21

Angular derivatives of $\mathcal{H}(b)$ functions

In the previous chapter, we characterized the boundary points where functions in $\mathcal{H}(b)$ admit an analytic continuation. In this chapter, we pursue this study and we characterize boundary points where functions in $\mathcal{H}(b)$ admit an angular derivative up to a certain order.

In Section 21.1, we start by characterizing those points $\zeta \in \mathbb{T}$ such that, for all functions $f \in \mathcal{H}(b)$, the nontangential limit

$$f(\zeta) = \lim_{\substack{z \to \zeta \\ \sphericalangle}} f(z)$$

exists. As we will see, this is connected to the well-known Julia–Carathéodory theorem. In fact, we recover this result using a Hilbert space approach based on $\mathcal{H}(b)$ spaces. We also show how to deduce Julia's inequality from the Cauchy–Schwarz inequality. In Section 21.2, we study the connection between angular derivatives and Clark measures. In Section 21.3, we give a simple sufficient condition for a Blaschke product and its derivatives up to a fixed order to admit radial limits at a boundary point. Then, in Section 21.4, we generalize this result to arbitrary functions in the closed unit ball of H^∞. In Section 21.5, we study an approximation problem by Blaschke products that will be useful in our studies on boundary derivatives of functions in $\mathcal{H}(b)$.

In Section 21.6, we give some interesting formulas for the reproducing kernels of derivatives of functions in $\mathcal{H}(b)$. In Section 21.7, we establish the connection between the existence of boundary derivatives in $\mathcal{H}(B)$, where B is a Blaschke product, and an interpolation problem. In Section 21.8, we give a nice characterization for the existence of boundary derivatives for functions in $\mathcal{H}(b)$. This explicit characterization is expressed in terms of the zeros of b, the singular measure associated with b and $\log |b|$.

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21.1 Derivative in the sense of Carathéodory

In Section 3.2, we studied the angular derivative of analytic functions on the open unit disk $\mathbb{D}$. In this section we consider the smaller class of analytic functions $f : \mathbb{D} \longrightarrow \bar{\mathbb{D}}$, i.e. the elements of the closed unit ball of $H^\infty(\mathbb{D})$. We say that such a function has *angular derivative* in the sense of *Carathéodory* at $\zeta_0 \in \mathbb{T}$ if it has an angular derivative at ζ_0 and moreover $|f(\zeta_0)| = 1$. By the maximum principle, for some $z \in \mathbb{D}$, $f(z) \in \mathbb{T}$ happens only if f is a constant function of modulus one. Hence, from now on, we consider functions that map $\mathbb{D}$ into $\mathbb{D}$.

Theorem 21.1 *Let $b : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, let $\zeta \in \mathbb{T}$, and put*

$$c = \liminf_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|}.$$

Then the following are equivalent.

(i) *The constant c is finite, i.e.*

$$c < \infty.$$

(ii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{b(z) - \lambda}{z - \zeta} \in \mathcal{H}(b).$$

(iii) *For all functions $f \in \mathcal{H}(b)$,*

$$f(\zeta) = \lim_{\substack{z \to \zeta \\ \sphericalangle}} f(z)$$

exists.

(iv) *The function b has an angular derivative in the sense of Carathéodory at ζ.*

Moreover, under the preceding equivalent conditions, the following results hold.

(a) *The constant c is not zero, i.e.*

$$c > 0.$$

(b) *We have $|b(\zeta)| = 1$, $c = |b'(\zeta)|$ and*

$$b'(\zeta) = \frac{b(\zeta)}{\zeta} |b'(\zeta)|.$$

(c) *We have*

$$k_\zeta^b(z) = \frac{1 - \overline{b(\zeta)} b(z)}{1 - \bar{\zeta} z} \in \mathcal{H}(b).$$

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(d) *For each $f \in \mathcal{H}(b)$,*

$$f(\zeta) = \langle f, k_\zeta^b \rangle_b.$$

(e) *We have*

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \|k_z^b - k_\zeta^b\|_b = 0.$$

(f) *We have*

$$|b'(\zeta)| = k_\zeta^b(\zeta) = \|k_\zeta^b\|_b^2 = c.$$

(g) *We have*

$$c = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|}{1 - |z|}.$$

Proof Our plan is to show that

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(i)}$$

and then

$$\text{(i), (ii), (iii)} \implies \text{(iv)} \implies \text{(i)}.$$

The properties (a)–(g) will be obtained at different steps of the proof.

(i) $\implies$ (ii) If $c < \infty$, then there is a sequence $(z_n)_{n \geq 1}$ in $\mathbb{D}$ converging to ζ such that

$$c = \lim_{n \to \infty} \frac{1 - |b(z_n)|}{1 - |z_n|} < \infty.$$

Hence, we necessarily have $\lim_{n \to \infty} |b(z_n)| = 1$. Therefore, we can write

$$c = \lim_{n \to \infty} \frac{1 - |b(z_n)|^2}{1 - |z_n|^2}.$$

In the light of Theorem 18.11, this means that

$$c = \lim_{n \to \infty} \|k_{z_n}^b\|_b^2.$$

This is the main observation, due to Sarason, that allows us to use Hilbert space techniques. By Theorem 1.27, $(k_{z_n}^b)_{n \geq 1}$ has a weakly convergent subsequence in $\mathcal{H}(b)$. Since $(b(z_n))_{n \geq 1}$ is bounded, it also has a convergent subsequence in the closed unit disk. Hence, replacing $(z_n)_{n \geq 1}$ by a subsequence if

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needed, we assume there are $\lambda \in \bar{\mathbb{D}}$ and $k \in \mathcal{H}(b)$ such that $b(z_n) \longrightarrow \lambda$ and that $k_{z_n}^b \xrightarrow{w} k$. Therefore, for each $z \in \mathbb{D}$,

$$\begin{aligned} k(z) &= \langle k, k_z^b \rangle_b \\ &= \lim_{n\to\infty} \langle k_{z_n}^b, k_z^b \rangle_b \\ &= \lim_{n\to\infty} k_{z_n}^b(z) \\ &= \lim_{n\to\infty} \frac{1-\overline{b(z_n)}b(z)}{1-\bar{z}_n z} \\ &= \frac{1-\bar{\lambda}b(z)}{1-\bar{\zeta}z}. \end{aligned}$$

Since $k \in H^2(\mathbb{D})$ and $1/(1-\bar{\zeta}\,z) \notin H^2(\mathbb{D})$, we must have $|\lambda| = 1$ and thus

$$\lambda\bar{\zeta}k(z) = \frac{b(z)-\lambda}{z-\zeta} \in \mathcal{H}(b).$$

Clearly $k \not\equiv 0$ and, by (1.30), the condition $k_{z_n}^b \xrightarrow{w} k$ implies that

$$0 < \|k\|_b^2 \le \liminf_{n\to\infty} \|k_{z_n}^b\|_b^2 = c. \tag{21.1}$$

This also establishes part (a).

(ii) $\Longleftrightarrow$ (iii) By assumption, $k \in \mathcal{H}(b)$. Hence,

$$b(z) = \lambda + \lambda\bar{\zeta}(z-\zeta)k(z),$$

which, by (4.15) and the fact that $k \in H^2(\mathbb{D})$, implies that

$$|b(z)-\lambda| \le |z-\zeta|\,\|k\|_2\,\|k_z\|_2 = \|k\|_2\,\frac{|z-\zeta|}{(1-|z|^2)^{1/2}}.$$

Thus, if $z \in S_C(\zeta)$, we have

$$|b(z)-\lambda| \le C\,\|k\|_2(1-|z|^2)^{1/2},$$

and the last quantity tends to zero when z tends to ζ from within $S_C(\zeta)$. Therefore,

$$\lim_{\substack{z\to\zeta \\ \sphericalangle}} b(z) = \lambda.$$

Let us write $b(\zeta)$ for λ, and k_ζ^b for k, i.e.

$$k_\zeta^b(z) = \frac{1-\overline{b(\zeta)}b(z)}{1-\bar{\zeta}z}.$$

With the new notation, we have $k_\zeta^b \in \mathcal{H}(b)$. This is part (c). We also have

$$k_\zeta^b(z) = \langle k_\zeta^b, k_z^b \rangle_b \qquad (z \in \mathbb{D}).$$

Moreover, by the Cauchy–Schwarz inequality,

$$|k_\zeta^b(z)| \le \|k_\zeta^b\|_b \, \|k_z^b\|_b.$$

Since

$$|k_\zeta^b(z)| = \frac{|1-\overline{b(\zeta)}b(z)|}{|1-\bar{\zeta}z|} \ge \frac{1-|b(z)|}{|z-\zeta|} = \frac{(1-|z|^2)\,\|k_z^b\|_b^2}{(1+|b(z)|)\,|z-\zeta|},$$

the preceding two inequalities imply that

$$\|k_z^b\|_b \le \|k_\zeta^b\|_b \frac{1+|b(z)|}{1+|z|} \, \frac{|z-\zeta|}{1-|z|}.$$

Hence, in each Stolz domain $S_C(\zeta)$,

$$\|k_z^b\|_b \le 2C\, \|k_\zeta^b\|_b \qquad (z \in S_C(\zeta)). \tag{21.2}$$

This inequality means that $\|k_z^b\|_b$ stays bounded as z tends nontangentially to ζ. This fact is exploited below.

For each fixed $w \in \mathbb{D}$,

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} k_z^b(w) = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1-\overline{b(z)}b(w)}{1-\bar{z}w} = \frac{1-\overline{b(\zeta)}b(w)}{1-\bar{\zeta}w} = k_\zeta^b(w).$$

We can rewrite this relation in the form

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \langle k_z^b, k_w^b \rangle_b = \langle k_\zeta^b, k_w^b \rangle_b.$$

Therefore,

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \langle f, k_z^b \rangle_b = \langle f, k_\zeta^b \rangle_b, \tag{21.3}$$

where $f \in \mathcal{H}(b)$ is any element of the form $f = \alpha_1 k_{w_1}^b + \cdots + \alpha_n k_{w_n}^b$. But the collection of such elements is dense in $\mathcal{H}(b)$, and thus, by (21.2), the identity (21.3) holds for all $f \in \mathcal{H}(b)$. At the same time, (21.3) shows that

$$f(\zeta) = \lim_{\substack{z \to \zeta \\ \sphericalangle}} f(z) = \langle f, k_\zeta^b \rangle_b \qquad (f \in \mathcal{H}(b)).$$

This is part (d). In particular, with $f = k_\zeta^b$, we obtain $k_\zeta^b(\zeta) = \|k_\zeta^b\|_b^2$. This is partially part (f). The relation (21.3) also implies that $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ.

(iii) $\Longrightarrow$ (i) Fix any Stolz domain $S_C(\zeta)$. Consider k_z^b as an element of the dual space of $\mathcal{H}(b)$. Then the relation $f(z) = \langle f, k_z^b \rangle_b$ along with our assumption imply that

$$\sup_{z \in S_C(\zeta)} |\langle f, k_z^b \rangle_b| = C(f) < \infty.$$

Thus, by the uniform boundedness principle,

$$C' = \sup_{z \in S_C(\zeta)} \|k_z^b\|_b < \infty.$$

Take $z_n = (1 - 1/n)\zeta$, $n \geq 1$. Since $z_n \in S_C(\zeta)$ for sufficiently large n, we have

$$\frac{1 - |b(z_n)|^2}{1 - |z_n|^2} = \|k_{z_n}^b\|_b^2 \leq C'^2 \qquad (n \geq N),$$

which implies in particular that $\lim_{n \to \infty} |b(z_n)| = 1$. Moreover,

$$c \leq \liminf_{n \to \infty} \frac{1 - |b(z_n)|^2}{1 - |z_n|^2} = \liminf_{n \to \infty} \|k_{z_n}^b\|_b^2 \leq C'^2.$$

(i), (ii), (iii) $\Longrightarrow$ (iv) Since $k_\zeta^b \in \mathcal{H}(b)$ we have

$$\frac{b(z) - b(\zeta)}{z - \zeta} = \frac{k_\zeta^b(z) b(\zeta)}{\zeta} = \langle k_\zeta^b, k_z^b \rangle_b \frac{b(\zeta)}{\zeta} \qquad (z \in \mathbb{D}).$$

On the other hand, we know that $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ. Hence,

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{b(z) - b(\zeta)}{z - \zeta} = \|k_\zeta^b\|_b^2 \frac{b(\zeta)}{\zeta},$$

which, by Theorem 3.1, means that

$$b'(\zeta) = \|k_\zeta^b\|_b^2 \frac{b(\zeta)}{\zeta}. \tag{21.4}$$

Thus, $|b'(\zeta)| = \|k_\zeta^b\|_b$. This is partially part (f).

By (21.1), $c \geq \|k_\zeta^b\|_b^2$. To show the reverse inequality, we prove that

$$\|k_z^b\|_b \longrightarrow \|k_\zeta^b\|_b$$

as z tends nontangentially to ζ. This fact has three consequences. First, it implies $c \leq \|k_\zeta^b\|_b^2$, and thus we indeed have $c = \|k_\zeta^b\|_b^2$. This is partially part (f). Second, since $k_z^b \xrightarrow{w} k_\zeta^b$, as z tends nontangentially to ζ, we have $\|k_z^b - k_\zeta^b\|_b \longrightarrow 0$. This is part (e). Third,

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|}{1 - |z|} = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|^2}{1 - |z|^2} = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \|k_z^b\|_b^2 = \|k_\zeta^b\|_b^2 = c.$$

This is part (g).

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To prove that $\|k_z^b\|_b \longrightarrow \|k_\zeta^b\|_b$, as z tends nontangentially to ζ, let

$$g(z) = \frac{b(z) - b(\zeta)}{z - \zeta} - b'(\zeta) \qquad (z \in \mathbb{D}).$$

Thus

$$b(z) = b(\zeta) + b'(\zeta)(z - \zeta) + (z - \zeta)g(z) \qquad (z \in \mathbb{D})$$

and, by (21.4),

$$|b(z)|^2 = 1 - 2\,\|k_\zeta^b\|_b^2\,\Re(1 - \bar{\zeta}z) + h(z) \qquad (z \in \mathbb{D}),$$

where

$$\begin{aligned} h(z) &= (|b'(\zeta)|^2 + |g(z)|^2)\,|z - \zeta|^2 \\ &\quad + 2\,\Re\Big(g(z)(z - \zeta)\overline{(b(\zeta) + b'(\zeta)(z - \zeta))} \Big). \end{aligned}$$

The only important fact about h that we need is that

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{h(z)}{1 - |z|} = 0.$$

It is also elementary to verify that

$$\frac{\Re(1 - \bar{\zeta}z)}{1 - |z|^2} = \frac{1}{2} + \frac{1}{2}\frac{|z - \zeta|^2}{1 - |z|^2},$$

which immediately gives

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{\Re(1 - \bar{\zeta}z)}{1 - |z|^2} = \frac{1}{2}.$$

Therefore,

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \|k_z^b\|_b^2 = \lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1 - |b(z)|^2}{1 - |z|^2} = \|k_\zeta^b\|_b^2.$$

(iv) $\Longrightarrow$ (i) If b has an angular derivative in the sense of Carathéodory at ζ, then the inequality

$$\frac{1 - |b(r\zeta)|}{1 - r} \le \left| \frac{b(r\zeta) - b(\zeta)}{r\zeta - \zeta} \right|$$

implies that

$$c = \liminf_{z \to \zeta} \frac{1 - |b(z)|}{1 - |z|} \le \lim_{r \to 1} \left| \frac{b(r\zeta) - b(\zeta)}{r\zeta - \zeta} \right| = |b'(\zeta)| < \infty. \qquad \square$$

Remark 21.2 It is trivial to see that, if

$$d = \sup_{r < 1} \frac{1 - |b(r\zeta)|}{1 - r} < \infty,$$

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then the quantity

$$c = \liminf_{z \to \zeta} \frac{1-|b(z)|}{1-|z|}$$

is finite. The converse is also true. Indeed, if $c < \infty$, then, by Theorem 21.1, we know that k_z^b tends to k_ζ^b in norm as $z \longrightarrow \zeta$ nontangentially. In particular, we have that $\|k_{r\zeta}^b\|_b \longrightarrow \|k_\zeta^b\|_b$ as $r \longrightarrow 1$. Hence, the norms $\|k_{r\zeta}^b\|_b$ are uniformly bounded with respect to r, which precisely means that $d < \infty$.

