colon B \to \mathcal{O}(\mathcal{U})^{\times}$ is a locally analytic character such that $\chi(\mathrm{mod}\ \mathfrak{m}_{k_{\mathfrak{p}}}) = \mathbb{1}_{k_{\mathfrak{p}}}$. Then, for an element ν in the tangent space of $\mathcal{U}$ at $k_{\mathfrak{p}}$ we have $\chi_{\nu} = \mathbb{1}_{k_{\mathfrak{p}}}(1 + \partial_{\nu}(\chi)\varepsilon)$. Consider the map induced in cohomology by the reduction of χ modulo $\mathfrak{m}_{k_{\mathfrak{p}}}$:

$$\mathrm{red}_{\chi}^{\epsilon}\colon \mathrm{H}^{q}_{\mathcal{O}(\mathcal{U}),\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\chi), V_{k^{\mathfrak{p}}}(\mathcal{O}(\mathcal{U}))^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} \longrightarrow \mathrm{H}^{q}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}}.$$

Proposition 4.2. *Let $\epsilon\colon \pi_{0}(G_{\infty}) \to \{\pm 1\}$ be a sign character. If $\mathrm{red}_{\chi}^{\epsilon}$ is surjective, then $\partial_{\nu}(\chi)$ belongs to $\mathcal{L}_{\mathfrak{p}}(\pi)^{\epsilon}$ for every element ν of the tangent space of $\mathcal{U}$ at k.*

Proof. The locally analytic character χ_{ν} of B over $\Omega[\varepsilon]$ can be seen as a two-dimensional representation $\tau_{\chi_{\nu}}$ of B over Ω. It is in fact the representation that we denoted by $\tau_{\partial_{\nu}(\chi),k_{\mathfrak{p}}}$ in Section 3.1. It follows from the discussion in Section 3.1 that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^{q}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} & \xrightarrow{\hat{c}^{\epsilon}_{\partial_{\nu}(\chi)}} & \mathrm{H}^{q+1}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} \\
\uparrow & & \downarrow \\
\mathrm{H}^{q}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} & \xrightarrow{c^{\epsilon}_{\partial_{\nu}(\chi)}} & \mathrm{H}^{q+1}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, V_{k_{\mathfrak{p}}}(\Omega), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}}
\end{array},$$

where $\hat{c}^{\epsilon}_{\partial_{\nu}(\chi)}$ and $c^{\epsilon}_{\partial_{\nu}(\chi)}$ are the boundary maps induced by the dual of the short exact sequences

$$0 \longrightarrow \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}) \longrightarrow \mathbb{I}_{B}(\tau_{\chi_{\nu}}) \longrightarrow \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}) \longrightarrow 0, \qquad (4.3)$$

and

$$0 \longrightarrow \mathrm{St}_{k_{\mathfrak{p}}}^{\mathrm{an}}(\Omega) \longrightarrow \mathcal{E}_{\partial_{\nu}(\chi),k_{\mathfrak{p}}} \longrightarrow V_{k_{\mathfrak{p}}}(\Omega) \longrightarrow 0,$$

respectively. It is sufficient to prove that $\hat{c}^{\epsilon}_{\partial_{\nu}(\chi)}$ is the zero map, which would follow from the surjectivity of the homomorphism

$$\mathrm{H}^{q}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\tau_{\chi_{\nu}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} \longrightarrow \mathrm{H}^{q}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} \qquad (4.4)$$

induced by the dual of Equation (4.3). Our assumption on the surjectivity of $\mathrm{red}_{\chi}^{\epsilon}$ immediately implies that the map

$$\mathrm{H}^{q}_{\Omega[\varepsilon],\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\chi_{\nu}), V_{k^{\mathfrak{p}}}(\Omega[\varepsilon])^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}} \longrightarrow \mathrm{H}^{q}_{\Omega,\mathrm{ct}}(X_{K^{\mathfrak{p}}}^{\mathfrak{p}}, \mathbb{I}_{B}(\mathbb{1}_{k_{\mathfrak{p}}}), V_{k^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}}$$

is surjective. The surjectivity of Equation (4.4) now follows from [25, Lemma 2.1]. $\qquad \square$

