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AN ELEMENTARY PROBABILISTIC COMPUTATION OF THE POISSON KERNEL FOR THE $n=2$ AND 3 EUCLIDEAN BALL

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ABSTRACT. Direct and elementary derivation of the classical Poisson kernel for the ball in $n=2$ or $n=3$ dimensions starting with the usual expression $\hat{f}(x)=E_x(f(b_\omega(T)))$ involving the brownian motion b_ω, the stopping time T on the boundary, and E_x the conditional expectation on paths starting at x.

It came out as a surprise when Peter Greiner told me that no probabilist he knew was able to give him a reference, and neither could I, of where to find the derivation of the classical Poisson kernel in $n=2$ or $n=3$ dimension starting with the usual compact formula

$$\hat{f}(x)=E_x(f(b_\omega(T))).$$

Here f is given on the boundary of the ball, b_ω is the usual brownian motion in $n=2$ or $n=3$ dimension; T stands for the stopping time of b_ω on the boundary, E_x is the conditional expectation for the paths starting at x.

I want, in this note, to give a direct procedure to find the Poisson kernel. Besides the fun, I hope to emphasize the use of the so called MacKean martingale which has been so accurate in the study of the Pauli's equation [1].

In order to be complete, I shall divide the paper in two: one concerning $n=2$ and the other for $n=3$.

I. The Poisson kernel in $n=2$. D is the unit disc, b_ω the brownian motion in $\mathbb{R}^2$ starting at x_0 in D. T_ω is the stopping time on the boundary ∂D of D:

$$T_\omega=\inf\{t \mid b_\omega(t)\in\partial D\}.$$

We shall use polar coordinates which means for the brownian that instead of using the Laplacian

$$\frac{1}{2}\Delta=\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

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We shall split it into radial and tangential parts:

$$\frac{1}{2}\Delta = \underbrace{\frac{1}{2}\frac{\partial^2}{\partial r^2}+\frac{1}{2r}\frac{\partial}{\partial r}}_{\text{radial}}+\underbrace{\frac{1}{2r^2}\frac{\partial^2}{\partial\theta^2}}_{\text{tangential.}}$$

This gives the polar decomposition of b_ω:

$$b_\omega \equiv (r_\omega, \theta_{\tilde\omega})$$

where r_ω and $\theta_{\tilde\omega}$ are given by the following stochastic equations:

$$dr_\omega = d\beta_\omega + \frac{ds}{r_\omega}$$

$$d\theta_{\tilde\omega} = d\beta_{\tilde\omega}/r_\omega$$

$d\beta_\omega$, $d\beta_{\tilde\omega}$ are usual brownians on the line $\mathbb{R}$, $d\beta_\omega$ and $d\beta_{\tilde\omega}$ being independent. So:

$$r_\omega(t) = r_0 + \beta_\omega(t) + \int_0^t \frac{ds}{r_\omega(s)}$$

$$\theta_{\tilde\omega}(t) = \theta_0 + \int_0^t \frac{d\beta_{\tilde\omega}(s)}{r_\omega(s)}$$

if $x_0 \equiv (r_0, \theta_0)$.

We are now in a position to get the Poisson kernel for D. As we know [3], the so called harmonic measure is the law of the hittings of b_ω on ∂D and the solution to the Dirichlet problem is given by

$$\hat f(x_0) = E_{x_0}(f(b_\omega(T_\omega)))$$

f is the data on ∂D, $\hat f$ the harmonic extension in D, E_{x_0} the expectation on the paths starting at x_0 in D.

Expand f in Fourier series and use the polar coordinates described before for b_ω (observe that T_ω is a stopping time for r_ω and not for $\theta_{\tilde\omega}$):

$$\begin{aligned}\hat f(x_0) &= E_{r_0}E_{\theta_0}\sum_{-\infty}^{+\infty} a_n(f)\exp in\,\theta_{\tilde\omega}(T_\omega)\\ &= \sum_{-\infty}^{+\infty} a_n(f)E_{r_0}E_{\theta_0}\exp\left\{in\left(\theta_0+\int_0^{T_\omega}\frac{d\beta_{\tilde\omega}}{r_\omega}\right)\right\}\\ &= \sum_{-\infty}^{+\infty} a_n(f)\exp in\,\theta_0 E_{r_0}E_{\theta_0}\left\{\exp\left(in\int_0^{T_\omega}\frac{d\beta_{\tilde\omega}}{r_\omega}\right)\right\}.\end{aligned}$$

Here we use the main tool in this subject: the MacKean martingale which gives

$$E_{\theta_0}\left(\exp in\int_0^{T_\omega}\frac{d\beta_{\tilde\omega}}{r_\omega}\right) = \exp -\frac{n^2}{2}\int_0^{T_\omega}\frac{ds}{r_\omega^2}.$$

because r_ω is independent from $\beta_{\tilde{\omega}}$ and appears here as a *sure function of* $\tilde{\omega}$([3] p. 22).