Corollary 21.3 *Let $b_1, b_2 : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, let $\zeta \in \mathbb{T}$, and assume that b_1 and b_2 have angular derivatives in the sense of Carathéodory at ζ. Then $b = b_1 b_2$ also has an angular derivative in the sense of Carathéodory at ζ and, moreover,*

$$|b'(\zeta)| = |b_1'(\zeta)| + |b_2'(\zeta)|.$$

Proof In the proof, we repeatedly appeal to several parts of Theorem 21.1. Since b_1 and b_2 have unimodular nontangential limits at ζ, then so does b. Write

$$\frac{1-|b(z)|}{1-|z|} = \frac{1-|b_1(z)|}{1-|z|} + |b_1(z)| \frac{1-|b_2(z)|}{1-|z|}.$$

Upon letting $z = r\zeta \longrightarrow \zeta$, the result follows. □

According to Theorem 21.1, the condition

$$\lim_{\substack{z \to \zeta \\ \sphericalangle}} \frac{1-|b(z)|}{1-|z|} < \infty \tag{21.5}$$

is equivalent to $(b(z)-\lambda)/(z-\zeta) \in \mathcal{H}(b)$ for some $\lambda \in \mathbb{T}$. Knowing this fact, one may naturally wonder if the condition

$$\frac{b(z)-\lambda}{z-\zeta} \in H^2$$

is still strong enough to imply (21.5). The following example provides a negative answer. Fix a number $p \in (1/2, 2/3)$ and let

$$b(z) = 1 - 2^{-p}(1-z)^p.$$

It is clear that b has the nontangential limit 1 at the point $\zeta = 1$ and, due to the assumption $p > 1/2$, that

$$\frac{1-b(z)}{1-z} \in H^2.$$

Moreover, since

$$\frac{1-b(r)}{1-r} = 2^{-p}(1-r)^{p-1}$$

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and thus the quotient tends to ∞ as $r \longrightarrow 1$, the condition (21.5) fails for b. It just remains to show that b is in the closed unit ball of H^∞. For that, it suffices to show that the mapping $z \longmapsto z^p$ (where we take the principal branch) sends the disk $|z-1/2| \le 1/2$ into the disk $|z-1| \le 1$. To verify this fact, note that the boundary of the disk $|z-1/2| \le 1/2$ is parameterized by

$$\begin{array}{rcl} [-\pi/2, \pi/2] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & e^{it}\cos(t). \end{array}$$

Hence, its image under the mapping $z \longmapsto z^p$ is given by

$$\begin{array}{rcl} [-\pi/2, \pi/2] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & e^{ipt}\cos^p(t). \end{array}$$

Therefore, we need to verify that

$$[1-\cos^p(t)\cos(pt)]^2 + [\cos^p(t)\sin(pt)]^2 \le 1 \qquad (0 \le t \le \pi/2).$$

This can be rewritten as

$$\cos^p(t) \le 2\cos(pt) \qquad (0 \le t \le \pi/2),$$

which is an elementary inequality. The assumption $p < 2/3$ is exploited here.

However, despite the above example for the general case, whenever $b = \Theta$ is an inner function, then the assumption

$$\frac{\Theta(z)-\lambda}{z-\zeta} \in H^2, \tag{21.6}$$

where $\lambda \in \mathbb{T}$, is enough to ensure (21.5). In fact, by (4.15), we easily see that condition (21.6) implies that $\Theta(z)$ tends to λ as z nontangentially tends to ζ. Hence, we can write $\lambda = \Theta(\zeta)$ and

$$\frac{\Theta(z)-\lambda}{z-\lambda} = \frac{\Theta(z)-\Theta(\zeta)}{z-\zeta} = \frac{\Theta(\zeta)}{\zeta}\,\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z}$$

or equivalently

$$\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z} = \zeta\,\overline{\Theta(\zeta)}\,\frac{\Theta(z)-\lambda}{z-\lambda} \in H^2.$$

For almost all $z \in \mathbb{T}$, we also have

$$\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z} = \bar{z}\,\Theta(z)\overline{\left(\frac{\Theta(z)-\Theta(\zeta)}{z-\zeta}\right)}.$$

Therefore, the function

$$\frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z}$$

actually belongs to $K_\Theta = H^2 \cap \Theta\overline{H_0^2}$. We can now apply Theorem 21.1 (implication (ii) $\Longrightarrow$ (i)) to conclude that Θ satisfies the condition (21.5).

A function that has an angular derivative in the sense of Carathéodory has an interesting geometrical property, which was discovered by Julia.

Theorem 21.4 *Let $b : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic, and let $\zeta \in \mathbb{T}$. Suppose that b has an angular derivative in the sense of Carathéodory at ζ. Then*

$$\frac{|b(z) - b(\zeta)|^2}{1 - |b(z)|^2} \le |b'(\zeta)| \frac{|z - \zeta|^2}{1 - |z|^2} \qquad (z \in \mathbb{D}).$$

Moreover, the equality holds if and only if b is a Möbius transformation.

Proof By the Cauchy–Schwarz inequality,

$$|\langle k_\zeta^b, k_z^b \rangle_b|^2 \le \|k_\zeta^b\|_b^2 \, \|k_z^b\|_b^2.$$

But, by Theorem 21.1, this is exactly the required inequality. To see when equality holds, note that Julia's inequality can be rewritten as

$$\Re\left(\frac{z + \zeta}{z - \zeta} - c \, \frac{b(z) + b(\zeta)}{b(z) - b(\zeta)} \right) \ge 0,$$

where $c = |b'(\zeta)|$. A positive harmonic function either identically vanishes or has no zeros. Hence, if equality holds even at one point inside $\mathbb{D}$, then we must have

$$\Re\left(\frac{z + \zeta}{z - \zeta} - c \, \frac{b(z) + b(\zeta)}{b(z) - b(\zeta)} \right) = 0 \qquad (z \in \mathbb{D}).$$

Therefore, we have

$$\frac{z + \zeta}{z - \zeta} - c \, \frac{b(z) + b(\zeta)}{b(z) - b(\zeta)} = i\gamma \qquad (z \in \mathbb{D}),$$

where $\gamma \in \mathbb{R}$. This identity shows that b is a Möbius transformation. That the equality holds for a Möbius transformation is easy to verify directly. □

Julia's inequality has a geometrical interpretation. The relation

$$\frac{|1 - z|^2}{1 - |z|^2} \le r \quad \Longleftrightarrow \quad \left| z - \frac{1}{1 + r} \right| \le \left(\frac{r}{1 + r} \right)^2$$

reveals that the set

$$\left\{ z \in \mathbb{C} : \frac{|z - \zeta|^2}{1 - |z|^2} \le r \right\}$$

is a disk of radius $r/(1+r)$ in $\mathbb{D}$ whose center is on the ray $[0, \zeta]$ and is tangent to the unit circle $\mathbb{T}$ at the point ζ. Julia's inequality say that this disk is mapped into a similar disk of radius $rc/(1 + rc)$ that is tangent to $\mathbb{T}$ at the point $b(\zeta)$; see Figure 21.1.

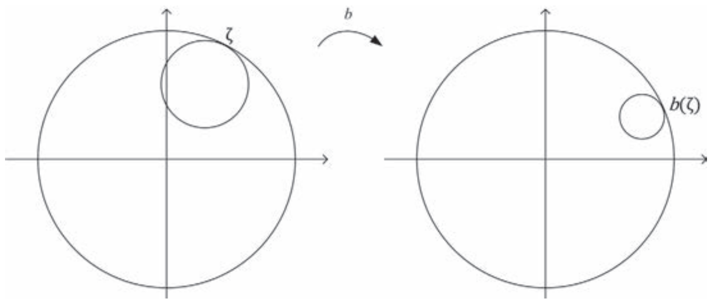

Figure 21.1 The geometric interpretation of Julia's inequality.

Exercises

Exercise 21.1.1 Let b be a function in the unit ball of H^∞ that is not the identity and not a constant. We say that a point $z_0 \in \bar{\mathbb{D}}$ is a fixed point of b if

$$\lim_{r \to 1^-} b(rz_0) = z_0.$$

Furthermore, a fixed point z_0 of b will be called a Denjoy–Wolff point of b if either $z_0 \in \mathbb{D}$ or $z_0 \in \mathbb{T}$ and b has an angular derivative at z_0 satisfying $b'(z_0) \le 1$.

(i) Show that b can have at most one fixed point in $\mathbb{D}$ and furthermore that $|b'(z)| \le 1$ at such a point.
Hint: Use the fact that we have equality in the Schwarz–Pick inequality if and only if b is a Möbius transformation.

(ii) Show that, if $\|b\|_\infty < 1$, then b does have a fixed point in $\mathbb{D}$.
Hint: Apply Rouché's theorem.

(iii) In this exercise, we would like to prove that b has at most one Denjoy–Wolff point.

(a) Assume first that b has two distinct Denjoy–Wolff points $z_0 \in \mathbb{T}$ and $z_1 \in \mathbb{D}$.

(1) Show that $k_{z_0}^b$ and $k_{z_1}^b$ are linearly dependent.
Hint: Show that

$$\det \begin{pmatrix} k_{z_0}^b(z_0) & k_{z_0}^b(z_1) \\ k_{z_1}^b(z_0) & k_{z_1}^b(z_1) \end{pmatrix} = b'(z_0) - 1.$$

(2) Prove that

$$\frac{1 - \bar{z}_1 b(z)}{1 - \bar{z}_1 z} = \frac{1 - \bar{z}_0 b(z)}{1 - \bar{z}_0 z}$$

and conclude that b is the identity function, which is a contradiction with the hypothesis.

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(b) Assume now that b has two distinct Denjoy–Wolff points $z_0, z_1 \in \mathbb{T}$. Argue as before also to get a contradiction.

(c) Conclude that b has at most one Denjoy–Wolff point.

(iv) In this exercise, we would like to prove that b has a unique Denjoy–Wolff point. Assume that b has no fixed point in $\mathbb{D}$. For $0 < r < 1$, let z_r be the fixed point in $\mathbb{D}$ of the function rb (note that $\|rb\|_\infty < 1$).

(a) Show that there exists a sequence $z_n = z_{r_n}$ that converges to a point $z_0 \in \mathbb{T}$.

(b) Show that b has an angular derivative in the sense of Carathéodory at z_0 with $0 < b'(z_0) \le 1$.
Hint: Apply Theorem 21.1.

(c) Show that z_0 is a Denjoy–Wolff point of b.

(d) Conclude.

Exercise 21.1.2 Let b be a function in the unit ball of H^∞ that is not the identity and not a constant. Let $z_0, z_1, \dots, z_n$ be distinct fixed points of b in $\bar{\mathbb{D}}$. Assume that z_0 is the Denjoy–Wolff point of b and assume that b has an angular derivative at $z_1, \dots, z_n$.

(i) Justify that necessarily $b'(z_j) > 1$, $j = 1, \dots, n$.

(ii) Assume that $z_0 = 0$ and $|b'(z_0)| \le 1$. Show that

$$\sum_{j=1}^{n} \frac{1}{b'(z_j) - 1} \le \Re\left(\frac{1 + b'(0)}{1 - b'(0)}\right). \tag{21.7}$$

Hint: Define $B(z) = b(z)/z$, $z \in \mathbb{D} \setminus \{0\}$ and $B(0) = b'(0)$. For $j \ge 1$, the functions $k_{z_j}^B$ are mutually orthogonal in $\mathcal{H}(B)$. Also verify that $\|k_{z_j}^B\|_B^2 = b'(z_j) - 1$, $\|k_{z_0}^B\|_B^2 = 1 - |b'(0)|^2$ and $\langle k_{z_0}^B, k_{z_j}^B\rangle_B = 1 - \overline{b'(0)}$. Then apply Bessel's inequality.

(iii) Assume that $z_0 = 1$ and $b'(1) < 1$. Show that

$$\sum_{j=1}^{n} \frac{1}{b'(z_j) - 1} \le \frac{b'(1)}{1 - b'(1)}. \tag{21.8}$$

Hint: Note that all the fixed points are on $\mathbb{T}$ and $\|k_{z_j}^b\|_b^2 = b'(z_j)$. Also check that, for $j \ne \ell$, $\langle k_{z_j}^b, k_{z_\ell}^b\rangle_b = 1$. Denote by $G(k_{z_0}^b, \dots, k_{z_n}^b)$ the determinant of the Gram matrix whose (i,j) entry equals $\langle k_{z_i}^b, k_{z_j}^b\rangle_b$. Then show that

$$\begin{aligned} &G(k_{z_0}^b, \dots, k_{z_n}^b) \\ &\quad = (1 - b'(1)) \prod_{i=1}^{n} (b'(z_i) - 1) \left(\frac{b'(1)}{1 - b'(1)} - \sum_{j=1}^{n} \frac{1}{b'(z_j) - 1} \right). \end{aligned}$$

(iv) Assume that $z_0 = 1$ and $b'(1) = 1$. Show that

$$\sum_{j=1}^{n} \frac{|1-z_j|^2}{b'(z_j)-1} \le 2\,\Re\left(\frac{1}{b(0)} - 1\right). \tag{21.9}$$

Hint: Note that

$$G(k_0^b, k_{z_0}^b, \dots, k_{z_n}^b) = |b(0)|^2 \prod_{i=1}^{n}(b'(z_i)-1)\left(2\,\Re\left(\frac{1}{b(0)}-1\right) - \sum_{j=1}^{n}\frac{|1-z_j|^2}{b'(z_j)-1}\right).$$

21.2 Angular derivatives and Clark measures

In this section, we explore the connection between the angular derivative in the sense of Carathéodory and the Clark measures μ_λ, which were introduced in Section 13.7. See also Section 20.11.

Theorem 21.5 *The function b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$ if and only if there is a point $\lambda \in \mathbb{T}$ such that the Clark measure μ_λ has an atom at z_0. In that case, we necessarily have $\lambda = b(z_0)$ and $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$.*

Proof According to Theorem 21.1, b has an angular derivative in the sense of Carathéodory at z_0 if and only if there is $\lambda \in \mathbb{T}$ such that

$$\frac{b(z)-\lambda}{z-\zeta} \in \mathcal{H}(b).$$

Since $\mathcal{H}(b) = \mathcal{H}(\bar\lambda b)$, we can say that b has an angular derivative in the sense of Carathéodory at z_0 if and only if there is $\lambda \in \mathbb{T}$ such that

$$\frac{\bar\lambda b(z)-1}{z-\zeta} \in \mathcal{H}(\bar\lambda b).$$

By Corollary 20.29, this happens if and only if $\mu_\lambda(\{z_0\}) > 0$. Under the above conditions, Theorem 21.1 also says that $\lambda = b(z_0)$.

It remains to show that $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$. Put

$$g_r = (1-r)k_{rz_0} \qquad (0<r<1).$$

For each $\zeta \in \mathbb{T}$, we have

$$|g_r(\zeta)| = \frac{1-r}{|1-r\bar z_0 \zeta|} \le 1 \qquad (0<r<1).$$

Moreover,

$$g_r(z_0) = \frac{1-r}{1-r\bar z_0 z_0} = 1,$$

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while, for each $\zeta \in \mathbb{T} \setminus \{z_0\}$,

$$\lim_{r\to 1} |g_r(\zeta)| = \lim_{r\to 1} \frac{1-r}{|1-r\bar{z}_0\zeta|} = 0.$$

In short, we can write

$$\lim_{r\to 1} |g_r(\zeta)| = \chi_{\{z_0\}}(\zeta) \qquad (\zeta \in \mathbb{T}).$$

Hence, by the dominated convergence theorem,

$$\lim_{r\to 1} \|g_r\|^2_{L^2(\mu)} = \lim_{r\to 1} \int_{\mathbb{T}} |g_r(\zeta)|^2 \, d\mu_\lambda(\zeta) = \int_{\mathbb{T}} \chi_{\{z_0\}}(\zeta) \, d\mu_\lambda(\zeta) = \mu_\lambda(\zeta). \tag{21.10}$$

But, according to Theorem 20.5, the mapping $\mathbf{V}_{\bar{\lambda}b}$ is a partial isometry from $L^2(\mu_\lambda)$ onto $\mathcal{H}(\bar{\lambda}b)$. Moreover,

$$\begin{aligned}
\mathbf{V}_{\bar{\lambda}b} g_r &= (1-r)\mathbf{V}_{\bar{\lambda}b} k_{rz_0} \\
&= (1-r)\frac{k^{\bar{\lambda}b}_{rz_0}}{1-\overline{\bar{\lambda}b(rz_0)}} \\
&= \frac{1-r}{1-\lambda\overline{b(rz_0)}} \, \frac{1-\lambda\overline{b(rz_0)}\bar{\lambda}b}{1-r\bar{z}_0 z} \\
&= \frac{1-r}{1-\lambda\overline{b(rz_0)}} k^b_{rz_0}.
\end{aligned}$$

Remember that $\mathcal{H}(\bar{\lambda}b) = \mathcal{H}(b)$ and that $\lambda = b(z_0)$. Hence,

$$\begin{aligned}
\|\mathbf{V}_{\bar{\lambda}b} g_r\|^2_{\bar{\lambda}b} &= \|\mathbf{V}_{\bar{\lambda}b} g_r\|^2_b \\
&= \frac{(1-r)^2}{|1-\lambda\overline{b(rz_0)}|^2} \|k^b_{rz_0}\|^2_b \\
&= \left| \frac{rz_0 - z_0}{b(rz_0)-b(z_0)} \right|^2 \frac{1-|b(rz_0)|^2}{1-r^2}.
\end{aligned}$$

Theorem 21.1 ensures that $|b'(z_0)| > 0$ and, by (21.10), that

$$\begin{aligned}
\mu_\lambda(\zeta) &= \lim_{r\to 1} \|g_r\|^2_{L^2(\mu)} \\
&= \lim_{r\to 1} \|\mathbf{V}_{\bar{\lambda}b} g_r\|^2_{\bar{\lambda}b} \\
&= |b'(z_0)|^{-2} \, |b'(z_0)| = 1/|b'(z_0)|.
\end{aligned}$$

This completes the proof. □

Assuming that b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$, then it follows immediately from Theorem 21.5 that the measure μ_λ, where $\lambda \in \mathbb{T}$, has an atom at z_0 if and only if $\lambda = b(z_0)$. Now, we show that, for other values of λ, a certain integrability condition holds.

