CORRECTIONS TO AND REMARKS ON SOME RESULTS FOR THE GENERALIZED LOTOTSKY TRANSFORM

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1. Introduction. One purpose of this note is to make some corrections to **(1)**. The authors are indebted to Professors A. E. Livingston and A. Meir for pointing out certain errors in **(1)** and for suggesting possible corrections. The definitions and notation used in this note may be found in **(1)**. First, we would like to list some corrections which are essentially trivial. On page 421, line 14 from the top, the formula for b_{nk} should contain $e^{-i(n-k)\alpha}$ instead of $e^{i(n-k)\alpha}$. In Corollary 4.2, we should have

$$\lim_{n\to\infty} \rho_n\theta_n = -\infty \qquad \text{instead of} \qquad \lim_{n\to\infty} \rho_n\theta_n = +\infty.$$

On page 429, line 5 from the top should read $(4\pi/9) < \alpha < \pi/2$ instead of $0 < \alpha < \pi/2$. On page 430, line 7 from the top should read $(1 + \rho)^2 - 2\alpha$ instead of $(1 + \rho)^2 - \alpha$.

This brings us to the first correction with non-trivial consequences. The last part of the first paragraph of Section 2 should read "$|d_n| = \rho_n(\rho_n \geqslant \gamma > 0)$ and $\arg d_n = \theta_n(-\pi < \theta_n \leqslant \pi)$" instead of "$|d_n| = \rho_n$ and $\arg d_n = \theta_n(-\pi < \theta_n \leqslant \pi)$." The condition $\rho_n \geqslant \gamma > 0$ is specifically needed only in the proofs of Theorems 2.2, 2.3, 2.4, and 3.1. It obviously does not apply to Theorems 4.1, 4.2, 5.1, and their corollaries; and the case $\rho = 0$ is of trivial interest in Theorems 4.3 and 4.4. As for the remaining Theorem 2.1, it should be fairly obvious from the notation that ρ_n is not to be restricted in this theorem. It should be noted that $\rho_n \geqslant \gamma > 0$ is just a convenient sufficient conditions for Theorems 2.2, 2.3, and 2.4 in **(1)**. As was pointed out to us, a slight generalization of these theorems can be obtained by inserting the hypothesis

$$\sum_{n=1}^{\infty} \rho_n(1 + \rho_n)^{-2} = +\infty$$

in each theorem instead of requiring the condition $\rho_n \geqslant \gamma > 0$.

The proofs of Theorem 3.1, Theorem 4.1, Corollary 4.1, and Corollary 4.2 are incorrect. The proofs may be made correct and the text made logical and correct by adding the hypothesis

$$\lim_{n\to\infty} \theta_n = 0$$

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to both theorems and both corollaries, and changing the first sentence on page 425 to read as follows: "By assumption

$$\lim_{n\to\infty} \theta_n = 0."$$

2. Remarks. Theorem 3.1, Theorem 4.1, Corollary 4.1, and Corollary 4.2, however, are correct as stated in **(1)** even without the restriction $\rho_n \geqslant \gamma > 0$. We shall first consider the case of Theorem 3.1 of **(1)**.

THEOREM 1. *If*

$$\sum_{n=1}^{\infty} \theta_n^2 \rho_n^{-1} < +\infty,$$

then

$$\sum_{n=1}^{\infty} \frac{\rho_n \sin^2(\theta_n/2)}{|1 + d_n|^2} < +\infty.$$

Proof. Since

$$\sum_{n=1}^{\infty} \theta_n^2 \rho_n^{-1} < +\infty,$$

we have

$$\lim_{n\to\infty} \theta_n^2 \rho_n^{-1} = 0.$$

Therefore, if $\{\rho_{nk}\}$ is a bounded sub-sequence of $\{\rho_n\}$, then

$$\lim_{k\to\infty} \theta_{nk} = 0.$$

Consequently only a finite number of the d_n lie inside the circle $|1 + z| = |z|/2$. For all other d_n we have $|1 + d_n| \geqslant |d_n|/2$, and so

$$\frac{\rho_n \sin^2(\theta_n/2)}{|1 + d_n|^2} \leqslant \frac{\rho_n(\theta_n/2)^2}{\rho_n^2/4} = \frac{\theta_n^2}{\rho_n} \tag{2.1}$$

for n sufficiently large. The theorem now follows.