Then

$$\Psi(r)=E_r\left(\exp -\frac{n^2}{2}\int_0^{T_\omega}\frac{ds}{r_\omega^2}\right)$$

is the solution to the following differential equation:

$$\begin{cases}\frac{1}{2}\Psi''+\frac{1}{2r}\Psi'-\frac{n^2}{2r^2}\Psi=0\\ \Psi(0)=0, \qquad \Psi(1)=1.\end{cases}$$

It is very easy to solve it and find

$$\Psi(r)=r^{|n|}.$$

So

$$\hat{f}(x_0)=\sum_{-\infty}^{+\infty} a_n(f)e^{\text{in}\,\theta_0}\,r_0^{|n|}$$

but

$$a_n(f)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)e^{-\text{in}\,\phi}\,d\phi$$

therefore

$$\hat{f}(x_0)=\frac{1}{2\pi}\int_0^{2\pi}\sum_{-\infty}^{+\infty} e^{\text{in}(\theta_0-\phi)}r_0^{|n|}f(\phi)\,d\phi$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r_0^2}{1-2r_0\cos(\theta_0-\phi)+r_0^2}f(\phi)\,d\phi$$

and here is the Poisson kernel.

II. The Poisson kernel in $n=3$. In $n=3$ the method though not very different is slightly more complicated, the system $\{e^{\text{in}\,\theta}\}$ being replaced by spherical harmonics. Namely, any L^2 function on the sphere Σ is the L^2 limit of

$$\sum_0^{\infty}\tau^n c_n\int_\Sigma P_n(\vec{\xi}\cdot\vec{\eta})f(\vec{\eta})\,d\sigma_{\vec{\eta}}$$

when $\tau\to 1^-$. P_n is the usual Legendre polynomial, c_n is a constant depending only on n. If f is continuous, the convergence is uniform and one can write

$$f(x)=\sum_0^{+\infty} c_n\int_0^{\pi}\int_{-\pi}^{+\pi}P_n(\cos\gamma)f(\alpha,\beta)\sin\alpha\,d\alpha\,d\beta;$$

α is the colatitude, β the longitude and $\cos\gamma$ is computed according to the rule with $x\equiv(\theta,\phi)$,

$$\cos\gamma=\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\beta-\phi).$$

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The Legendre's polynomial has a remarkable addition property if one introduces the $A_{n,j}(t)$, associated Legendre functions, which are up to a coefficient

$$\left[\sqrt{\left(\frac{(n-j)!}{n!}\frac{\Gamma(n+1)}{\Gamma(n+j+3)}c_n\right)}\right]:(1-t^2)^{j/2}\frac{d^j}{dt^j}P_n(t).$$

Namely, one has:

$$c_nP_n(\cos\gamma)=\frac{1}{2\pi}\sum_0^n A_{n,j}(\cos\alpha)A_{n,j}(\cos\theta)\cos j(\beta-\phi).$$

I shall proceed now to the probabilistic frame: We introduce radial (r), colatitude (θ) and longitude (ϕ) coordinates for x in the unit ball B. T_ω is the hitting time for the brownian traveller starting at $x_0\equiv(r_0,\theta_0,\phi_0)$ on the sphere Σ. The splitting of the Laplacian in spherical coordinates

$$\frac{1}{2}\Delta=\left[\frac{1}{2}\frac{\partial^2}{\partial r^2}+\frac{2}{2r}\frac{\partial}{\partial r}\right]+\left[\frac{1}{2r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot an\,\theta}{2r^2}\frac{\partial}{\partial\theta}\right]+\left[\frac{1}{2r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

gives the following

$$d\phi_{\omega_1}=\frac{d\beta_{\omega_1}}{r\sin\theta}$$

$$d\theta_{\omega_2}=\frac{d\beta_{\omega_2}}{r}+\frac{\cot an\,\theta_{\omega_2}}{2r^2}\,ds$$

$$dr_{\omega_3}=d\beta_{\omega_3}+\frac{ds}{2r_{\omega_3}}$$

with β_{ω_1}, β_{ω_2}, β_{ω_3} three independent browian motions on the line.