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Theorem 21.6 *Assume that b has an angular derivative in the sense of Carathéodory at the point $z_0 \in \mathbb{T}$. Then, for every $\lambda \in \mathbb{T} \setminus \{b(z_0)\}$, we have*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2} \, d\mu_\lambda(e^{i\theta}) = \frac{|b'(z_0)|}{|\lambda - b(z_0)|^2}.$$

Proof Put $h_r = r z_0 k_{r z_0}$, where $0 < r < 1$. Hence, for each fixed $\zeta \in \mathbb{T}$,

$$|h_r(\zeta)|^2 = \frac{r^2}{|1 - r\bar{z}_0\zeta|^2} = \frac{r^2}{1 + r^2 - 2rC},$$

where $C = \Re(\bar{z}_0\zeta)$. A simple computation shows that

$$\frac{d}{dr}|h_r(\zeta)|^2 = \frac{2r(1 - rC)}{(1 + r^2 - 2rC)^2} \geq 0.$$

Therefore, the mapping $r \longmapsto |h_r(\zeta)|^2$ is increasing. Knowing this fact, by the monotone convergence theorem, we deduce that

$$\begin{aligned}
\lim_{r\to 1} \|h_r\|^2_{L^2(\mu_\lambda)} &= \lim_{r\to 1} \int_{\mathbb{T}} |h_r(\zeta)|^2 \, d\mu_\lambda(\zeta) \\
&= \int_{\mathbb{T}} \lim_{r\to 1} |h_r(\zeta)|^2 \, d\mu_\lambda(\zeta) \\
&= \int_{\mathbb{T}} \frac{1}{|1 - \bar{z}_0\zeta|^2} \, d\mu_\lambda(\zeta) \\
&= \int_{\mathbb{T}} \frac{1}{|e^{i\theta} - z_0|^2} \, d\mu_\lambda(e^{i\theta}).
\end{aligned}$$

Now, we use the same techniques as in the proof of Theorem 21.5. According to Theorem 20.5, the mapping $\mathbf{V}_{\bar{\lambda}b}$ is a partial isometry from $L^2(\mu_\lambda)$ onto $\mathcal{H}(\bar{\lambda}b)$. Moreover,

$$\begin{aligned}
\mathbf{V}_{\bar{\lambda}b} h_r &= r z_0 \mathbf{V}_{\bar{\lambda}b} k_{r z_0} \\
&= r z_0 \frac{k^{\bar{\lambda}b}_{r z_0}}{1 - \overline{\bar{\lambda}b(r z_0)}} \\
&= \frac{r z_0}{1 - \lambda\overline{b(r z_0)}} \, \frac{1 - \lambda\overline{b(r z_0)}\bar{\lambda}b}{1 - r\bar{z}_0 z} \\
&= \frac{r z_0}{1 - \lambda\overline{b(r z_0)}} \, k^b_{r z_0}.
\end{aligned} \tag{21.11}$$

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Remember that $\mathcal{H}(\bar{\lambda}b) = \mathcal{H}(b)$. Hence,

$$\begin{aligned}\|h_r\|^2_{L^2(\mu_\lambda)} &= \|\mathbf{V}_{\bar{\lambda}b} h_r\|^2_{\bar{\lambda}b} \\ &= \|\mathbf{V}_{\bar{\lambda}b} h_r\|^2_b \\ &= \frac{r^2}{|1 - \lambda\overline{b(rz_0)}|^2}\, \|k^b_{rz_0}\|^2_b \\ &= \frac{r^2}{|\lambda - b(rz_0)|^2}\, \|k^b_{rz_0}\|^2_b.\end{aligned}$$

Theorem 21.1 ensures that $|b'(z_0)| = \lim_{r\to 1} \|k^b_{rz_0}\|^2_b$. Moreover, if b has a derivative in the sense of Carathéodory at z_0, it surely has a radial limit at this point too. Thus,

$$\lim_{r\to 1} |\lambda - b(rz_0)| = |\lambda - b(z_0)| \qquad (\lambda \in \mathbb{T} \setminus \{b(z_0)\}).$$

Therefore, we finally deduce that

$$\lim_{r\to 1} \|h_r\|^2_{L^2(\mu_\lambda)} = \frac{|b'(z_0)|}{|\lambda - b(z_0)|^2}. \qquad \square$$

Theorem 21.7 *Let $z_0 \in \mathbb{T}$. Suppose that there exists a point $\lambda \in \mathbb{T}$ such that*

$$\int_{\mathbb{T}} |e^{i\theta} - z_0|^{-2}\, d\mu_\lambda(e^{i\theta}) < \infty.$$

Then b has an angular derivative in the sense of Carathéodory at z_0.

Proof As in the proof of Theorem 21.6, put $h_r = rz_0 k_{rz_0}$. Then, by (21.11),

$$rz_0 k^b_{rz_0} = (1 - \lambda\overline{b(rz_0)})^2 V_{\bar{\lambda}b} h_r.$$

Hence,

$$\begin{aligned} r^2\|k^b_{rz_0}\|^2_b &= |1 - \lambda\overline{b(rz_0)}|^2\, \|V_{\bar{\lambda}b} h_r\|^2_b \\ &= |\lambda - b(rz_0)|^2\, \|h_r\|^2_{L^2(\mu_\lambda)} \\ &= |\lambda - b(rz_0)|^2 \int_{\mathbb{T}} \frac{r^2}{|1 - r\bar{z}_0\zeta|^2}\, d\mu_\lambda(\zeta). \end{aligned} \tag{21.12}$$

In the proof of Theorem 21.6, we also saw that the mapping $r \longmapsto |h_r(\zeta)|^2$ is increasing. Therefore, since, by assumption,

$$\int_{\mathbb{T}} \frac{d\mu_\lambda(\zeta)}{|\zeta - z_0|^2} < \infty,$$

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by the monotone convergence theorem, we deduce that

$$\sup_{0\le r<1} \int_{\mathbb{T}} \frac{r^2}{|1 - r\bar{z}_0\zeta|^2}\, d\mu_\lambda(\zeta) < \infty.$$

Since $|b(rz_0) - \lambda| \le 2$, by (21.12), the above growth restriction actually implies that

$$\sup_{0\le r<1} \|k^b_{rz_0}\|^2_b < \infty.$$

In particular,

$$\liminf_{z\to z_0} \|k^b_z\|^2_b \sup_{0\le r<1} \|k^b_{rz_0}\|^2_b < \infty.$$

But a simple computation shows that

$$\|k^b_z\|^2_b = \frac{1-|b(z)|^2}{1-|z|^2} \ge \frac{1}{2}\,\frac{1-|b(z)|}{1-|z|}.$$

Hence, we can say

$$\liminf_{z\to z_0} \frac{1-|b(z)|}{1-|z|} < \infty.$$

Therefore, by Theorem 21.1, b has an angular derivative in the sense of Carathéodory at z_0. □

21.3 Derivatives of Blaschke products

Let $(a_n)_{n\ge1}$ be a Blaschke sequence in $\mathbb{D}$, and let B be the corresponding Blaschke product. Fix a point ζ on the boundary $\mathbb{T}$. If ζ is not an accumulation point of the sequence $(a_n)_{n\ge1}$, then B is actually analytic at this point, and hence, in particular, for any value of $j \ge 0$, both limits

$$\lim_{r\to1^-} B^{(j)}(r\zeta) \quad \text{and} \quad \lim_{R\to1^+} B^{(j)}(R\zeta)$$

exist and are equal. What is more interesting is that ζ might be an accumulation point of the sequence $(a_n)_{n\ge1}$ and yet some of the above properties still hold.

Theorem 21.8 *Let $(a_n)_{n\ge1}$ be a Blaschke sequence in $\mathbb{D}$, and let B be the corresponding Blaschke product. Assume that, for an integer $N \ge 0$ and a point $\zeta \in \mathbb{T}$, we have*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|^{N+1}} \le A. \tag{21.13}$$

Then the following hold.

(i) *For each* $0 \le j \le N$*, both limits*

$$B^{(j)}(\zeta) := \lim_{r \to 1^-} B^{(j)}(r\zeta) \quad \textit{and} \quad \lim_{R \to 1^+} B^{(j)}(R\zeta)$$

exist and are equal.

(ii) *There is a constant* $C = C(N, A)$ *such that the estimation*

$$|B^{(j)}(r\zeta)| \le C$$

uniformly holds for $r \in [0, 1]$ *and* $0 \le j \le N$*.*

Proof The essential case is $N = 0$. The rest follows by induction.

Case $N = 0$. Our strategy is to show that, under the proposed condition, $|B(r\zeta)|$ and $\arg B(r\zeta)$ have both finite limits as r tends to 1^-. For the simplicity of notation, without loss of generality, assume that $\zeta = 1$.

In the course of the proof, we repeatedly use the inequalities

$$|1 - \bar{a}_n r| > 1 - r \quad \text{and} \quad |1 - \bar{a}_n r| > \tfrac{1}{2}|1 - a_n|,$$

for $r \in (0, 1)$, which are elementary to establish. As the first application, note that

$$\frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} \le 2\frac{(1 - r^2)(1 - |a_n|^2)}{(1 - r)|1 - a_n|} \le 8\frac{1 - |a_n|}{|1 - a_n|}.$$

Therefore, the Weierstrass M-test shows that the series

$$\sum_{n \ge 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2}$$

converges uniformly in $r \in [0, 1]$, and thus

$$\lim_{r \to 1^-} \sum_{n \ge 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} = 0.$$

But we have

$$\begin{aligned} |B(r)|^2 &= \prod_{n \ge 1} \frac{|a_n - r|^2}{|1 - \bar{a}_n r|^2} \\ &= \prod_{n \ge 1} \left(1 - \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2}\right) \\ &\ge 1 - \sum_{n \ge 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2}, \end{aligned}$$

and this estimation enables us to deduce that

$$\liminf_{r \to 1^-} |B(r)|^2 \ge 1 - \lim_{r \to 1^-} \sum_{n \ge 1} \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \bar{a}_n r|^2} = 1.$$

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Since $|B(z)| < 1$, we conclude that

$$\lim_{r\to 1^-} |B(r)| = 1.$$

To deal with the argument, write

$$\frac{\bar{a}_n}{|a_n|}\,\frac{a_n - r}{1-\bar{a}_n r} = \frac{1}{|a_n|}\,\frac{|a_n|^2 - 1 + 1 - r\bar{a}_n}{1-\bar{a}_n r} = \frac{1}{|a_n|}\left(1 - \frac{1-|a_n|^2}{1-\bar{a}_n r}\right).$$

Thus

$$\arg\left(\frac{\bar{a}_n}{|a_n|}\,\frac{a_n - r}{1-\bar{a}_n r}\right) = \arg\left(1 - \frac{1-|a_n|^2}{1-\bar{a}_n r}\right),$$

and, for large enough n for which the combination $(1-|a_n|)/(|1-a_n|)$ is small, we have

$$\left|\arg\left(1 - \frac{1-|a_n|^2}{1-\bar{a}_n r}\right)\right| \le M\frac{1-|a_n|^2}{|1-\bar{a}_n r|} \le 4M\frac{1-|a_n|}{|1-a_n|},$$

where M is a positive constant. Thus the series

$$\arg B(r) = \sum_{n\ge 1} \arg\left(1 - \frac{1-|a_n|^2}{1-\bar{a}_n r}\right)$$

converges absolutely and uniformly on $[0,1]$, which proves that $\lim_{r\to 1^-} \arg B(r)$ exists.

The preceding two discussions together show that $L = \lim_{r\to 1^-} B(r)$ exists and has modulus one, i.e $|L| = 1$. The estimation in part (ii) trivially holds with $C = 1$. Finally, the Blaschke product satisfies the functional equation

$$B(z)\overline{B(1/\bar{z})} = 1.$$

Therefore,

$$\lim_{R\to 1^+} B(R) = \frac{1}{\lim_{R\to 1^+} \overline{B(1/R)}} = \frac{1}{\lim_{r\to 1^-} \overline{B(r)}} = \frac{1}{\bar{L}} = L.$$

This argument also shows that, if $\varepsilon > 0$ is such that $[1-\varepsilon, 1)$ is free from the zeros of B, then B is actually continuous on $[1-\varepsilon, 1+\varepsilon]$.

Case $N \ge 1$. Fix $1 \le j \le N$, and suppose that the result holds for $0, 1, \dots, j-1$. Using the formula for B and taking the logarithmic derivative of both sides gives us

$$\frac{B'(z)}{B(z)} = \sum_{n\ge 1} \frac{(1-|a_n|^2)}{(z-a_n)(1-\bar{a}_n z)}. \tag{21.14}$$

Thus,

$$B'(z) = \sum_{n\ge 1} B_n(z)\,\frac{(1-|a_n|^2)}{(1-\bar{a}_n z)^2}, \tag{21.15}$$

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where

$$B_n(z) = \frac{B(z)(1-\bar{a}_n z)}{(z-a_n)} \qquad (n \geq 1) \tag{21.16}$$

is the subproduct formed with all zeros except a_n. Now, we use the formula for B' and take the derivative of both sides $j-1$ times. Then Leibniz's formula tells us that

$$B^{(j)}(z) = \sum_{k=0}^{j-1} \binom{j-1}{k} \sum_{n\geq 1} B_n^{(j-1-k)}(z)\, \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n z)^{k+2}}.$$

Note that, on the right-hand side, we have $B_n^{(\ell)}$, where ℓ runs between 0 and $j-1$. Hence, the induction hypothesis applies. To deal with the other term, we consider $r<1$ and $R>1$ separately.

If $r<1$, then

$$\begin{aligned} \left| \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n r)^{k+2}} \right| &\leq \frac{(k+1)!\,(1-|a_n|^2)}{|(1-a_n)/2|^{k+2}} \\ &\leq \frac{2(k+1)!\,(1-|a_n|)}{|(1-a_n)/2|^{N+1}} \\ &= 2^{N+2}(k+1)!\, \frac{(1-|a_n|)}{|1-a_n|^{N+1}}. \end{aligned}$$

But, for $R>1$, we have

$$\begin{aligned} \left| \frac{(k+1)!\, \bar{a}_n^k (1-|a_n|^2)}{(1-\bar{a}_n R)^{k+2}} \right| &\leq \frac{(k+1)!\,(1-|a_n|^2)}{|R^{-1}-a_n|^{k+2}} \\ &\leq M\, \frac{(1-|a_n|)}{|1-a_n|^{N+1}}, \end{aligned}$$

where M is a constant. This is because the condition (21.13) ensures that any Stolz domain anchored at ζ can only contain a finite number of zeros a_n. Take any of these domains anchored at $\zeta = 1$, e.g. the one with opening $\pi/2$ or more explicitly the domain $|\Im z| \leq 1 - \Re z$. Then, for a_n that are not in this domain but are close to $\zeta = 1$, say at a distance at most 1, we have

$$|R^{-1} - a_n| \leq |1-a_n|/\sqrt{2}.$$

Thus,

$$\begin{aligned} \frac{(k+1)!\,(1-|a_n|^2)}{|R^{-1}-a_n|^{k+2}} &\leq \frac{2^{(k+2)/2}(k+1)!\,(1-|a_n|^2)}{|1-a_n|^{k+2}} \\ &\leq \frac{2^{(N+1)/2} N!\,(1-|a_n|^2)}{|1-a_n|^{N+1}}. \end{aligned}$$

The other points rest at a uniform positive distance from $\zeta = 1$.

Based on the above discussion and the induction hypothesis, if $\delta > 0$ is such that $[1-\delta, 1)$ is free from the zeros of B, then all the series

$$\sum_{n\geq 1} B_n^{(j-1-k)}(z)\,\frac{(k+1)!\,\bar{a}_n^k(1-|a_n|^2)}{(1-\bar{a}_n z)^{k+2}} \qquad (0 \leq k \leq j-1)$$

are uniformly and absolutely convergent for $z \in [1-\delta, 1+\delta]$. Hence, $B^{(j)}(z)$ is also a continuous function on this interval, which can be equally stated as in the theorem based on the right and left limits at $\zeta = 1$.