In **(1)** there is a mistake in the proof of Theorem 3.1 at the top of page 425. It is only possible to conclude that

$$\liminf_{n\to\infty} \theta_n = 0, \qquad \text{not} \qquad \lim_{n\to\infty} \theta_n = 0$$

as stated. Consequently, the inequality in line 5, page 425 of **(1)** is not established in **(1)**. This inequality, however, is now established as it follows from (2.1) above. Hence, if either the condition $\rho_n \geqslant \gamma > 0$ is satisfied or the hypothesis

$$\sum_{n=1}^{\infty} \rho_n(1 + \rho_n)^{-2} = +\infty$$

is added, the proof of Theorem 3.1 is now complete. However, neither of these conditions is necessary to the proof of Theorem 3.1. It is trivial to see that (3.5) of **(1)** can be replaced by

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$$\left|\frac{t + d_n}{1 + d_n}\right| \leqslant \exp\left\{\frac{-8\rho_n - 15}{32(1 + \rho_n)^2}\right\};$$

and that (3.6) of (1) can be replaced by

$$|P_{nk}| \leqslant M4^k \exp\left\{-\sum_{j=N}^{n} \frac{8\rho_j + 15}{32(1 + \rho_j)^2}\right\}$$

for $n \geqslant N$. Now

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$$

implies that

$$\sum_{j=N}^{\infty} \frac{8\rho_j + 15}{32(1 + \rho_j)^2} = +\infty$$

for $\rho_n \geqslant 0$. This completes a proof of Theorem 3.1 as it is stated in **(1)** without even the assumption $\rho_n \geqslant \gamma > 0$. Thus it is not necessary to make the changes in Theorem 3.1 given above in Section 1. Instead by making a few alterations in the proof of Theorem 3.1 given in **(1)** we can obtain a correct proof of this theorem. For easy reference we now state the following theorem.

THEOREM 3.1. *Suppose that*

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \theta_n^2 \rho_n^{-1} < +\infty,$$

then the $[F, d_n]$ *transform is regular.*

If we do not make use of (2.1) above on page 425, line 5 of **(1)**, but instead replace $\theta_n^2 \rho_n^{-1}$ by $\rho_n \sin^2(\theta_n/2)\ |1 + d_n|^{-2}$ in the remaining equations in that paragraph, we obtain the following theorem, which is Theorem A of **(2)**.

THEOREM A. *If*

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{\rho_n \sin^2(\theta_n/2)}{|1 + d_n|^2} < +\infty,$$

then the $[F, d_n]$ *matrix is regular.*

THEOREM 2. *If* $\rho_n \geqslant \gamma > 0,\ -\pi < \theta_n \leqslant \pi$, *and*

$$\sum_{n=1}^{\infty} \frac{\rho_n \sin^2(\theta_n/2)}{|1 + d_n|^2} < +\infty,$$

then

$$\sum_{n=1}^{\infty} \theta_n^2 \rho_n^{-1} < +\infty.$$

Proof. The proof of this theorem is similar to the proof of Theorem 1. We need only interchange the roles of the two series involved.

From our proof of Theorem A we easily see that Theorem A implies Theorem

3.1. Theorems 1 and 2 show that if $\rho_n \geqslant \gamma > 0$, then Theorem A of **(2)** and Theorem 3.1 of **(1)** are equivalent.

Theorem 4.1 of **(1)** is correct as stated. The proof of Theorem 4.1 given in **(1)** is incorrect because the sentence following inequality (4.5) in **(1)** is incorrect. To correct this proof we must show that

$$\prod_{j=1}^{\infty} \frac{z + d_j}{1 + d_j} = 0$$

follows from the inequality (4.5) of **(1)** whenever $x - 1 < 0$. This will be accomplished if we prove the following theorem.

THEOREM 3. *Let $z = x + iy$. If $x < 1$, $-\pi < \theta_n \leqslant \pi$,*

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty, \qquad \sum_{n=1}^{\infty} \theta_n^{\,2} \rho_n^{-1} < +\infty, \qquad \text{and} \quad \lim_{n\to\infty} \rho_n = +\infty,$$

then the series

$$\sum_{n=1}^{\infty} \frac{2\rho_n[(x-1)\cos\theta_n + y\sin\theta_n] + |z|^2 - 1}{1 + 2\rho_n\cos\theta_n + \rho_n^{\,2}} \tag{2.2}$$

diverges to $-\infty$.

Proof. Let $z(x < 1)$ be given. Set $-\delta = x - 1$ and $\tan\beta = \delta(2|y|)^{-1}, 0 < \beta < \pi/2$. Our hypotheses imply that $|\theta_n| < \beta$ for infinitely many values of n. Let $\{n_k\}$ denote the sub-sequence of integers such that $|\theta_{n_k}| < \beta$; then

$$|y \sin\theta_{n_k}| < 2^{-1}\delta\cos\beta.$$

Since $|\theta_{n_k}| < \beta$ and $\delta > 0$, we have $-\delta\cos\theta_{n_k} < -\delta\cos\beta$. It now follows that

$$-\delta\cos\theta_{n_k} + y\sin\theta_{n_k} < -2^{-1}\delta\cos\beta.$$

Multiplying both sides of this inequality by $2\rho_{n_k}$, we obtain

$$2\rho_{n_k}[(x-1)\cos\theta_{n_k} + y\sin\theta_{n_k}] < -\rho_{n_k}\delta\cos\beta.$$

Since

$$\lim_{k\to\infty} \rho_{n_k} = +\infty,$$

there exists N_1 such that if $k \geqslant N_1$, then

$$\rho_{n_k}\delta\cos\beta > 2(|z|^2 - 1).$$

Hence if $n \geqslant N_1$, then

$$2\rho_{n_k}[(x-1)\cos\theta_{n_k} + y\sin\theta_{n_k}] + |z|^2 - 1 < -2^{-1}\rho_{n_k}\delta\cos\beta.$$