For the sake of better clarity, I shall divide the subsequent computations into three lemmas which consist in the computation of E_ϕ, E_θ, E_r respectively in the following formula

$$\hat{f}(x_0)=\hat{f}(r_0,\theta_0,\phi_0)=E_{r_0}E_{\theta_0}E_{\phi_0}[f(b_\omega(T_\omega))]$$

expressing the solution of the Dirichlet kernel given f on the boundary Σ.

LEMMA 1.

$$E_{\phi_0}[\cos j(\beta-\phi_{\omega_1}(T_\omega))]=\exp\left(-\frac{j^2}{2}\int_0^{T_\omega}\frac{ds}{r_{\omega_3}^2\sin^2\theta_{\omega_2}}\right)\cos j(\beta-\phi_0).$$

Proof.

$$\cos j(\beta-\phi_{\omega_1}(T_\omega))=\tfrac{1}{2}[\exp(ij\beta)\exp(-ij\phi_{\omega_1})+\exp(-ij\beta)\exp(+ij\phi_{\omega_1})].$$

By *MacKean's martingale*, one has

$$E_{\phi_0}(\exp \pm ij\phi_{\omega_1}(T_\omega)) = \exp\left(-\frac{j^2}{2}\int_0^{T_\omega} \frac{ds}{r^2 \sin^2\theta}\right) \times \exp(\pm ij\phi_0)$$

and this gives Lemma 1.

LEMMA 2.

$$E_{\theta_0}\left[A_{n_{1j}}(\cos\theta_{\omega_2})\exp -\frac{j^2}{2}\int_0^{T_\omega}\frac{ds}{r^2\sin^2\theta}\right]$$
$$= (2n+1)A_{n_{1j}}(\cos\theta_0)\exp\left(-\frac{n(n+1)}{2}\int_0^{T_\omega}\frac{ds}{r_\omega^2}\right).$$

Proof. Recall that T_ω depends only on r_ω, so in the preceding equality T_ω is *fixed.*

The fundamental solution for

$$\frac{\partial\psi}{\partial t} = \frac{1}{2r_\omega^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\operatorname{cotan}^2\theta}{2r_\omega^2}\frac{\partial\psi}{\partial\theta} - \frac{j^2}{2r_\omega^2\sin^2\theta}\Psi$$

is given by Paul Levy in his famous book on brownian motion [2]:

$$\Psi(t,\theta) = \sum_{p=0}^{+\infty}(2p+1)\exp\left(-p(p+1)\int_0^t \frac{ds}{2r_\omega^2}\right)A_{p,j}(\cos\theta).$$

So the left hand side to the above equality is given by

$$A_{n,j}(\cos\theta_0)\int_0^\pi \sum_{p=0}^{+\infty}(2p+1)\exp\left(-p(p+1)\int_0^{T_\omega}\frac{ds}{2r_\omega^2}\right)\times\cdots$$
$$\times A_{p,j}(\cos\alpha)A_{n,j}(\cos\alpha)\sin\alpha\, d\alpha.$$

But the $A_{p,j}$ are orthogonal so $\int_0^\pi A_{p,j}A_{n,j}\sin\alpha\, d\alpha$ is null unless $n=p$ for which it equals 1. This gives Lemma 2.

LEMMA 3.

$$E_{r_0}\left(\exp -\frac{n(n+1)}{2}\int_0^T \frac{ds}{r_\omega^2}\right) = r_0^n, \qquad n \geq 0.$$

Proof. Same as in dimension 2, now with the equation

$$\begin{cases} \dfrac{1}{2}y'' + \dfrac{1}{r}y' - \dfrac{n(n+1)}{2r^2}y = 0 \\ y(0)=0, \qquad y(1)=1. \end{cases}$$

Summing up Lemmas 1 to 3:

PROPOSITION:

$$E_{r_0}E_{\theta_0}E_{\phi_0}A_{n,j}(\cos\theta_{\omega_2}(T_\omega))\cos j(\beta - \phi_{\omega_1}(T_\omega))$$
$$= (2n+1)r_0^n A_{n,j}(\cos\theta_0)\cos j(\beta-\phi_0)$$

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But now this gives

$$\hat{f}(r_0, \theta_0; \phi_0) = \sum_{n \geq 0} (2n+1)r^n \int_0^{\pi} \int_{-\pi}^{+\pi} P_n(\cos\gamma) f(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta$$

with $\cos\gamma = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\beta-\phi)$. The Poisson's kernel is

$$\sum_{n \geq 0} (2n+1)r^n P_n(\cos\gamma) = \frac{1-r^2}{(1-2r\cos\gamma+r^2)^{3/2}}. \quad \text{cqfd.}$$

QUESTION. Is it possible to do the same thing for the Heisenberg group? This will answer a question raised in [4].

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