Appealing to the induction hypothesis, assume that the estimation in part (ii) holds for derivatives up to order $j-1$. Then the above calculation for $r < 1$ shows that

$$\begin{aligned} |B^{(j)}(r)| &\leq \sum_{k=0}^{j-1} \binom{j-1}{k} \sum_{n\geq 1} |B_n^{(j-1-k)}(r)|\, \frac{2^{N+2}(k+1)!\,(1-|a_n|)}{|1-a_n|^{N+1}} \\ &\leq \left(\sum_{k=0}^{j-1} \binom{j-1}{k} 2^{N+2}(k+1)! \right) CA. \end{aligned}$$

Hence, with a bigger constant, the result holds for the derivative of order j. We choose the largest constant corresponding to the derivative of order N as the constant C. This completes the proof of Theorem 21.8. □

The mere usefulness of the estimation in Theorem 21.8(ii) is that the constant C does not depend on the distribution of zeros. It just depends on the upper bound A and the integer N. Hence, it is equally valid for all the subproducts of B.

Theorem 21.8 is also valid if $\zeta \in \mathbb{D}$. In fact, the proof is simpler in this case, since part (i) is trivial. Hence, we can say that, if $\zeta \in \bar{\mathbb{D}}$ and

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n\zeta|^{N+1}} \leq A,$$

then there is a constant $C = C(N, A)$ such that the estimation

$$|B^{(j)}(r\zeta)| \leq C$$

uniformly holds for $r \in [0, 1]$ and $0 \leq j \leq N$.

Corollary 21.9 *Let $(a_n)_{n\geq 1}$ be a Blaschke sequence in $\mathbb{D}$, and let B be the corresponding Blaschke product. Let $\zeta \in \mathbb{T}$ be such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|^2} < \infty.$$

Then B has a derivative in the sense of Carathéodory at ζ and

$$|B'(\zeta)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\zeta - a_n|^2}.$$

Proof That B has a derivative in the sense of Carathéodory at ζ is a direct consequence of Theorems 21.1 and 21.8. To obtain the formula for $|B'(\zeta)|$, we use (21.15). Note that our condition implies that the subproducts B_n have radial limits at ζ. Hence, we can let $r \longrightarrow 1$ in

$$B'(r\zeta) = \sum_{n=1}^{\infty} B_n(r\zeta)\, \frac{(1-|a_n|^2)}{(1-\bar{a}_n r\zeta)^2}$$

to obtain

$$B'(\zeta) = \sum_{n=1}^{\infty} B_n(\zeta)\, \frac{(1-|a_n|^2)}{(1-\bar{a}_n \zeta)^2}.$$

The upper bound

$$\left| \frac{(1-|a_n|^2)}{(1-\bar{a}_n r\zeta)^2} \right| \leq \frac{4(1-|a_n|)}{|\zeta - a|^2} \qquad (0 < r < 1)$$

allows one to pass to the limit inside the sum. But, according to (21.16), we have

$$B_n(\zeta) = \frac{B(\zeta)(1-\bar{a}_n\zeta)}{(\zeta - a_n)} \qquad (n \geq 1).$$

Plugging this back in to the formula for $B'(\zeta)$ gives

$$B'(\zeta) = \bar{\zeta} B(\zeta) \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|a_n - \zeta|^2}.$$

By taking the absolute values of both sides, the result follows. □

21.4 Higher derivatives of *b*

According to the canonical factorization theorem, b can be decomposed as

$$b(z) = B(z)S(z)O(z) \qquad (z \in \mathbb{D}), \tag{21.17}$$

where

$$B(z) = \gamma \prod_n \left(\frac{|a_n|}{a_n}\, \frac{a_n - z}{1 - \bar{a}_n z} \right),$$

$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) \right)$$

and

$$O(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|b(\zeta)|\, dm(\zeta)\right).$$

We can also extend the function b outside the unit disk by the identity (21.17) and the formulas provided for B, S and O. The extended function is analytic for $|z| > 1$, $z \neq 1/\bar{a}_n$. At $1/\bar{a}_n$ it has a pole of the same order as a_n, as a zero of B. We denote this function also by b, and it is easily verified that it satisfies the functional identity

$$b(z)\,\overline{b(1/\bar{z})} = 1. \tag{21.18}$$

One should be careful in dealing with function b inside and outside the unit disk. For example, if

$$b(z) = \tfrac{1}{2} z^n \qquad (|z| < 1),$$

it is natural to use the same nice formula for $|z| > 1$. However, the functional equation (21.18) says that

$$b(z) = 2z^n \qquad (|z| > 1).$$

Hence, b and its derivatives up to order n show a different behavior if we approach a point $\zeta_0 \in \mathbb{T}$ from within $\mathbb{D}$ or from outside. In Theorem 21.10 below, we show that, under certain circumstances, this can be avoided.

For our application in this section, we can merge $S(z)$ and $O(z)$ and write

$$b(z) = B(z) f(z), \tag{21.19}$$

where

$$f(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right) \tag{21.20}$$

and μ is the positive measure $d\mu(\zeta) = -\log|b(\zeta)|\, dm(\zeta) + d\sigma(\zeta)$. Now, Leibniz's formula says that

$$b^{(j)}(z) = \sum_{k=0}^{j} B^{(k)}(z) f^{(j-k)}(z).$$

For the derivatives of B on a ray, we have already established Theorem 21.8. However, a similar result holds for function f, and thus similar statements actually hold for b, i.e. for any function in the closed unit ball of H^∞.

Theorem 21.10 *Let b be in the closed unit ball of H^∞ with the decomposition* (21.17). *Assume that, for an integer $N \geq 0$ and a point $\zeta_0 \in \mathbb{T}$, we have*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta_0 - a_n|^{N+1}} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|\zeta_0 - \zeta|^{N+1}} + \int_0^{2\pi} \frac{\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^{N+1}}\, dm(\zeta) \leq A. \tag{21.21}$$

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Then the following hold.

(i) *For each* $0 \le j \le N$, *both limits*

$$b^{(j)}(\zeta_0) := \lim_{r \to 1^-} b^{(j)}(r\zeta_0) \quad \textit{and} \quad \lim_{R \to 1^+} b^{(j)}(R\zeta)$$

exist and are equal.

(ii) *There is a constant* $C = C(N, A)$ *such that the estimation*

$$|b^{(j)}(r\zeta_0)| \le C$$

uniformly holds for $r \in [0, 1]$ *and* $0 \le j \le N$.

Proof As discussed before the statement of the theorem, it is enough to establish the result just for the function $f = SO$ given by (21.20). The proof has the same flavor as the proof of Theorem 21.8. We first consider the case $N = 0$, and then the rest follows by induction.

Case $N = 0$. We show that, under the condition (21.21), which now translates as

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta_0 - \zeta|} \le A, \tag{21.22}$$

$|f(r\zeta_0)|$ and $\arg f(r\zeta_0)$ have both finite limits as r tends to 1^-. For simplicity of notation, without loss of generality, assume that $\zeta_0 = 1$.

A simple computation shows that

$$f(r) = \exp\left(-\int_{\mathbb{T}} \frac{1 - r^2}{|\zeta - r|^2} d\mu(\zeta)\right) \exp\left(-i \int_{\mathbb{T}} \frac{r\, \Im(\zeta)}{|\zeta - r|^2}\, d\mu(\zeta)\right).$$

Therefore, we have explicit formulas for $|f(r)|$ and $\arg f(r)$.

The assumption (21.22) implies that there is no Dirac mass at $\zeta_0 = 1$, i.e. $\mu(\{1\}) = 0$. Therefore,

$$\lim_{r \to 1^-} \frac{1 - r^2}{|\zeta - r|^2} = 0$$

for μ-almost every $\zeta \in \mathbb{T}$. Moreover, we have the upper bound estimation

$$\frac{1 - r^2}{|\zeta - r|^2} \le \frac{2}{|1 - \zeta|} \qquad (\zeta \in \mathbb{T}),$$

which holds uniformly for all values of the parameter $r \in (0, 1)$. The condition (21.22) means that the function on the right-hand side belongs to $L^1(\mu)$. Hence, by the dominated convergence theorem, we get

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \frac{1 - r^2}{|\zeta - r|^2}\, d\mu(\zeta) = 0.$$

In return, this observation implies that

$$\lim_{r \to 1^-} |f(r)| = 1.$$

In a similar manner,

$$\lim_{r\to 1^-} \frac{r\,\Im(\zeta)}{|\zeta - r|^2} = \frac{\Im(\zeta)}{|1-\zeta|^2}$$

for μ-almost all $\zeta \in \mathbb{T}$. We also have the upper bound estimation

$$\frac{r|\Im(\zeta)|}{|\zeta - r|^2} \le \frac{2}{|1-\zeta|} \qquad (\zeta \in \mathbb{T}),$$

which holds uniformly for all values of the parameter $r \in (0,1)$. Finally, again by the dominated convergence theorem, we see that the limit

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \frac{r\,\Im(\zeta)}{|\zeta - r|^2}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{\Im(\zeta)}{|\zeta - 1|^2}\, d\mu(\zeta)$$

exists and is a finite real number. In return, this implies that

$$\lim_{r\to 1^-} \arg f(r)$$

also exists and is a finite real number. Therefore, $L := \lim_{r\to 1^-} f(r)$ exists and, moreover, $|L| = 1$.

Put $L = \lim_{r\to 1^-} f(r)$. By (21.18), the function f satisfies the functional equation

$$f(z)\overline{f(1/\bar{z})} = 1.$$

Therefore,

$$\lim_{R\to 1^+} f(R) = \frac{1}{\lim_{R\to 1^+} \overline{f(1/R)}} = \frac{1}{\lim_{r\to 1^-} \overline{f(r)}} = \frac{1}{\bar{L}} = L.$$

This argument also shows that f is actually bounded on $[0, +\infty)$. The estimation in part (ii) trivially holds with $C = 1$.

Case $N \ge 1$. Fix $1 \le j \le N$, and suppose that the result holds for $0, 1, \ldots,$ $j-1$. The condition (21.21) is rewritten as

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|1-\zeta|^{N+1}} \le A. \tag{21.23}$$

Using the formula for f and taking the derivative of both sides gives us

$$f'(z) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - z)^2}\, d\mu(\zeta) \right) f(z). \tag{21.24}$$

Now, take the derivative of both sides $j-1$ times. Then Leibniz's formula tells us that

$$f^{(j)}(z) = \sum_{k=0}^{j-1} \left(\int_{\mathbb{T}} \frac{-2(k+1)!\,\zeta}{(\zeta - z)^{k+2}}\, d\mu(\zeta) \right) f^{(j-1-k)}(z). \tag{21.25}$$

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On the right-hand side, we have $f^{(\ell)}$, where ℓ runs between 0 and $j-1$. Hence, the induction hypothesis applies. To deal with the other term, note that, for $z = r < 1$ and also $z = R > 1$, we have

$$\frac{1}{|\zeta - z|} \le \frac{2}{|\zeta - 1|} \qquad (\zeta \in \mathbb{T}).$$

Thus, for all $z \in (0, \infty) \setminus \{1\}$ and all k with $0 \le k \le j-1 \le N-1$,

$$\begin{aligned} \left| \frac{-2(k+1)!\,\zeta}{(\zeta - z)^{k+2}} \right| &\le \frac{2(k+1)!}{|(\zeta-1)/2|^{k+2}} \\ &\le \frac{2N!}{|(\zeta-1)/2|^{N+1}} \\ &= \frac{2^{N+2}N!}{|\zeta - 1|^{N+1}} \qquad (\zeta \in \mathbb{T}). \end{aligned} \tag{21.26}$$

Therefore, by (21.23), (21.26) and the dominated convergence theorem,

$$\lim_{r \to 1^{\pm}} \int_{\mathbb{T}} \frac{-2(k+1)!\,\zeta}{(\zeta - z)^{k+2}}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{-2(k+1)!\,\zeta}{(\zeta - 1)^{k+2}}\, d\mu(\zeta).$$

Note that again we have implicitly used the fact that $\mu(\{1\}) = 0$. Thus, by the induction hypothesis and (21.25), part (i) follows. Moreover, again by the induction hypothesis, assume that the estimation in part (ii) holds for derivatives up to order $j-1$. Then, by (21.25) and (21.26),

$$\begin{aligned} |f^{(j)}(r)| &\le \sum_{k=0}^{j-1} \left(\int_{\mathbb{T}} \left| \frac{-2(k+1)!\,\zeta}{(\zeta - r)^{k+2}} \right| d\mu(\zeta) \right) |f^{(j-1-k)}(r)| \\ &\le \left(\int_{\mathbb{T}} \frac{2^{N+2}N!}{|\zeta - 1|^{N+1}}\, d\mu(\zeta) \right) \sum_{k=0}^{j-1} |f^{(j-1-k)}(r)| \le j 2^{N+2} N!\, AC. \end{aligned}$$

Hence, with a bigger constant, the result holds for the derivative of order j. We choose the largest constant corresponding to the derivative of order N as the constant C. This completes the proof of Theorem 21.10. □

We highlight one property that was explicitly mentioned in the proof of Theorem 21.8 for Blaschke products, but also holds for an arbitrary b. Under the hypothesis of Theorem 21.10, there is a $\delta > 0$ (which depends on b) such that $b^{(j)}(z)$, for $0 \le j \le N$, is a continuous function on the ray $[(1-\delta)\zeta_0, (1+\delta)\zeta_0]$.

Corollary 21.11 *Let b be in the closed unit ball of H^∞ with the decomposition* (21.17). *Let $\zeta_0 \in \mathbb{T}$ be such that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{|\zeta_0 - \zeta|^2} + \int_0^{2\pi} \frac{\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^2}\, dm(\zeta) < \infty.$$

Then b has a derivative in the sense of Carathéodory at ζ_0 and

$$|b'(\zeta_0)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{2\,d\sigma(\zeta)}{|\zeta_0 - \zeta|^2} + \int_0^{2\pi} \frac{2\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^2}\,dm(\zeta).$$

Proof As we did in (21.19), write $b = Bf$. Corollary 21.9 treats the Blaschke product B and gives a formula (the first term appearing in $|b'(\zeta_0)|$ above). Hence, in the light of Corollary 21.3, we just need to study f and prove that $|f'(\zeta_0)|$ is precisely the remaining two terms in the formula for $|b'(\zeta_0)|$.

That f has a derivative in the sense of Carathéodory at ζ_0 is a direct consequence of Theorems 21.1 and 21.10. To obtain the formula for $|f'(\zeta_0)|$, we use (21.24), i.e.

$$f'(r\zeta_0) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - r\zeta_0)^2}\,d\mu(\zeta) \right) f(r\zeta_0).$$

Now, let $r \longrightarrow 1$ to obtain

$$f'(\zeta_0) = \left(\int_{\mathbb{T}} \frac{-2\zeta}{(\zeta - \zeta_0)^2}\,d\mu(\zeta) \right) f(\zeta_0).$$

The upper bound

$$\frac{1}{|\zeta - r\zeta_0|} \le \frac{2}{|\zeta - \zeta_0|} \qquad (\zeta \in \mathbb{T},\ 0 < r < 1)$$

allows one to pass to the limit inside the integral. Now, note that

$$\frac{-2\zeta}{(\zeta - \zeta_0)^2} = \frac{2\zeta}{(\zeta - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)\zeta\zeta_0} = \frac{2\bar{\zeta}_0}{|\zeta - \zeta_0|^2}.$$

Hence, we rewrite the formula for $f'(\zeta_0)$ as

$$f'(\zeta_0) = \left(\int_{\mathbb{T}} \frac{2}{|\zeta - \zeta_0|^2}\,d\mu(\zeta) \right) \bar{\zeta}_0 f(\zeta_0).$$

By taking the absolute values of both sides, the result follows. □

21.5 Approximating by Blaschke products

According to (21.19), an arbitrary element of the closed unit ball of H^∞ may be decomposed as $b = Bf$, where B is a Blaschke product and f is a nonvanishing function given by (21.20). Generally speaking, since B is given by a product of some simple fraction of the form $(az+b)/(cz+d)$, it is easy to handle and study its properties. That is why in this section we explore the possibility of approximating f by some Blaschke products. This will enable us to establish certain properties for the family of Blaschke products first, and then extend them to the whole closed unit ball of H^∞.

Given a Blaschke product B with zeros $(a_n)_{n\geq 1}$, we define the measure σ_B on $\bar{\mathbb{D}}$ by

$$\sigma_B = \sum_{n=1}^{\infty} (1-|a_n|)\, \delta_{\{a_n\}},$$

where $\delta_{\{z\}}$ is the Dirac measure anchored at the point z. We consider σ_B as an element of $\mathcal{M}(\bar{\mathbb{D}})$, the space of finite complex Borel measures on $\bar{\mathbb{D}}$. This space is the dual of $\mathcal{C}(\bar{\mathbb{D}})$. Hence, we equip it with the weak-star topology. Since $\mathcal{C}(\bar{\mathbb{D}})$ is separable, this topology is first countable on $\mathcal{M}(\bar{\mathbb{D}})$. More specifically, this means that each measure has a countable local basis. Naively speaking, this implies that we just need to consider sequences of measures to study the properties of this topology.

In the following, we assume that the Blaschke products are normalized so that $B(0) > 0$.

