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MATHEMATICAL NOTES

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HYPERSURFACES FIXED BY EQUIAFFINITIES

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1. **Introduction.** In this note we state and prove the following

THEOREM. *Any equiaffinity acting on the points of an n-dimensional vector space* ($n \geq 2$) *leaves invariant the members of a one parameter family of hypersurfaces defined by polynomials* $p(x_1, \ldots, x_n) = c$ *of degree* $m \leq n$.

REMARK. The theorem, restricted to the real plane, appears to have been discovered almost simultaneously by Coxeter [4] and Komissaruk [5]. The former paper presents an elegant geometric argument, showing that the result follows from the converse of Pascal's theorem. The present approach is more closely related to that of [5], in which the transformations are reduced to a canonical form.

DEFINITIONS AND NOTATION. Let F^n denote an n-dimensional vector space over the commutative field F. An *equiaffinity* $\phi: F^n \to F^n$ is defined as the product of a matrix transformation of determinant 1 (usually called a *unimodular matrix*) and a translation. The image of a point $P:(x_1, \ldots, x_n)$ under ϕ is written as $P^\phi = PA + L$ where A is an $n \times n$ matrix whose determinant $|A| = 1$, and L is a row vector (a $1 \times n$ matrix that represents the translation from the origin to the point with the coordinates of L). In other words, the equiaffinities form a group that has as subgroups the special linear group and the group of translations. A discussion of affine transformations can be found in [1] or [2].

Alternatively, ϕ can be defined on an affine Pappian space of n dimensions as a product of shears. Details of the geometric approach can be found in [3] or [4].

LEMMAS. (1.1) *When A has all its eigenvalues in F, the coordinates of F^n can be chosen so that the matrix A is in its Jordan Canonical form; specifically,*

$$A = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & A_k \end{bmatrix}, \qquad \textit{where } A_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 & 0 \\ 0 & 0 & \lambda_i & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix}$$

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(for $1 \le i \le k$) is an $e_i \times e_i$ matrix, 0 are zero matrices of appropriate sizes, λ_i are eigenvalues of A (not necessarily distinct), and $\sum_{i=1}^{k} e_i = n$.

A proof can be found in [2].

(1.2) *When A has no eigenvalues equal to one, ϕ has a fixed point, namely*

$$P = L(I-A)^{-1},$$

where I is the $n \times n$ identity matrix.

2. **Proof of the theorem.** The theorem will be proved in three steps, each conceptually easy, complicated only by the notation.

(i) The case when $\lambda_i \in F$, $\lambda_i \neq 1$, $1 \le i \le k$.

There is a fixed point by (1.2) which we can take to be the origin; hence ϕ can be represented by the matrix A in (1.1).

A family of invariant polynomials of degree n and parameter c would be

$$x_1^{e_1} x_{e_1+1}^{e_2} x_{e_1+e_2+1}^{e_3} \cdots x_{s_k}^{e_k} = c \tag{2.1}$$

where $s_k = 1 + \sum_{i=1}^{k-1} e_i$. To verify this claim we need only check that $P:(x_1, \ldots, x_n)$ satisfies (2.1) if and only if $P^\phi = (\lambda_1 x_1, \ldots, \lambda_2 x_{e_1+1}, \ldots, \lambda_3 x_{e_1+e_2+1}, \ldots)$ does:

$$\begin{aligned}(\lambda_1 x_1)^{e_1} (\lambda_2 x_{e_1+1})^{e_2} (\lambda_3 x_{e_1+e_2+1})^{e_3} \cdots (\lambda_k x_{s_k})^{e_k} &= \left(\prod_{i=1}^{k} \lambda_i^{e_i}\right) (x_1^{e_1} x_{e_1+1}^{e_2} x_{e_1+e_2+1}^{e_3} \cdots x_{s_k}^{e_k}) \\ &= |A|\, c = c.\end{aligned}$$

(ii) The case when some of the λ_i are not in F and $\lambda_i \neq 1$, $1 \le i \le k$.

Extend F to the splitting field $K = F(\lambda_1, \ldots, \lambda_n)$ of the characteristic equation of A so that this case reduces to (i). Since the equiaffinity induced by ϕ on the extended space K^n leaves the original space invariant, the intersection of the invariant family of polynomial hypersurfaces given in (i) with the smaller space would be an invariant family of polynomial hypersurfaces.

(iii) The case when $\lambda_i = 1$ for at least one i, $1 \le i \le k$.

Let us assume that $\lambda_1 = 1$. We complete the proof by induction on n.

The case $n=2$ was dealt with in [4] and [5]. These papers, however, were concerned only with the real affine plane; for completeness I prove it again here:

Since $|A|=1$ both eigenvalues are 1; thus there are two possible forms for A and four for ϕ. When L is the nonzero vector $[l_1, l_2]$, these are

$$\underset{\text{(a)}}{P^\phi = P\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \quad \underset{\text{(b)}}{P^\phi = P\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}} \quad \underset{\text{(c)}}{P^\phi = P\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + L} \quad \underset{\text{(d)}}{P^\phi = P\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} + L}$$

(a) is the identity for which the theorem is trivially true, (b) is a shear and (c) is a translation so that there are families of invariant lines, $x=c$ and $l_1 y - l_2 x = c$ respectively. In (d) when $l_1 \neq 0$, there is a family of fixed parabolas, $x^2 + (2l_2 - l_1)x - 2l_1 y = c$. (When $l_1 = 0$, (d) is a shear.)

Assume now that the theorem is true in $n-1$ dimensions. Since A_1 in (1.1) has e_1 rows and columns, it has all zeros in the $e_1{}^{st}$ row except in the $e_1{}^{st}$ column. We define an affinity ϕ' of $n-1$ dimensions as follows: delete the $e_1{}^{st}$ row and column in A to get A' and delete the $e_1{}^{st}$ entry in L. ϕ' is an equiaffinity since $1=|A|=\prod_{i=1}^{k} \lambda_i^{e_i}=\lambda_1^{e_1-1} \prod_{i=2}^{k} \lambda_i^{e_i}=|A'|$ [the third equality is a consequence of $\lambda_1=1$]. By the induction hypothesis ϕ' leaves invariant the members of a one parameter family of hypersurfaces defined by polynomials in $n-1$ variables x_i, $i\neq e_1$. Their equations represent cylinders in n dimensions. These cylinders are left invariant by ϕ since none of the coordinates of the image of an arbitrary point are affected by the value of x_{e_1}, the point's $e_1{}^{st}$ coordinate, except possibly the $e_1{}^{st}$ coordinate (which does not appear in the equations). This completes the proof.

REMARKS. (3.1) The converse of the theorem is not true. For example, it is pointed out in [1, p. 96] that $A=\left[\begin{smallmatrix} a^2 & 0 \\ 0 & a \end{smallmatrix}\right]$ sends the parabolas $x=cy^2$ into themselves. $|A|\neq 1$ so that ϕ is not an equiaffinity.

(3.2) There may be more than one family of polynomials fixed by an equiaffinity. For example, a rotation of Euclidean 3-space about the z-axis leaves invariant the cylinders $x^2+y^2=c$ as well as the spheres $x^2+y^2+z^2=c$.

(3.3) Two equiaffinities that leave invariant the same family of polynomials are not necessarily similar. For example if ϕ_1 and ϕ_2 are represented by the matrices

$$\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \text{ and } \begin{bmatrix} b & 0 \\ 0 & 1/b \end{bmatrix}$$

respectively, they both leave invariant the hyperbolas $xy=c$. However, they are not similar unless $a=b$ or $a=1/b$.

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