Recall that we denoted by $\alpha_{\mathfrak{p}}^{\epsilon} \in \mathcal{O}(\mathcal{U})_{k_{\mathfrak{p}}}^{\times}$ the eigenvalue of the Hecke operator $U_{\mathfrak{p}}^{\circ}$ acting on $\mathrm{H}^{q}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}_{\mathcal{U},k_{\mathfrak{p}}}^{n})^{\epsilon}_{(\mathfrak{m}_{\pi}^{S}, U_{\mathfrak{p}}-1)}$. Up to shrinking $\mathcal{U}$, we can assume that $\alpha_{\mathfrak{p}}^{\epsilon} \in \mathcal{O}(\mathcal{U})^{\times}$. Let $\chi_{\alpha_{\mathfrak{p}}^{\epsilon}}\colon B \to \mathcal{O}(\mathcal{U})^{\times}$ be the character defined by

$$\chi_{\alpha_{\mathfrak{p}}^{\epsilon}}|_{B \cap I_{\mathfrak{p}}} = \kappa_{\mathcal{U}}^{\mathrm{un}}$$
$$\chi_{\alpha_{\mathfrak{p}}^{\epsilon}}(u_{\mathfrak{p}}) = \alpha_{\mathfrak{p}}^{\epsilon}.$$

Now, we are ready to prove the main result of this section.

Theorem 4.3. *Let $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ be a sign character. For every element $\mathbf{v}$ of the tangent space of $\mathcal{U}$ at $\mathbf{k}_{\mathfrak{p}}$, we have*

$$\partial_{\mathbf{v}}(\chi_{\alpha_{\mathfrak{p}}^\epsilon}) \in \mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon.$$

Proof. By Proposition 4.2, it is enough to prove that $\mathrm{red}^\epsilon_{\chi_{\alpha_{\mathfrak{p}}^\epsilon}}$ is surjective. By Theorem 2.11, with the same arguments as in the proof of Proposition 2.13, the map in cohomology

$$\mathrm{H}^q_{\mathcal{O}(\mathcal{U}),\mathrm{ct}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\chi_{\alpha_{\mathfrak{p}}^\epsilon}), V_{\mathbf{k}^{\mathfrak{p}}}(\mathcal{O}(\mathcal{U}))^\vee)_{\mathfrak{m}^S} \longrightarrow \mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\mathcal{U},\mathbf{k}^{\mathfrak{p}}})_{(\mathfrak{m}^S_\pi, U_{\mathfrak{p}} - \alpha_{\mathfrak{p}}^\epsilon)}$$

induced by Equation (2.5) is an isomorphism. By Theorem 4.1, the reduction modulo $\mathfrak{m}_{\mathbf{k}_{\mathfrak{p}}}$ yields a surjective map

$$\mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\mathcal{U},\mathbf{k}^{\mathfrak{p}}})^\epsilon_{(\mathfrak{m}^S_\pi, U_{\mathfrak{p}} - 1)} \longrightarrow \mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, V_{\mathbf{k}}(\Omega)^\vee)^\epsilon_{(\mathfrak{m}^S_\pi, U_{\mathfrak{p}} - 1)}.$$

Furthermore, one has the isomorphisms

$$\mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, V_{\mathbf{k}}(\Omega)^\vee)^\epsilon_{(\mathfrak{m}^S_\pi, U_{\mathfrak{p}} - 1)} \cong \mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, i_B(\mathbb{1}_{\mathbf{k}_{\mathfrak{p}}}), V_{\mathbf{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi}$$
$$\cong \mathrm{H}^q_{\Omega,\mathrm{ct}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\mathbb{1}_{\mathbf{k}_{\mathfrak{p}}}), V_{\mathbf{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi},$$

where the first isomorphism can be deduced from the arguments in the proof of Proposition 2.13, and the second one follows from the noncriticality of π at $\mathfrak{p}$. Recollecting all the maps, one deduces the claim. $\square$

4.4. Relation with Galois representations

Let $\rho = \rho_\pi\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\Omega)$ be the two-dimensional Galois representation attached to π, and let $\rho_{\mathfrak{p}}$ be its restriction to a decomposition group $\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})$ at $\mathfrak{p}$. As local-global compatibility is known in this case by Saito (cf. [35]), the representation $\rho_{\mathfrak{p}}$ is semistable, noncrystalline, that is:

$$\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}}) = (\rho_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})}$$

is a free $\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p},0}$-module (where $F_{\mathfrak{p},0}$ denotes the maximal unramified subfield of $F_{\mathfrak{p}}$) and the nilpotent linear map $N_{\mathfrak{p}}$ inherited from the corresponding map on Fontaine's semistable period ring B_{st} is nonzero. Moreover, the kernel of $N_{\mathfrak{p}}$ is a free $\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p},0}$-module of rank one. It follows that there exists a basis $\{e_1, e_2\}$ of Frobenius eigenvectors such that $e_1 = N_{\mathfrak{p}}(e_2)$. Furthermore, it is known that the zeroth step of the de Rham filtration