Theorem 21.12 *Let f be given by* (21.20)*, and let $(B_n)_{n\geq 1}$ be a sequence of Blaschke products. Then B_n converges uniformly to f on compact subsets of $\mathbb{D}$ if and only if $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$.*

Proof Assume that $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$, and denote the zeros of B_n by $(a_{nm})_{m\geq 1}$. Since μ is supported on $\mathbb{T}$, the zeros of B_n must tend to $\mathbb{T}$. In fact, fix any $r < 1$ and consider a continuous positive function φ that is identically 1 on $|z| \leq r$, and identically 0 on $|z| > (1+r)/2$. In between, it has a continuous transition from 1 to 0. Since $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology, we have

$$\int_{\bar{\mathbb{D}}} \varphi \, d\sigma_{B_n} \longrightarrow \int_{\bar{\mathbb{D}}} \varphi \, d\mu = 0.$$

But we also have

$$\begin{aligned} \int_{\bar{\mathbb{D}}} \varphi \, d\sigma_{B_n} &\geq \int_{|z|\leq r} \varphi \, d\sigma_{B_n} \\ &= \sum_{|a_{nm}|\leq r} (1-|a_{nm}|) \\ &\geq (1-r) \times \text{Card}\{m : |a_{nm}| \leq r\}. \end{aligned}$$

Therefore, for each $r < 1$, there is an $N = N(r)$ such that

$$|a_{nm}| > r \qquad (n \geq N,\ m \geq 1). \tag{21.27}$$

We now further explore our assumption to show that

$$B_n(0) \longrightarrow f(0). \tag{21.28}$$

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Since $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology, we have

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} \longrightarrow \int_{\bar{\mathbb{D}}} d\mu.$$

By (21.20), $f(0)$ is a positive real number and

$$\int_{\bar{\mathbb{D}}} d\mu = -\log f(0).$$

But the left-hand side is

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} = \sum_{m=1}^{\infty} (1 - |a_{nm}|),$$

which is not precisely $-\log B_n(0)$. The actual formula is

$$-\log B_n(0) = -\sum_{m=1}^{\infty} \log |a_{nm}|.$$

However, thanks to (21.27), this difference can be handled. It is elementary to verify that

$$0 \le t - 1 - \log t \le (1-t)^2 \qquad (1/2 \le t \le 1).$$

Hence, for $n \ge N(r)$,

$$(1 - |a_{nm}|) \le -\log |a_{nm}| \le (1+r)(1 - |a_{nm}|) \qquad (m \ge 1).$$

Summing over m gives

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} \le -\log B_n(0) \le (1+r) \int_{\bar{\mathbb{D}}} d\sigma_{B_n} \qquad (n \ge N(r)). \tag{21.29}$$

Let $n \longrightarrow \infty$ to deduce that

$$\begin{aligned} -\log f(0) &\le \liminf_{n\to\infty} -\log B_n(0) \\ &\le \limsup_{n\to\infty} -\log B_n(0) \le -(1+r) \log f(0). \end{aligned}$$

Now, let $r \longrightarrow 1$ to conclude that

$$\lim_{n\to\infty} \log B_n(0) = \log f(0).$$

The next step is to show that B_n actually uniformly converges to f on any compact subset of $\mathbb{D}$. In the language of σ_{B_n}, formula (21.14) can be rewritten as

$$\frac{B_n'(z)}{B_n(z)} = \int_{\bar{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}\, d\sigma_{B_n}(\zeta).$$

At first glance, it seems that the function

$$\zeta \longmapsto \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}$$

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is not continuous on $\bar{\mathbb{D}}$ and thus we cannot appeal to the weak-star convergence. However, we fix a compact set $|z| \leq r$ and, as we saw above, after a finite number of indices, the support of σ_{B_n} is in $|z| > (1+r)/2$. Hence, we can multiply the above function by a transient function that is 1 on $|z| \geq (1+r)/2$ and 0 on $|z| \leq 1$. This operation, on the one hand, will not change the value of the integrals and, on the other, will create a genuine continuous function on $\bar{\mathbb{D}}$. Therefore, we can surely say

$$\begin{aligned}\int_{\bar{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}\, d\sigma_{B_n}(\zeta) \longrightarrow \int_{\bar{\mathbb{D}}} \frac{(1+|\zeta|)}{(z-\zeta)(1-\bar{\zeta}z)}\, d\mu(\zeta) \\ = \int_{\mathbb{T}} \frac{-2\zeta}{(\zeta-z)^2}\, d\mu(\zeta),\end{aligned}$$

which translates as

$$\frac{B_n'(z)}{B_n(z)} \longrightarrow \frac{f'(z)}{f(z)} \tag{21.30}$$

as $n \longrightarrow \infty$.

Since $(B_n)_{n\geq 1}$ is uniformly bounded by 1 on $\mathbb{D}$, it is a normal family. Let g be any pointwise limit of a subsequence of $(B_n)_{n\geq 1}$. Then, by (21.28) and (21.30), we must have

$$g(0) = f(0) \quad \text{and} \quad \frac{f'(z)}{f(z)} = \frac{f'(z)}{f(z)} \qquad (z \in \mathbb{D}).$$

Thus, $g = f$, which means that the whole sequence converges uniformly to f on compact sets.

To prove the other way around, assume that B_n converges uniformly to f on compact sets. Thus, $B_n(0) \longrightarrow f(0)$ and, since f has no zeros on $\mathbb{D}$, for each r, (21.27) must hold. Hence, if we let $n \longrightarrow \infty$ in (21.29), we obtain

$$\limsup_{n\to\infty} \int_{\bar{\mathbb{D}}} d\sigma_{B_n} \leq -\log f(0) \leq (1+r) \liminf_{n\to\infty} \int_{\bar{\mathbb{D}}} d\sigma_{B_n}.$$

Let $r \longrightarrow 1$ to deduce that

$$\int_{\bar{\mathbb{D}}} d\sigma_{B_n} \longrightarrow -\log f(0).$$

Hence, $(\sigma_{B_n})_{n\geq 1}$ is a bounded sequence in $\mathcal{M}(\bar{\mathbb{D}})$, and any weak-star limit of this sequence must be a positive measure supported on $\mathbb{T}$. But the sequence has just one weak-star limit, i.e. μ. This is because, if ν is any weak-star limit of the sequence, the first part of the proof shows that a subsequence of B_n converges to f_ν, where f_ν is given by (21.20) (with μ replaced by ν). Therefore, $f_\nu = f$ on $\mathbb{D}$ and, using the uniqueness theorem for Fourier coefficients of measures, we conclude that $\nu = \mu$. □

To establish our next approximation theorem, we need a result of Frostman, which by itself is interesting and has numerous other applications. Let Θ be an inner function and, for each $w \in \mathbb{D}$, define

$$\Theta_w(z) = \frac{w - \Theta(z)}{1 - \bar{w}\Theta(z)}.$$

The function Θ_w is called a *Frostman shift* of Θ. It is easy to verify that Θ_w is an inner function for each $w \in \mathbb{D}$. However, a lot more is true. Define the *exceptional set* of Θ to be

$$\mathcal{E}(\Theta) = \{w \in \mathbb{D} : \Theta_w \text{ is not a Blaschke product}\}.$$

Frostman showed that $\mathcal{E}(u)$ is a very small set.

Lemma 21.13 *Let Θ be a nonconstant inner function, and let $0 < \rho < 1$. Define*

$$\mathcal{E}_\rho(\Theta) = \{\zeta \in \mathbb{T} : \Theta_{\rho\zeta} \textit{ is not a Blaschke product}\}.$$

Then $\mathcal{E}_\rho(\Theta)$ has one-dimensional Lebesgue measure zero.

Proof For each $\alpha \in \mathbb{D}$, we have

$$\int_{\mathbb{T}} \log \left| \frac{\rho\xi - \alpha}{1 - \rho\bar{\xi}\alpha} \right| dm(\xi) = \max(\log \rho, \log |\alpha|). \tag{21.31}$$

This is an easy consequence of the mean value property of harmonic functions. In (21.31) replace α by $\Theta(r\zeta)$ and then integrate with respect to ζ to get

$$\int_{\mathbb{T}} \left(\int_{\mathbb{T}} \log |\Theta_{\rho\xi}(r\zeta)| \, dm(\xi) \right) dm(\zeta) = \int_{\mathbb{T}} \max(\log \rho, \log |\Theta(r\zeta)|) \, dm(\zeta).$$

The collection

$$f_r(\zeta) = \max(\log \rho, \log |\Theta(r\zeta)|), \qquad r \in [0, 1),$$

satisfies $\log \rho \le f_r(\zeta) \le 0$ and, moreover,

$$\lim_{r \to 1} f_r(\zeta) = \max\left(\log \rho, \lim_{r \to 1} \log |\Theta(r\zeta)| \right) = \max(\log \rho, 0) = 0$$

for almost every $\zeta \in \mathbb{T}$. Therefore, by the dominated convergence theorem,

$$\lim_{r \to 1} \int_{\mathbb{T}} f_r(\zeta) \, dm(\zeta) = 0.$$

We rewrite this identity as

$$\lim_{r \to 1} \int_{\mathbb{T}} \left(\int_{\mathbb{T}} -\log |\Theta_{\rho\xi}(r\zeta)| \, dm(\xi) \right) dm(\zeta) = 0.$$

Since the integrand $-\log|\Theta_{\rho\xi}|$ is positive, Fubini's theorem can be applied. The outcome is

$$\lim_{r\to 1}\int_{\mathbb{T}}\left(\int_{\mathbb{T}}-\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)\right)dm(\xi)=0.$$

Now, Fatou's lemma implies that

$$\int_{\mathbb{T}}\left(\liminf_{r\to 1}\int_{\mathbb{T}}-\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)\right)dm(\xi)=0.$$

Hence, we must have

$$\liminf_{r\to 1}\int_{\mathbb{T}}\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)=0 \tag{21.32}$$

for almost all $\xi\in\mathbb{T}$, and this precisely means that, for such values of ξ, the Frostman shift $\Theta_{\rho\xi}$ is a Blaschke product. Indeed, if we consider the canonical factorization $\Theta_{\rho\xi}=BS$, where B is a Blaschke product and S is the singular measure

$$S(z)=\exp\left(-\int_{\mathbb{T}}\frac{w+z}{w-z}\,d\sigma(w)\right),$$

then

$$\int_{\mathbb{T}}\log|\Theta_{\rho\xi}(r\zeta)|\,dm(\zeta)\le\int_{\mathbb{T}}\log|S(r\zeta)|\,dm(\zeta)=-\sigma(\mathbb{T}).$$

which, by (21.32), implies that $\sigma\equiv 0$. □

Among numerous applications of Lemma 21.13, we single out the one which states that Blaschke products are uniformly dense in the family of inner functions. A variation of the technique used in the proof of the following result will be exploited in establishing Theorem 21.15.

Corollary 21.14 *Let Θ be an inner function, and let $\varepsilon>0$. Then there is a Blaschke product B such that*

$$\|\Theta-B\|_\infty<\varepsilon.$$

Proof We have

$$\Theta(z)+\Theta_w(z)=\Theta(z)+\frac{w-\Theta(z)}{1-\bar{w}\Theta(z)}=\frac{w-\bar{w}\Theta^2(z)}{1-\bar{w}\Theta(z)},$$

and thus

$$|\Theta(z)+\Theta_w(z)|\le\frac{2|w|}{1-|w|}.$$

On the one hand, this shows that $-\Theta_w\longrightarrow\Theta$ in the H^∞ norm as $w\longrightarrow 0$ and, on the other, Lemma 21.13 ensures that there are numerous choices of w for which $-\Theta_w$ is a Blaschke product. □

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In the following result, we again use $\mathcal{M}(\bar{\mathbb{D}})$, equipped with the weak-star topology. We recall that it is first countable, i.e. each point has a countable local basis of open neighborhood.

Theorem 21.15 *Let $\lambda \in \bar{\mathbb{D}}$, let $N \geq 1$, and let μ be a positive measure on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|1 - \bar{\lambda}\zeta|^N} < \infty.$$

Then there is a sequence of Blaschke products $(B_n)_{n\geq 1}$ such that $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$ and, moreover,

$$\sum_{m=1}^{\infty} \frac{1 - |a_{nm}|^2}{|1 - \bar{\lambda} a_{nm}|^N} \longrightarrow \int_{\mathbb{T}} \frac{2\, d\mu(\zeta)}{|1 - \bar{\lambda}\zeta|^N}$$

as $n \longrightarrow \infty$.

Proof First, note that the growth restriction on μ implies that μ cannot have a Dirac mass at $1/\bar{\lambda}$. Our strategy is to prove the theorem for discrete measures with finitely many Dirac masses and then appeal to a limiting argument to extend it for the general case.

Assume that $\sigma = \alpha\delta_{\{1\}}$, where $\alpha > 0$. Construct f according to the recipe (21.20), i.e.

$$f(z) = \exp\left(-\alpha\, \frac{1+z}{1-z}\right).$$

By Lemma 21.13, the function

$$B_c(z) = \gamma_c\, \frac{f(z) - c}{1 - \bar{c} f(z)}$$

is a Blaschke product for values of c through a sequence that tends to zero and avoids the exceptional set of f. The unimodular constant

$$\gamma_c = \frac{f(0) - \bar{c}}{|f(0) - c|}\, \frac{|1 - \bar{c} f(0)|}{1 - c f(0)}$$

is added to ensure that $B_c(0) > 0$. The precise value of γ_c is not used below. We just need to know that $\gamma_c \longrightarrow 1$ as $c \longrightarrow 0$. The formula for B_c implies that

$$|f(z) - B_c(z)| \leq |1 - \gamma_c| + \frac{2|c|}{1 - |c|} \qquad (z \in \mathbb{D}).$$

Thus, B_c converges uniformly to f on $\mathbb{D}$ (even uniform convergence on compact sets is enough for us). Therefore, by Theorem 21.12, σ_{B_c} tends to μ in the weak-star topology. The zeros of B_c are

$$\{z : f(z) = c\} = \left\{ a_{cm} = \frac{\alpha + \log c + i2\pi m\alpha}{-\alpha + \log c + i2\pi m\alpha} : m \in \mathbb{Z} \right\},$$

which clearly cluster at 1 as $c \longrightarrow 0$. In this case, $\lambda \neq 1$, and thus the function

$$\zeta \longmapsto \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}$$

can be considered as a continuous function on $\bar{\mathbb{D}}$ when we deal with measures μ and σ_{B_c} (at least for small values of c). Hence,

$$\int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\sigma_{B_c}(\zeta) \longrightarrow \int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\mu(\zeta),$$

but

$$\int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\sigma_{B_c}(\zeta) = \sum_{m=1}^{\infty} \frac{1-|a_{cm}|^2}{|1-\bar{\lambda}a_{cm}|^N}$$

and

$$\int_{\bar{\mathbb{D}}} \frac{1+|\zeta|}{|1-\bar{\lambda}\zeta|^N}\, d\mu(\zeta) = \int_{\mathbb{T}} \frac{2}{|1-\bar{\lambda}\zeta|^N}\, d\mu(\zeta).$$

Therefore, the result follows.

If μ consists of a finite sum of Dirac masses, the result still holds by induction. Now, we turn to the general situation. Assume that μ is an arbitrary positive Borel measure on $\mathbb{T}$, fulfilling the above-mentioned growth restriction. Put

$$d\tau(z) = \frac{d\mu(z)}{|1-\bar{\lambda}z|^N}.$$

Again, note that μ cannot have a Dirac mass at $1/\bar{\lambda}$, and this property persists for all measures considered below. The family of discrete measures with finite number of Dirac masses is dense in $\mathcal{M}(\bar{\mathbb{D}})$. Hence, there is a sequence τ_n of such measures so that $\tau_n \longrightarrow \tau$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$. Therefore, for each $f \in \mathcal{C}(\bar{\mathbb{D}})$,

$$\int_{\bar{\mathbb{D}}} f(z)|1-\bar{\lambda}z|^N\, d\tau_n(z) \longrightarrow \int_{\bar{\mathbb{D}}} f(z)|1-\bar{\lambda}z|^N\, d\tau(z) = \int_{\bar{\mathbb{D}}} f(z)\, d\mu(z).$$

This means that $\sigma_n \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$, where σ_n is the discrete measure

$$d\sigma_n(z) = |1-\bar{\lambda}z|^N\, d\tau_n(z) \qquad (n \geq 1).$$

We appeal to the first part and find a Blaschke product B_n such that σ_{B_n} is close enough to σ_n in the weak-star topology and also

$$\left| \int_{\bar{\mathbb{D}}} \frac{1+|z|}{|1-\bar{\lambda}z|^N}\, d\sigma_{B_n}(z) - \int_{\mathbb{T}} \frac{2}{|1-\bar{\lambda}\zeta|^N}\, d\sigma_n(\zeta) \right| < \frac{1}{n}.$$

The result thus follows. Our choice of B_n also implies that $\sigma_{B_n} \longrightarrow \mu$ in the weak-star topology of $\mathcal{M}(\bar{\mathbb{D}})$. □

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21.6 Reproducing kernels for derivatives

Let $\mathcal{H}$ be a reproducing kernel of functions that are analytic on the domain Ω. The kernels of evaluation at point $z \in \Omega$ form a two-parameter family of functions $k_z^{\mathcal{H}}(w)$, where z and w run through Ω and $k_z^{\mathcal{H}}(w)$ is analytic with respect to w and conjugate analytic with respect to z. The essential property of $k_z^{\mathcal{H}}(z)$ is

$$f(z) = \langle f, k_z^{\mathcal{H}} \rangle_{\mathcal{H}} \qquad (f \in \mathcal{H},\ z \in \Omega). \tag{21.33}$$

For further information, see Chapter 9.

If we successively take the derivative of f with respect to z, we see that the evaluation functional $f \longmapsto f^{(n)}(z)$ is given by

$$f^{(n)}(z) = \langle f, \partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n \rangle_{\mathcal{H}} \qquad (f \in \mathcal{H}). \tag{21.34}$$

But we need to show that $\partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n \in \mathcal{H}$ and also that taking the derivative operator inside the inner product is legitimate. We verify this for $n = 1$. For higher derivatives, a similar argument works.

For simplicity, write k_z for $k_z^{\mathcal{H}}$. Put $\delta = (1 - |z|)/2$. Then, for each $f \in \mathcal{H}$ and each Δ with $0 < |\Delta| < \delta$, we have

$$\left| \left\langle f,\ \frac{k_{z+\Delta} - k_z}{\bar{\Delta}} \right\rangle_{\mathcal{H}} \right| = \left| \frac{f(z+\Delta) - f(z)}{\Delta} \right| = \left| \frac{1}{\Delta} \int_z^{z+\Delta} f'(\zeta)\, d\zeta \right| \le C_f,$$

where C_f is the maximum of f' on the disk with center z and radius δ. Therefore, by the uniform boundedness principle (Theorem 1.19), there is a constant C such that

$$\left\| \frac{k_{z+\Delta} - k_z}{\bar{\Delta}} \right\| \le C \qquad (0 < |\Delta| < \delta).$$

Let $g \in \mathcal{H}$ be a weak limit of this fraction as $\Delta \longrightarrow 0$. Then, on the one hand, for each $f \in \mathcal{H}$ we have

$$\langle f, g \rangle = \lim_{\Delta \to 0} \left\langle f,\ \frac{k_{z+\Delta} - k_z}{\bar{\Delta}} \right\rangle = \lim_{\Delta \to 0} \frac{f(z+\Delta) - f(z)}{\Delta} = f'(z).$$

On the other hand,

$$\begin{aligned} g(\zeta) = \langle g, k_\zeta \rangle &= \lim_{\Delta \to 0} \left\langle \frac{k_{z+\Delta} - k_z}{\bar{\Delta}},\ k_\zeta \right\rangle \\ &= \lim_{\Delta \to 0} \frac{k_{z+\Delta}(\zeta) - k_z(\zeta)}{\bar{\Delta}} = \frac{\partial k_z}{\partial \bar{z}}(\zeta). \end{aligned}$$

In short, $g = \partial k_z / \partial \bar{z}$.

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In the light of relation (21.34), we define the notation

$$k_{z,n}^{\mathcal{H}} = \partial^n k_z^{\mathcal{H}} / \partial \bar{z}^n, \tag{21.35}$$

i.e. the kernel of the evaluation functional of the nth derivative at $z \in \Omega$. The relation (21.34) can be rewritten as

$$f^{(n)}(z) = \langle f, k_{z,n}^{\mathcal{H}} \rangle_{\mathcal{H}} \qquad (f \in \mathcal{H}). \tag{21.36}$$

In the above formula, if we replace f by $k_{z,n}^{\mathcal{H}}$, we obtain

$$(k_{z,n}^{\mathcal{H}})^{(n)}(z) = \|k_{z,n}^{\mathcal{H}}\|_{\mathcal{H}}^2. \tag{21.37}$$

There are some other formulas for $k_{z,n}^{\mathcal{H}}$ and each has its merits and uses in applications. We treat some of them below.

For the space $\mathcal{H}(b)$, instead of $k_{z,n}^{\mathcal{H}(b)}$ we will write $k_{z,n}^b$. Our first formula for $k_{z,n}^b$ is based on the operator X_b (see Section 18.7).

Lemma 21.16 *We have*

$$k_{z,n}^b = n!(I - \bar{z}X_b^*)^{-(n+1)} X_b^{*n} k_0^b.$$

Proof According to Theorem 18.21,

$$k_z^b = (I - \bar{z}X_b^*)^{-1} k_0^b.$$

Hence, using the definition (21.35), we get

$$\begin{aligned} k_{z,n}^b &= \frac{\partial^n k_z^b}{\partial \bar{z}^n} \\ &= \frac{\partial^n}{\partial \bar{z}^n} ((I - \bar{z}X_b^*)^{-1} k_0^b) \\ &= n!(I - \bar{z}X_b^*)^{-(n+1)} X_b^{*n} k_0^b. \end{aligned}$$

This completes the proof. □

According to Theorem 18.11, the formula for $k_z^b = k_{z,0}^b$ is

$$k_z^b(w) = \frac{1 - \overline{b(z)}b(w)}{1 - \bar{z}w} \qquad (z, w \in \mathbb{D}).$$

Using Leibniz's rule, by straightforward computations, we obtain

$$k_{z,n}^b(w) = \frac{\partial^n k_z^b(w)}{\partial \bar{z}^n} = \frac{h_{z,n}^b(w)}{(1 - \bar{z}w)^{n+1}}, \tag{21.38}$$

where $h_{z,n}^b$ is the function

$$h_{z,n}^b(w) = n!\, w^n - b(w) \sum_{j=0}^{n} \binom{n}{j} \overline{b^{(j)}(z)} (n-j)!\, w^{n-j} (1 - \bar{z}w)^j. \tag{21.39}$$

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Lemma 21.17 *Let $z_0 \in \mathbb{D}$ with $b(z_0) \neq 0$. Then*

$$h^b_{z_0,n}(1/\bar{z}_0) = (h^b_{z_0,n})'(1/\bar{z}_0) = \cdots = (h^b_{z_0,n})^{(n)}(1/\bar{z}_0) = 0.$$

Proof The functional 21.18 shows that b is analytic in a neighborhood of the point $1/\bar{z}_0$. (If $b(z_0) = 0$, then b has pole at $1/\bar{z}_0$ of the same order as the order of b at z_0.) Therefore, the formula for $k^b_{z_0}(w)$ shows that this kernel is a meromorphic function on $|w| > 1$ with poles as described above and a possible pole at $1/\bar{z}_0$. However, again by (21.18), we have $b(z_0)\overline{b(1/\bar{z}_0)} = 1$ and thus the pole is removable. In short, $w \longmapsto k^b_{z_0}(w)$ is analytic at $1/\bar{z}_0$. Therefore, the same is true for the application $w \longmapsto k^b_{z_0}(w)$. Respecting this property, the representation (21.38) implies that $h^b_{z_0,n}$ must have a zero of order $n+1$ at $1/\bar{z}_0$. □

We finish this section by studying $k^B_{z,n}$, where B is a Blaschke product formed with zeros $(a_n)_{n\geq 1}$. We recall that, by Theorem 14.7,

$$h_j(z) = \left(\prod_{k=1}^{j-1} \frac{a_k - z}{1 - \bar{a}_k z} \right) \frac{(1-|a_j|^2)^{1/2}}{1 - \bar{a}_j z} \qquad (j \geq 1) \tag{21.40}$$

is an orthonormal basis for $K_B = \mathcal{H}(B)$. Sometimes, we will write

$$h_j(z) = (1-|a_j|^2)^{1/2} \frac{B_{j-1}(z)}{1 - \bar{a}_j z} \qquad (j \geq 1), \tag{21.41}$$

where B_j is the finite product formed with the first j zeros.

Lemma 21.18 *Let B be a Blaschke product with zeros $(a_n)_{n\geq 1}$. Let $z \in \mathbb{D}$. Then*

$$k^B_{z,n} = \sum_{j=1}^{\infty} \overline{h_j^{(n)}(z)} h_j.$$

The series converges in H^2 norm.

Proof Since $(h_j)_{j\geq 1}$ forms an orthonormal basis for K_B, there are coefficients c_j, $j \geq 1$, such that

$$k^B_{z,n} = \sum_{j=1}^{\infty} c_j h_j,$$

where the series converges in H^2 norm. Moreover, thanks to orthonormality, c_j is given by

$$c_j = \langle k^B_{z,n}, h_j \rangle_2.$$

But the formula 21.34 immediately implies that $\bar{c}_j = h_j^{(n)}(z)$. □

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For a Blaschke product, $X_B = S^*|K_B$ and $k_0^B = P_B 1$. Thus, $X_B^* = P_B S = M_B$ is the compressed shift on K_B. Therefore, by Lemma 21.16, we have

$$k_{z,n}^B = n!(I - \bar{z}M_B)^{-(n+1)} M_B^n P_B 1. \tag{21.42}$$

Lemma 21.19 *Let $z_0 \in \mathbb{D}$ and $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n\geq 1}$. Assume that there are functions $f, g \in H^2$ such that*

$$z^N = (1 - \bar{z}_0 z)^{N+1} f(z) + B(z)g(z) \qquad (z \in \mathbb{D}).$$

Then we have $P_B f = k_{z_0,N}^B / N!$.

Proof We write the above equation for f and g as

$$S^N 1 = (1 - \bar{z}_0 S)^{N+1} f + Bg.$$

Since M_B is the compression of S, if we apply P_B to both sides, we obtain

$$M_B^N P_B 1 = (1 - \bar{z}_0 M_B)^{N+1} P_B f,$$

and thus

$$P_B f = (1 - \bar{z}_0 M_B)^{-N-1} M_B^N P_B 1,$$

and the result follows from (21.42). □

21.7 An interpolation problem

There is a close relation between the existence of derivatives of elements of $\mathcal{H}(b)$ at the boundary and the containment of $X_b^{*N} k_0^b$ to the range of $(I - \overline{\zeta_0} X_b^*)^{N+1}$. This is fully explored in Theorem 21.26. But, to reach that general result, we need to pave the road by studying some special cases. We start doing this by considering Blaschke products. First, a technical lemma.

Lemma 21.20 *Let $S, (S_n)_{n\geq 1} \in \mathcal{L}(\mathcal{H})$ with the following properties.*

(i) *Each S_n is invertible.*
(ii) *S is injective.*
(iii) *$S_n \longrightarrow S$ in the norm topology.*
(iv) *There is a constant M such that*

$$\|S_n^{-1} S\| \leq M \qquad (n \geq 1).$$

Let $y \in \mathcal{H}$. Then $(S_n^{-1} y)_{n\geq 1}$ is a bounded sequence in $\mathcal{H}$ if and only if $y \in \mathcal{R}(S)$. Moreover, if this holds, we actually have $S_n^{-1} y \longrightarrow S^{-1} y$ in the weak topology.

Proof Assume that $(S_n^{-1}y)_{n\geq 1}$ is a bounded sequence in $\mathcal{H}$. Hence, it has at least one weak limit point in $\mathcal{H}$. Let $x \in \mathcal{H}$ be a weak limit point of the sequence. Since $S_n \longrightarrow S$ in the norm topology, we surely have (at least for a subsequence) $S_n S_n^{-1} y \longrightarrow Sx$. Therefore, $y = Sx$, i.e. $y \in \mathcal{R}(S)$. But, since S is injective, the above argument shows that the sequence has precisely one weak point (if x' is another weak limit point, we would have $y = Sx = Sx'$). In other words, the whole sequence tends weakly to x.

To prove the other (easy) direction, assume that $y \in \mathcal{R}(S)$, i.e. $y = Sx$ for some $x \in \mathcal{H}$. Then

$$\|S_n^{-1}y\| \leq \|S_n^{-1}Sx\| \leq \|S_n^{-1}S\| \, \|x\| \leq M\|x\| \qquad (n \geq 1). \qquad \square$$

The following corollary is a realization of the preceding lemma. The assumptions are adjusted to fit our application in the study of derivatives of $\mathcal{H}(b)$ functions.

Corollary 21.21 *Let $T_k \in \mathcal{L}(\mathcal{H})$, $\zeta_k \in \mathbb{T}$ and $\lambda_{k,n} \in \mathbb{D}$, for $n \geq 1$ and $1 \leq k \leq p$, with the following properties.*

(i) *Each T_k is a contraction.*
(ii) *Each $I - \zeta_k T_k$ is one-to-one.*
(iii) *$T_k T_{k'} = T_{k'} T_k$ for $k, k' \in \{1, \dots, p\}$.*
(iv) *For each k, $\lambda_{k,n}$ tends nontangentially to ζ_k as $n \longrightarrow \infty$.*

Let $y \in \mathcal{H}$. Then the sequence

$$((I - \lambda_{1,n}T_1)^{-1} \cdots (I - \lambda_{p,n}T_p)^{-1} y)_{n\geq 1}$$

is uniformly bounded if and only if y belongs to the range of the operator $(I - \zeta_1 T_1) \cdots (I - \zeta_p T_p)$, in which case,

$$(I - \lambda_{1,n}T_1)^{-1} \cdots (I - \lambda_{p,n}T_p)^{-1} y \longrightarrow (I - \zeta_1 T_1)^{-1} \cdots (I - \zeta_p T_p)^{-1} y$$

in the weak topology.

Proof We apply Lemma 21.20 with

$$S_n = (I - \lambda_{1,n}T_1) \cdots (I - \lambda_{p,n}T_p)$$

and

$$S = (I - \zeta_1 T_1) \cdots (I - \zeta_p T_p).$$

The only nontrivial property is the boundedness of $S_n^{-1}S$. Since T_k are commuting, it is enough to verify that the sequence

$$((I - \lambda_{k,n}T_k)^{-1}(I - \zeta_k T_k))_{n\geq 1}$$

is bounded. But we have

$$\begin{aligned}\|(I-\lambda_{k,n}T_k)^{-1}(I-\zeta_k T_k)\| &= \|I+(\lambda_{k,n}-\zeta_k)(I-\lambda_{k,n}T_k)^{-1}T_k\|\\ &\le 1+|\lambda_{k,n}-\zeta_k|\,\|(I-\lambda_{k,n}T_k)^{-1}\|\\ &\le 1+|\lambda_{k,n}-\zeta_k|\,(1-|\lambda_{k,n}|)^{-1}\\ &\le 1+M_k.\end{aligned}$$

The last estimation holds since $\lambda_{k,n}$ tends nontangentially to ζ_k. The result thus follows. □

In fact, we even need a special case of Corollary 21.21 in which $T_1 = \cdots = T_p$.

Corollary 21.22 *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction, $\zeta \in \mathbb{T}$ and $(\lambda_n)_{n\ge1} \subset \mathbb{D}$, with the following properties.*

(i) *$I - \zeta T$ is one-to-one.*
(ii) *λ_n tends nontangentially to ζ as $n \longrightarrow \infty$.*

Let $y \in \mathcal{H}$. Then the sequence

$$((I-\lambda_n T)^{-p}y)_{n\ge1}$$

is uniformly bounded if and only if y belongs to the range of the operator $(I-\zeta T)^p$, in which case

$$(I-\lambda_n T)^{-p}y \longrightarrow (I-\zeta T)^{-p}y$$

in the weak topology.

Now we are ready to establish the connection between the existence of boundary derivatives in K_B and an interpolation problem.

Theorem 21.23 *Let $\zeta \in \mathbb{T}$, and let $N \ge 0$. Let B be a Blaschke product with zeros $(a_n)_{n\ge1}$ such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta-a_n|^{2N+2}} \le A.$$

Then there are functions $f, g \in H^2$ such that

$$z^N = (1-\bar{\zeta}z)^{N+1}f(z) + B(z)g(z) \qquad (z \in \mathbb{D})$$

with

$$\|f\|_2 \le C,$$

where $C = C(N, A)$ is a constant.