$$\mathrm{Fil}^0(\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}})) \subseteq \mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}}) \otimes_{F_{\mathfrak{p},0}} F_{\mathfrak{p}}$$

is a free $\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}}$-module of rank one. In particular there exist $a_1^{\rho_{\mathfrak{p}}}, a_2^{\rho_{\mathfrak{p}}} \in \Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}}$ such that $\mathrm{Fil}^0(\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}}))$ is generated by $a_1^{\rho_{\mathfrak{p}}} \cdot e_1 + a_2^{\rho_{\mathfrak{p}}} \cdot e_2$.

Definition 4.4. We call the local Galois representation $\rho_{\mathfrak{p}}$ noncritical if $a_2^{\rho_{\mathfrak{p}}} \in (\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}})^\times$.

It is expected that every $\rho_{\mathfrak{p}}$ coming from a Hilbert modular form as above is noncritical. If $F_{\mathfrak{p}} = \mathbb{Q}_p$ or $k_\sigma = 0$ for all $\sigma \in \Sigma_p$, the fact that $\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}})$ is weakly admissible implies that $\rho_{\mathfrak{p}}$ is noncritical. It seems to the authors of this article that in [14, Section 3.2] noncriticality of the Galois representation is assumed implicitly. Note that the main theorem of [43] states that a Galois representation as above is noncritical if $\mathfrak{p}$ is the only prime of F above p. But similar as on page 653 of [14] an Amice–Vélu and Vishik-type argument is used in the crucial Proposition 7.3 of *loc.cit.* and it is not clear to the authors of this article, whether such an argument is applicable here (see Remark 2.6 above).

To any tuple

$$a = (a_\sigma) \in \Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}} \cong \prod_{\sigma \in \Sigma_{\mathfrak{p}}} \Omega,$$

we attach a codimension one subspace $\mathcal{L}^a \subseteq \mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega)$ as follows: Define

$$\log_\sigma = \log_p \circ \sigma \colon F_{\mathfrak{p}}^\times \to \Omega,$$

where $\log_p$ is the usual branch of the p-adic logarithm fulfilling $\log_p(p) = 0$ and put

$$\mathcal{L}^a = \langle \log_\sigma - a_\sigma \, \mathrm{ord}_p \mid \sigma \in \Sigma_{\mathfrak{p}} \rangle \subseteq \mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega),$$

where ord_p denotes the p-adic valuation of $F_{\mathfrak{p}}^\times$ fulfilling $\mathrm{ord}_p(p) = 1$. Since the elements $\log_\sigma, \sigma \in \Sigma_{\mathfrak{p}}$, together with ord_p form a basis of $\mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega)$ one deduces that $\mathcal{L}^a \subseteq \mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega)$ is a subspace of codimension one that does not contain the subspace of smooth homomorphisms.

Definition 4.5. Suppose that $\rho_{\mathfrak{p}}$ is noncritical. The Fontaine–Mazur $\mathcal{L}$-invariant of $\rho_{\mathfrak{p}}$ is the codimension one subspace

$$\mathcal{L}^{FM}(\rho_{\mathfrak{p}}) = \mathcal{L}^{a_1^{\rho_{\mathfrak{p}}}/a_2^{\rho_{\mathfrak{p}}}} \subseteq \mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega).$$

Theorem 4.6. *Suppose that π is noncritical at $\mathfrak{p}$ and that $\rho_{\mathfrak{p}}$ is noncritical. Then the equality*

$$\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon = \mathcal{L}^{FM}(\rho_{\mathfrak{p}})$$

holds for every sign character $\epsilon \colon \pi_0(G_\infty) \to \{\pm 1\}$. In particular, the automorphic $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon$ does not depend on the sign character ϵ.

Proof. This follows directly by comparing Theorem 4.3 with the corresponding formula on the Galois side (cf. [44, Theorem 1.1]) for the family of Galois representations attached to the family passing through π. See [25, Theorem 4.1] for more details in case $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. □

In [16] respectively [17], Ding proves that in case D is split at exactly one Archimedean place the Fontaine–Mazur $\mathcal{L}$-invariant can be detected by completed cohomology of the associated Shimura curve. Thus, by the theorem above the automorphic $\mathcal{L}$-invariant can also be detected by completed cohomology in that case. For the modular curve, Breuil gives a direct proof of this consequence in [10]. It would be worthwhile to explore whether Breuil's proof extends to our more general setup.

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