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Proof According to Lemma 21.18,

$$k_{z,N}^B = \sum_{j=1}^{\infty} \overline{h_j^{(N)}(z)} h_j,$$

and hence

$$\|k_{z,n}^B\|^2 = \sum_{j=1}^{\infty} |h_j^{(n)}(z)|^2. \tag{21.43}$$

We rewrite the formula in (21.40) for h_j as

$$h_j(z) = (1-|a_j|^2)^{1/2} \frac{B_{j-1}(z)}{1-\bar{a}_j\, z}.$$

Hence, by Leibniz's formula,

$$h_j^{(N)}(z) = (1-|a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} B_{j-1}^{(N-k)}(z) \frac{k!(-\bar{a}_j)^k}{(1-\bar{a}_j z)^{k+1}}.$$

Therefore, by Theorem 21.8 and denoting the constant $C(N, A)$ of this theorem by C,

$$\begin{aligned} |h_j^{(N)}(r\zeta)| &\le (1-|a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} C \frac{k!}{|1-\bar{a}_j r\zeta|^{k+1}} \\ &\le C(1-|a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} \frac{2^{k+1} k!}{|1-\bar{a}_j\zeta|^{k+1}} \\ &\le C(1-|a_j|^2)^{1/2} \sum_{k=1}^{N} \binom{N}{k} \frac{2^{N+1} k!}{|1-\bar{a}_j\zeta|^{N+1}} \\ &= \left(2^{N+1} C \sum_{k=1}^{N} k! \binom{N}{k}\right) \frac{(1-|a_j|^2)^{1/2}}{|\zeta-a_j|^{N+1}} \\ &= C' \frac{(1-|a_j|^2)^{1/2}}{|\zeta-a_j|^{N+1}}. \end{aligned}$$

Considering (21.43), we conclude that

$$\|k_{r\zeta,N}^B\|^2 \le 2AC'^2 \qquad (0 < r < 1). \tag{21.44}$$

The next step is to appeal to the formula (21.42) and Corollary 21.22. Theorem 14.28 ensures that $\sigma_p(M_B) \subset \mathbb{D}$, and thus the operator $I - \bar{\zeta} M_B$ is injective. Hence, with $T = M_B$, $p = N+1$ and $y = M_B^N P_B 1$, we see that

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$M_B^N P_B 1$ belongs to the range of $(I - \bar{\zeta} M_B)^{N+1}$. This means that there is a function $f \in H^2$ such that

$$M_B^N P_B 1 = (I - \bar{\zeta} M_B)^{N+1} f.$$

Since M_B is the compressed shift, we can rewrite the preceding identity as

$$P_B(z^N) = P_B((1 - \bar{\zeta} z)^{N+1} f).$$

Hence, $z^N - (1 - \bar{\zeta} z)^{N+1} f \perp K_B$, or equivalently $z^N - (1 - \bar{\zeta} z)^{N+1} f \in BH^2$. Therefore, there is $g \in H^2$ such that

$$z^N - (1 - \bar{\zeta} z)^{N+1} f = Bg.$$

Finally, Corollary 21.22 also says that

$$f = (I - \bar{\zeta} M_B)^{-N-1} M_B^N P_B 1 = \lim_{r \to 1} (I - \overline{r\zeta} M_B)^{-N-1} M_B^N P_B 1.$$

Hence, by (21.42)

$$f = (I - \bar{\zeta} M_B)^{-N-1} M_B^N P_B 1 = \frac{1}{N!} \lim_{r \to 1} k_{r\zeta, N}^B,$$

and, by (21.44), the latter is uniformly bounded by a constant. □

The above result can be referred to as an interpolation problem since the equation

$$z^n = (1 - \bar{\zeta} z) f(z) + B(z) g(z)$$

has a solution if and only if there is a function $f \in H^2$ such that

$$f(a_n) = \frac{a_n}{(1 - \bar{\zeta} a_n)^{N+1}} \qquad (n \geq 1).$$

Since

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{(1 - \bar{\zeta} a_n)^{N+1}} \right|^2 (1 - |a_n|^2) < \infty,$$

if $(a_n)_{n \geq 1}$ was an interpolation sequence, then the function f trivially exists (see Section 15.6). The surprising feature of Theorem 21.23 is that it ensures that a solution, even with an additional growth restriction, always exists.

Theorem 21.23, in a sense, is reversible. Indeed, this is the version that we need in the proof of Theorem 21.26.

Theorem 21.24 *Let $N \geq 0$. Let B be a Blaschke product with zeros $(a_n)_{n \geq 1}$. Assume that there are functions $f, g \in H^2$ such that*

$$z^N = (1 - \bar{z}_0 z)^{N+1} f(z) + B(z) g(z) \qquad (z \in \mathbb{D}),$$

with

$$\|f\|_2 \leq C \quad \textit{and} \quad \left(1 - \frac{1}{2C^2}\right)^{1/2} \leq |z_0| < 1,$$

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where $C > 1$ is a constant. Then there is a constant $A = A(N, C)$ such that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - \bar{a}_n z_0|^{2N+2}} \le A.$$

Proof Since we appeal to induction, the functions f and g that appear in the Nth step will be denoted by f_N and g_N. Note that, by Lemma 21.19,

$$P_B f_N = \frac{k^B_{z_0,N}}{N!}. \tag{21.45}$$

Case $N = 0$. By Lemma 21.18,

$$k^B_{z_0} = \sum_{j=1}^{\infty} \overline{h_j(z_0)} h_j.$$

Hence, by (21.45), our condition $\|f_0\|_2 \le C$ translates as

$$\sum_{j=1}^{\infty} |h_j(z_0)|^2 \le C^2.$$

We use (21.41) to rewrite this estimation as

$$\sum_{j=1}^{\infty} |B_{j-1}(z_0)|^2 \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^2} \le C^2. \tag{21.46}$$

We just need to get rid of $|B_{j-1}(z_0)|^2$ to establish the result. To do so, just note that, since B_j is a subproduct of B, we have

$$P_{B_j} k^B_{z_0}(z) = k^{B_j}_{z_0}(z) = \frac{1 - \overline{B_j(z_0)} B_j(z)}{1 - \bar{z}_0 z}.$$

Hence,

$$\frac{1 - |B_j(z_0)|^2}{1 - |z_0|^2} = k^{B_j}_{z_0}(z_0) = \|k^{B_j}_{z_0}\|^2 \le \|k^{B_j}_{z}\|^2 \le C^2.$$

The restriction $1 - 1/(2C^2) \le |z_0|^2 < 1$ now implies that $|B_j(z_0)|^2 \ge 1/2$. Therefore, from (21.46), we conclude that

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \bar{a}_j z_0|^2} \le 2C^2.$$

This settles the case $N = 0$.

Case $N \ge 1$. Assume that the result holds for $N - 1$. Our assumption is that there are functions $f_N, g_N \in H^2$ such that

$$z^N = (1 - \bar{z}_0 z)^{N+1} f_N(z) + B(z) g_N(z) \qquad (z \in \mathbb{D}) \tag{21.47}$$

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with $\|f_N\|_2 \le C$. Write

$$1-(1-\bar{z}_0 z)^N = -\sum_{k=1}^{N}\binom{N}{k}(-\bar{z}_0)^k z^k.$$

Multiply by z^{N-1} to get

$$z^{N-1} = (1-\bar{z}_0 z)^N z^{N-1} - \left(\sum_{k=1}^{N}\binom{N}{k}(-\bar{z}_0)^k z^{k-1}\right) z^N.$$

Comparing this to (21.47) written for $N-1$ rather than N gives

$$z^{N-1} = (1-\bar{z}_0 z)^N f_{N-1}(z) + B(z) g_{N-1}(z) \qquad (z \in \mathbb{D}),$$

where

$$f_{N-1}(z) = z^{N-1} - \left(\sum_{k=1}^{N}\binom{N}{k}(-\bar{z}_0)^k z^{k-1}\right)(1-\bar{z}_0 z) f_N(z).$$

Hence,

$$\|f_{N-1}\|_2 \le 1 + 2^{N+1} C.$$

This means that all the required conditions are fulfilled and we can apply the induction for $N-1$. Thus, there is a constant A such that

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n z_0|^{2N}} \le A. \tag{21.48}$$

Theorem 21.8 now ensures that $B_j^{(k)}(z_0)$, $0 \le k \le 2N-1$, exist and are uniformly bounded by a constant A', where B_j is any subproduct of B.

If we take N times the derivative of both sides in (21.41), we obtain

$$h_j^{(N)}(z) = (1-|a_j|^2)^{1/2} \sum_{k=0}^{N}\binom{N}{k} B_{j-1}^{(k)}(z) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1-\bar{a}_j z)^{N-k+1}} \qquad (j \ge 1).$$

We rewrite this as

$$\begin{aligned}(1-|a_j|^2)^{1/2} B_{j-1}(z) &\frac{N!(-\bar{a}_j)^N}{(1-\bar{a}_j z)^{N+1}} \\ &= h_j^{(N)}(z) - (1-|a_j|^2)^{1/2} \sum_{k=1}^{N}\binom{N}{k} B_{j-1}^{(k)}(z) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1-\bar{a}_j z)^{N-k+1}}.\end{aligned} \tag{21.49}$$

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As we saw above, for $1 \le k \le N$,

$$\left| \binom{N}{k} B_{j-1}^{(k)}(z_0) \frac{(N-k)!(-\bar{a}_j)^{N-k}}{(1-\bar{a}_j z)^{N-k+1}} \right| \le \frac{A' N!}{|1-\bar{a}_j z_0|^{N-k+1}}$$
$$\le \frac{A' N!\, 2^N}{|1-\bar{a}_j z_0|^N}.$$

Thus, the right-hand side of (21.49) is bounded above by

$$|h_j^{(N)}(z)| + A' N N!\, 2^N \frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^N}.$$

The left-hand side of (21.49) is bounded below by

$$\frac{N!}{2^{N+1}} \frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^{N+1}}$$

for zeros $|a_j| \ge 1/2$. Hence, for such j, we have

$$\frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^{N+1}} \le 2^{N+1} |h_j^{(N)}(z_0)| + A' N 2^{2N+1} \frac{(1-|a_j|^2)^{1/2}}{|1-\bar{a}_j z_0|^N}.$$

Hence, by Minkowski's inequality, Lemmas 21.18 and 21.19, and (21.48), we find

$$\left(\sum_{|a_j| \ge 1/2} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N+2}} \right)^{1/2}$$
$$\le 2^{N+1} \left(\sum_{j=1}^{\infty} |h_j^{(N)}(z_0)|^2 \right)^{1/2} + A' N 2^{2N+1} \left(\sum_{j=1}^{\infty} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N}} \right)^{1/2}$$
$$\le 2^{N+1} \|k_{z_0,N}^B\| + A' A N 2^{2N+1}$$
$$\le 2^{N+1} N!\, \|P_B f_N\| + A' A N 2^{2N+1}$$
$$\le 2^{N+1} N!\, C + A' A N 2^{2N+1}.$$

For zeros with $|a_j| < 1/2$, we have

$$\sum_{|a_j|<1/2} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N+2}} \le 4 \sum_{|a_j|<1/2} \frac{1-|a_j|^2}{|1-\bar{a}_j z_0|^{2N}} \le 4A.$$

Hence, the result follows. □

21.8 Derivatives of $\mathcal{H}(b)$ functions

In Theorem 20.13, we saw the connection between the analytic continuation of b across a subarc of $\mathbb{T}$, on the one hand, and the analytic continuation of all functions of $\mathcal{H}(b)$ across the same subarc, on the other hand. In this section,

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we treat a similar result. While we are studying the derivative of elements in $\mathcal{H}(b)$, we are content with the existence of nontangential boundary values.

We begin with a simple lemma, which is a simple exercise from calculus and is interesting in its own right. We do not prove it in such a generality since in our application even the derivative of order $n+m+1$ exists at all points. However, the proof for the general case is essentially the same.

Lemma 21.25 *Let I be an open interval, and let $a, b \in I$. Suppose that the function $h : I \longrightarrow \mathbb{C}$ satisfies the following properties:*

(i) *h has $n+m$ continuous derivatives on I;*
(ii) *$h^{(n+m+1)}$ is continuous and bounded on $I \setminus \{a\}$;*
(iii) $h(b) = h'(b) = \cdots = h^{(n-1)}(b) = 0$.

Put

$$k(x) = \frac{h(x)}{(x-b)^n} \qquad (x \in I).$$

Then k is $m+1$ times differentiable on I and, moreover,

$$k^{(m+1)}(x) = \int_0^1 \cdots \int_0^1 h^{(m+n+1)}(b + t_1 \cdots t_n(x-b))v(t)\, dt_1 \cdots dt_n,$$

where $v(t) = t_1^{p_1} \cdots t_n^{p_n}$ is some monomial.

Proof Since $h(b) = 0$, the fundamental theorem of calculus says

$$\begin{aligned} h(x) &= \int_0^1 \frac{d}{dt_1}[h(b + t_1(x-b))]\, dt_1 \\ &= (x-b)\int_0^1 h'(b + t_1(x-b))\, dt_1. \end{aligned}$$

Applying the same result to the function $x \longmapsto h'(b + t_1(x-b))$ gives

$$h'(b + t_1(x-b)) = t_1(x-b)\int_0^1 h''(b + t_1 t_2(x-b))\, dt_2.$$

Therefore,

$$h(x) = (x-b)^2 \int_0^1 \int_0^1 t_1 h''(b + t_1 t_2(x-b))\, dt_1\, dt_2.$$

Continuing this process n times gives

$$k(x) = \int_0^1 \cdots \int_0^1 t_1^{n-1} t_2^{n-2} \cdots t_{n-1} h^{(n)}(b + t_1 \cdots t_n(x-b))\, dt_1 \cdots dt_n.$$

Write $m(t) = t_1^{n-1} t_2^{n-2} \cdots t_{n-1}$.

Since h has $n+m$ continuous derivatives on I, and $h^{(n+m+1)}$ is continuous and bounded on $I \setminus \{a\}$, the function k has $m+1$ continuous derivatives on I and

$$
\begin{aligned}
k^{(m+1)}(x) &= \int_0^1 \cdots \int_0^1 m(t) \frac{\partial^{m+1}}{\partial x^{m+1}} h^{(n)}(b + t_1 \cdots t_n(x-b))\, dt_1 \cdots dt_n \\
&= \int_0^1 \cdots \int_0^1 v(t) h^{(m+n+1)}(b + t_1 \cdots t_n(x-b))\, dt_1 \cdots dt_n,
\end{aligned}
$$

where $v(t) = t_1^{m+n} t_2^{m+n-1} \cdots t_{n-1}^{m+2} t_n^{m+1}$. □

The following result gives a criterion for the existence of the derivatives for functions of $\mathcal{H}(b)$.

Theorem 21.26 *Let b be a point in the closed unit ball of $H^\infty(\mathbb{D})$ with the canonical factorization* (21.17), *let $\zeta_0 \in \mathbb{T}$ and let N be a nonnegative integer. Then the following are equivalent.*

(i) *For every $f \in \mathcal{H}(b)$, the functions $f(z), f'(z), \ldots, f^{(N)}(z)$ have finite limits as z tends radially to ζ_0.*
(ii) *For every $f \in \mathcal{H}(b)$, the function $|f^{(N)}(z)|$ remains bounded as z tends radially to ζ_0.*
(iii) *$\|k_{z,N}^b\|_b$ is bounded on the ray $z \in [0, \zeta_0]$.*
(iv) *$X_b^{*N} k_0^b$ belongs to the range of $(I - \overline{\zeta_0} X_b^*)^{N+1}$.*
(v) *We have*

$$
\sum_n \frac{1-|a_n|^2}{|\zeta_0 - a_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta_0 - \zeta|^{2N+2}} + \int_{\mathbb{T}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta_0 - \zeta|^{2N+2}}\, dm(\zeta) < \infty.
$$

Moreover, we have

$$
(I - \overline{\zeta_0} X_b^*)^{N+1} k_{\zeta_0,N}^b = N! X_b^{*N} k_0^b, \tag{21.50}
$$

where $k_{\zeta_0,N}^b \in \mathcal{H}(b)$ and satisfies

$$
f^{(N)}(\zeta_0) = \langle f, k_{\zeta_0,N}^b \rangle_b \qquad (f \in \mathcal{H}(b)).
$$

Proof (i) $\Longrightarrow$ (ii) This is trivial.

(ii) $\Longrightarrow$ (iii) In the light of representation (21.36), this implication follows from the principle of uniform boundedness (Theorem 1.19).

(iii) $\Longrightarrow$ (iv) By Lemma 21.16,

$$
k_{z,N}^b = N!(I - \bar{z} X_b^*)^{-(N+1)} X_b^{*N} k_0^b. \tag{21.51}
$$

Theorem 18.26 ensures that $\sigma_p(X_b^*) \subset \mathbb{D}$, and thus the operator $I - \overline{\zeta_0} X_b^*$ is injective. By assumption, $(I - \overline{z_n} X_b^*)^{-(N+1)} X_b^{*N} k_0^b$ is uniformly bounded for any sequence $z_n \in \mathbb{D}$ tending radially to ζ_0. Now, we apply Corollary 21.22

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with $T = X_b^*$, $p = N+1$ and $y = X_b^{*N} k_0^b$ to conclude that $X_b^{*N} k_0^b$ belongs to the range of $(I - \overline{\zeta_0} X_b^*)^{N+1}$.

(iv) $\Longrightarrow$ (i) Using once more Corollary 21.22, we see that

$$(I - \overline{z_n} X_b^*)^{-(N+1)} X_b^{*N} k_0^b \longrightarrow (I - \overline{\zeta_0} X_b^*)^{-(N+1)} X_b^{*N} k_0^b$$

in the weak topology, for any sequence $z_n \in \mathbb{D}$ tending radially to ζ. But (21.51) says that the left-hand side is precisely $(1/N!) k_{z_n,N}^b$. Hence, in the light of (21.36), for every function f in $\mathcal{H}(b)$, the Nth derivative $f^{(N)}(z)$ has a finite limit as z tends radially to ζ_0. Moreover, the linear functional $f \longmapsto f^{(N)}(\zeta_0)$ is continuous on $\mathcal{H}(b)$ and thus it is induced by a kernel function $k_{\zeta_0,N}^b$, which should satisfy

$$(I - \overline{\zeta_0} X_b^*)^{-(N+1)} X_b^{*N} k_0^b = \frac{1}{N!}\, k_{\zeta_0,N}^b.$$

That proves (21.50).

The rest is by induction. We have

$$I - (I - \overline{\zeta_0} X_b^*)^N = -\sum_{\ell=1}^{N} \binom{N}{\ell} (-\zeta_0)^\ell X_b^{*\ell}.$$

Applying to both sides the function $X_b^{*(N-1)} k_0^b$ we get

$$\begin{aligned} X_b^{*(N-1)} k_0^b &= (I - \overline{\zeta_0} X_b^*)^N X_b^{*(N-1)} k_0^b \\ &\qquad - \sum_{\ell=1}^{N} \binom{N}{\ell} (-\zeta_0)^\ell X_b^{*(\ell-1)} X_b^{*N} k_0^b. \end{aligned}$$

Hence, $X_b^{*(N-1)} k_0^b$ belongs to the range of $(I - \overline{\zeta_0} X_b^*)^N$. The above argument applies with N replaced by $N-1$. We continue this process N times. Therefore, for every function f in $\mathcal{H}(b)$, $f^{(j)}(z)$, $0 \le j \le N$, has a finite limit as z tends radially to ζ_0.

(v) $\Longrightarrow$ (iii) Without loss of generality, we assume that $\zeta_0 = 1$. By Theorem 21.10, the condition (v) implies that

$$\lim_{r \to 1^-} b^{(j)}(r) \quad \text{and} \quad \lim_{R \to 1^+} b^{(j)}(R)$$

exist and are equal for $0 \le j \le 2N+1$. Moreover, since b can have only a finite number of real zeros, we can take $\delta > 0$ such that the interval $[1-\delta, 1)$ is free of zeros of b. Therefore, b has $2N+1$ continuous bounded derivatives on $[1-\delta, 1+\delta]$. Now, fix r in the interval $(1-\delta, 1)$.

We recall that, by (21.38) and (21.39), $k_{r,N}^b(x) = h_{r,N}^b(x)/(1-rx)^{N+1}$, where

$$h_{r,N}^b(x) = N!\, x^N - b(x) \sum_{j=0}^{N} \binom{N}{j} \overline{b^{(j)}(r)} (N-j)!\, x^{N-j} (1-rx)^j.$$

Hence, $h^b_{r,N}$ has $2N+1$ continuous bounded derivatives on $(1-\delta, 1+\delta)$. Moreover, by Lemma 21.17, we have

$$h^b_{r,N}(1/r) = (h^b_{r,N})'(1/r) = \cdots = (h^b_{r,n})^{(N)}(1/r) = 0.$$

We now apply Lemma 21.25 with $I = (1-\delta, 1+\delta)$, $a = 1$, $b = 1/r$, $n = N+1$, $m = N$ and $h = h^b_{r,N}$. Note that

$$\frac{h(x)}{(x-b)^n} = \frac{h^b_{r,N}(x)}{(x-1/r)^{N+1}} = (-r)^{N+1} k^b_{N,r}(x).$$

Thus, the lemma says that $(-r)^{N+1}(k^b_{N,r})^{(N)}(x)$ is equal to

$$\int_0^1 \cdots \int_0^1 (h^b_{r,N})^{(2N+1)} \left(\frac{1}{r} + t_1 \cdots t_{N+1} \left(x - \frac{1}{r} \right) \right) v(t)\, dt_1 \cdots dt_{N+1}.$$

Since there is an M such that

$$|(h^b_{r,N})^{(2N+1)}(s)| \le M \qquad (1-\delta < s < 1+\delta),$$

we deduce that

$$|(k^b_{r,N})^{(N)}(x)| \le M\delta^{-N-1} \qquad (1-\delta < x < 1).$$

In particular, $(k^b_{r,N})^{(N)}(r)$ is bounded as $r \longrightarrow 1^-$. But, according to (21.37),

$$\|k^b_{z,N}\|_b^2 = (k^b_{z,N})^{(N)}(z).$$

Thus, $\|k^b_{r,N}\|_b$ remains bounded as $r \longrightarrow 1^-$.

(iii) $\Longrightarrow$ (v) Again, without loss of generality, assume that $\zeta_0 = 1$. Fix $r \in (0,1)$. Considering the canonical factorization of b, since b is in the closed unit ball of H^∞, we have

$$\sum_n \frac{1-|a_n|^2}{|1-a_n r|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|r-\zeta|^{2N+2}} + \int_{\mathbb{T}} \frac{\big|\log|b(\zeta)|\big|}{|r-\zeta|^{2N+2}}\, dm(\zeta) < \infty.$$

For simplicity of formulas, denote the left-hand side by Δ_r. According to Theorem 21.15, there is a sequence $(B_j)_{j\ge1}$ of Blaschke products, with zeros $(a_{jk})_{k\ge1}$, converging uniformly to b on compact subsets of $\mathbb{D}$ and such that

$$\sum_{k=1}^{\infty} \frac{1-|a_{jk}|^2}{|1-ra_{jk}|^{2N+2}} \longrightarrow \Delta_r$$

as $j \longrightarrow \infty$. Hence, the formulas (21.38) and (21.39) show that $k^{B_j}_{w,N}$ tends to $k^b_{w,N}$ uniformly on compact subsets of $\mathbb{D}$. In particular, we must have

$$\lim_{j\to\infty} (k^{B_j}_{w,N})^{(N)}(w) = (k^b_{w,N})^{(N)}(w).$$

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In the light of (21.37), we can rewrite this identity as

$$\lim_{j\to\infty} \|k_{w,N}^{B_j}\|_2 = \|k_{w,N}^b\|_b.$$

The assumption (iii) implies that there is a $C > 0$ such that

$$\|k_{r,N}^b\|_b \le C \qquad (0 < r < 1).$$

Therefore, there is an index j_r such that

$$\|k_{r,N}^{B_j}\|_2 \le C + 1 \qquad (j \ge j_r).$$

The formulas (21.38) and (21.39) also show that

$$(1 - rz)^{N+1} k_{r,N}^{B_j}(z) = N!\, z^N - B_j(z) g_j(z),$$

where $g_j \in H^2$. Hence, it follows from Theorem 21.24 that there is a constant $A = A(C, N)$ (independent of r) such that

$$\sum_k \frac{1 - |a_{jk}|^2}{|1 - r a_{jk}|^{2N+2}} \le A \qquad (j \ge j_r).$$

Letting $j \longrightarrow \infty$, we obtain $\Delta_r \le A$ for all $r \in (0, 1)$. Finally, we let $r \longrightarrow 1^-$ to get the desired condition (v). This completes the proof of Theorem 21.26. □

The identity (21.38) provides an explicit formula for the kernel of the functional for a derivative at the point $z \in \mathbb{D}$. Using Theorem 21.26, it is easy to see that this formula can be extended for the kernel of the functional for a derivative at the point $\zeta_0 \in \mathbb{T}$ that satisfies one of the equivalent conditions (i)–(v); see Lemma 22.4.

Theorem 21.26 implies also a sufficient condition for the existence of the derivatives for functions in the range of a Toeplitz operator with co-analytic symbol.

Corollary 21.27 *Let a be a nonextreme point of the closed unit ball of H^∞, let $\zeta_0 \in \mathbb{T}$, and assume that there is a neighborhood I_{ζ_0} of ζ_0 on $\mathbb{T}$, a constant $c > 0$ and an integer $N \ge 0$ such that*

$$|a(\zeta)| \le c\,|\zeta - \zeta_0|^N \qquad (\zeta \in I_{\zeta_0}).$$

Then every function $f \in \mathcal{M}(\bar{a})$, as well as its derivatives up to order $N - 1$, have finite radial limits at ζ_0.

Proof Thanks to Lemma 17.3, we can assume that a is an outer function with $a(0) > 0$. Consider the outer function b such that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Then (a, b) is a pair and we have $\mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$. Thus it is sufficient to prove that every function $f \in \mathcal{H}(b)$, as well as its derivatives up to order $N-1$, has a

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finite radial limit at ζ_0. For this purpose, we may check the sufficient condition (v) of Theorem 21.26. Since b is outer, this condition is simply

$$\int_{\mathbb{T}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta-\zeta_0|^{2N}}\,dm(\zeta)<\infty. \tag{21.52}$$

Pick $\zeta \in I_{\zeta_0}$. Then,

$$\big|\log|b(\zeta)|\big| \asymp \big|\log|b(\zeta)|^2\big| = \big|\log(1-|a(\zeta)|^2)\big| \lesssim |a(\zeta)|^2 \le c^2|\zeta-\zeta_0|^{2N}.$$

Hence,

$$\int_{I_{\zeta_0}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta-\zeta_0|^{2N}}\,dm(\zeta) \le c^2 m(I_{\zeta_0})<\infty.$$

On the other hand, since b is nonextreme, we have

$$\int_{\mathbb{T}\setminus I_{\zeta_0}} \frac{\big|\log|b(\zeta)|\big|}{|\zeta-\zeta_0|^{2N}}\,dm(\zeta) \lesssim \int_{\mathbb{T}} \big|\log|b(\zeta)|\big|\,dm(\zeta)<\infty,$$

which proves (21.52). □

If we combine Theorems 21.1 and 21.26 and Remark 21.2, we get immediately the following result, which will be useful in Chapter 31.

Corollary 21.28 *Let $\zeta_0 \in \mathbb{T}$. Then the following assertions are equivalent:*

(i) *b has an angular derivative in the sense of Carathéodory at ζ_0;*
(ii) *k_0^b belongs to the range of $I-\overline{\zeta_0}X_b^*$.*

Moreover, in this case, we have $(I-\overline{\zeta_0}X_b^*)k_{\zeta_0}^b = k_0^b$.

Notes on Chapter 21

Section 21.1

Theorems 21.1 and 21.4 can be found under different names in the literature, e.g. the Julia–Carathéodory theorem, the Julia–Wolff–Carathéodory theorem and even the Julia–Wolff theorem. These results combine some celebrated results of Julia [111], Carathéodory [40–42] and also Wolff's boundary version of the Schwarz lemma [191]. The proof here is due to Sarason [161], who applied Hilbert space techniques to prove the existence of angular derivatives. Using the hyperbolic Poincaré metric, P. R. Mercer [132] gave a strengthened version of Julia's result. Potapov [145] extended Julia's result to matrix-valued holomorphic mappings of a complex variable. His results were generalized by Fan and Ando [20, 71–73] to operator-valued holomorphic mappings, and to holomorphic mappings of proper contractions on the unit Hilbert ball acting in the sense of functional calculus. Different generalizations of Theorem 21.1 for

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bounded domains in $\mathbb{C}^n$ are known, e.g. for the unit ball [103, 148, 156], for the polydisk [3, 110], and for strongly convex and strongly pseudoconvex domains [1, 2]. Abate and Tauraso [5] used the Kobayashi metric on a bounded domain in $\mathbb{C}^n$ to obtain a generalized version. There are various versions and proofs of this concept – see for example [40, 59, 95, 115, 132, 161, 173, 175, 192]. For a survey of work in higher dimensions, see [4, 5, 94, 131, 147, 156]. The proof of the existence and uniqueness of a Denjoy–Wolff point for a function b in the unit ball of H^∞ which is not the identity is known as the Denjoy–Wolff theorem. The proof of this fact presented in Exercise 21.1.1 is taken from [166]. Exercise 21.1.2 is taken from Li [118]. The inequalities (21.7), (21.8) and (21.9) are due to Cowen and Pommerenke [60], who established many inequalities for fixed points of holomorphic functions. For the proof of these inequalities, Cowen and Pommerenke used deep complex analysis and some Grunsky-type inequalities. In his paper, Li employed a new method (which is presented in Exercise 21.1.2) based on $\mathcal{H}(b)$ spaces. This new method not only provides simpler proofs but also leads to some improvements.

Section 21.2

The connection between angular derivatives and mass points on the boundary has a long history. It can be traced back to Nevanlinna [135]. The connection between angular derivatives and square summability is due to M. Riesz [153].

Section 21.3

The cases $N = 0$ and $N = 1$ of Theorem 21.8(i) are due to Frostman [83]. Frostman's results were generalized by Cargo [43]. The version presented here was obtained by Ahern and Clark [10, 11]. In fact, Ahern and Clark systematically studied the boundary behavior of analytic functions in a series of papers [7–13]. Some of their results are addressed in this chapter; see also [44, 45]. The monograph [129] treats a systematic study of this subject.

Section 21.4

A special case of Theorem 21.10 for $N = 0$ is given in [44] without proof. The general version was mentioned in [10, 11], again without proof.

Section 21.5

The approximation Theorems 21.12 and 21.15 are taken from [10]. Frostman shifts, exceptional sets, Lemma 21.13 and Corollary 21.14 were introduced in [84]. This result has several applications, in particular in Carleson's proof of the corona conjecture; see also [108, 109, 130, 133].

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Section 21.6

The results presented in this section are very general and considered as common knowledge. For example, Lemmas 21.18 and 21.19 are implicitly used in [10].

Section 21.7

Lemma 21.20 can be found in [10, 81]. Theorems 21.23 and 21.24 are due to Ahern and Clark [10].

Section 21.8

In the case where b is an inner function, Helson [100] studied the problem of analytic continuation across the boundary for functions in the model space K_b. Then, still when b is an inner function, Ahern and Clark [8] characterized those points x_0 of $\mathbb{R}$ where every function f of K_b and all its derivatives up to order n have a radial limit. These results were generalized in the form of Theorem 21.26 for an arbitrary element of the closed unit ball by Fricain and Mashreghi [81]. We also mention that Sarason has obtained another criterion in terms of the Clark measure μ_λ associated with b; see following theorem.

Theorem 21.29 (Sarason [166]) *Let ζ_0 be a point of $\mathbb{T}$ and let ℓ be a nonnegative integer. The following conditions are equivalent.*

(i) *Each function in $\mathcal{H}(b)$ and all its derivatives up to order ℓ have nontangential limits at ζ_0.*
(ii) *There is a point $\lambda \in \mathbb{T}$ such that*

$$\int_{\mathbb{T}} |e^{i\theta} - \zeta_0|^{-2\ell-2} \, d\mu_\lambda(e^{i\theta}) < \infty. \tag{21.53}$$

(iii) *The last inequality holds for all $\lambda \in \mathbb{T} \setminus \{b(\zeta_0)\}$.*
(iv) *There is a point $\lambda \in \mathbb{T}$ such that μ_λ has a point mass at ζ_0 and*

$$\int_{\mathbb{T}\setminus\{z_0\}} |e^{i\theta} - \zeta_0|^{-2\ell} \, d\mu_\lambda(e^{i\theta}) < \infty.$$

Recently, Bolotnikov and Kheifets [36] gave a third criterion (in some sense more algebraic) in terms of the Schwarz–Pick matrix. Recall that, if b is a function in the closed unit ball of H^∞, then the matrix $\mathbf{P}_\ell^\omega(z)$, which will be referred to as to a Schwarz–Pick matrix and defined by

$$\mathbf{P}_\ell^b(z) := \left[\frac{1}{i!j!} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} \frac{1 - |b(z)|^2}{1 - |z|^2} \right]_{i,j=0}^{\ell},$$

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is positive semidefinite for every $\ell \geq 0$ and $z \in \mathbb{D}$. One can extend this notion to boundary points as follows: given a point $\zeta_0 \in \mathbb{T}$, the boundary Schwarz–Pick matrix is

$$\mathbf{P}_\ell^b(\zeta_0) = \lim_{\substack{z \to \zeta_0 \\ \sphericalangle}} \mathbf{P}_\ell^b(z) \qquad (\ell \geq 0),$$

provided this nontangential limit exists; see following theorem.

Theorem 21.30 *Let b be a point in the closed unit ball of H^∞, let $\zeta_0 \in \mathbb{T}$ and let ℓ be a nonnegative integer. Assume that the boundary Schwarz–Pick matrix $\mathbf{P}_\ell^b(\zeta_0)$ exists. Then each function in $\mathcal{H}(b)$ and all its derivatives up to order ℓ have nontangential limits at ζ_0.*

Further it is shown in [36] that the boundary Schwarz–Pick matrix $\mathbf{P}_\ell^b(\zeta_0)$ exists if and only if

$$\lim_{\substack{z \to \zeta_0 \\ \sphericalangle}} d_{b,\ell}(z) < \infty, \tag{21.54}$$

where

$$d_{b,\ell}(z) := \frac{1}{(\ell!)^2} \frac{\partial^{2\ell}}{\partial z^\ell \partial \bar{z}^\ell} \frac{1 - |b(z)|^2}{1 - |z|^2}.$$

We should mention that, to date, there is no clear direct connection between conditions (21.53), (21.54) and condition (v) of Theorem 21.26.