In any case $1 + 2\rho_{n_k}\cos\theta_{n_k} + \rho_{n_k}^{\,2} \leqslant (1 + \rho_{n_k})^2$. Combining these last two inequalities, we obtain

$$\frac{2\rho_{n_k}[(x-1)\cos\theta_{n_k} + y\sin\theta_{n_k}] + |z|^2 - 1}{1 + 2\rho_{n_k}\cos\theta_{n_k} + \rho_{n_k}^{\,2}} < -\frac{\rho_{n_k}\delta\cos\beta}{2(1+\rho_{n_k})^2} \tag{2.3}$$

whenever $k \geqslant N_1$.

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We divide the remainder of the proof into two cases. First, suppose there are only a finite number of values of n such that $|\theta_n| > \beta$; then there exists $N (N \geqslant N_1)$ such that if $n \geqslant N$, then $|\theta_n| \leqslant \beta$. Therefore, if $n \geqslant N$

$$\frac{2\rho_n[(x-1)\cos\theta_n + y\sin\theta_n] + |z|^2 - 1}{1 + 2\rho_n\cos\theta_n + \rho_n^2} < -\frac{\rho_n\delta\cos\beta}{2(1+\rho_n)^2}.$$

Since

$$\lim_{n\to\infty}\rho_n = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\rho_n^{-1} = +\infty,$$

it follows from this inequality that the series (2.2) is divergent to $-\infty$.

Now suppose that $|\theta_n| \geqslant \beta$ for infinitely many values of n. Let $\{n_m\}$ denote the sub-sequence of integers such that $|\theta_{n_m}| \geqslant \beta$. Since

$$\theta_n^2 \rho_n^{-1} > 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\theta_n^2\rho_n^{-1} < +\infty,$$

it follows that

$$\sum_{m=1}^{\infty}\theta_{nm}^2\rho_{nm}^{-1} < +\infty.$$

Since $|\theta_{n_m}| \geqslant \beta$, we have

$$\sum_{m=1}^{\infty}\rho_{nm}^{-1} < +\infty.$$

Hence the assumption

$$\sum_{n=1}^{\infty}\rho_n^{-1} = +\infty$$

implies that

$$\sum_{k=1}^{\infty}\rho_{nk}^{-1} = +\infty.$$

Since

$$\lim_{n\to\infty}\rho_n = +\infty,$$

there exists N_1 such that if $m \geqslant N_1$, then

$$\rho_{n_m}^{-1}(|z|^2 - 1) < 2 \qquad \text{and} \qquad 1 + 2\rho_{n_m}\cos\theta_{n_m} + \rho_{n_m}^2 \geqslant 2^{-1}\rho_{n_m}^2.$$

Hence, if $m \geqslant N_1$, we have

$$\frac{2\rho_{nm}[(x-1)\cos\theta_{nm} + y\sin\theta_{nm}] + |z|^2 - 1}{1 + 2\rho_{nm}\cos\theta_{nm} + \rho_{nm}^2} \leqslant \frac{4(|x| + |y| + 2)}{\rho_{nm}}.$$

Since

$$\sum_{m=1}^{\infty}\rho_{nm}^{-1}$$

is convergent, it follows from this inequality that

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$$\sum_{m=1}^{\infty} \frac{2\rho_{nm}[(x-1)\cos\theta_{nm} + y\sin\theta_{nm}] + |z|^2 - 1}{1 + 2\rho_{nm}\cos\theta_{nm} + \rho_{nm}^{\;2}} < +\infty. \tag{2.4}$$

Since

$$\sum_{k=1}^{\infty} \rho_{nk}^{\;-1} = +\infty \quad \text{and} \quad \lim_{k\to\infty} \rho_{nk} = +\infty,$$

it follows that

$$\sum_{k=1}^{\infty} \frac{\rho_{nk}\,\delta\cos\beta}{4(1+\rho_{nk})^2} = +\infty. \tag{2.5}$$

It now follows from (2.3), (2.4), and (2.5) that the series (2.2) diverges to $-\infty$. This completes the proof.

Corollaries 4.1 and 4.2 as stated in **(1)** now follow from Theorem 4.1.

In conclusion, the Theorems 3.1 and 4.1 are correct as stated in **(1)** without additional assumptions, the proofs being now complete. Theorems 2.2, 2.3, and 2.4 are false without the correction $\rho_n \geqslant \gamma > 0$ or some additional hypothesis such as $\Sigma\rho_n(1+\rho_n)\pi^2 = +\infty$. The addition of either of these conditions makes the proofs given in **(1)** of Theorems 2.2, 2.3, and 2.4 correct.

References

1. V. F. Cowling and C. L. Miracle, *Some results for the generalized Lototsky transform,* Can. J. Math., *14* (1962), 418–435.
2. A. E. Livingston, *Regularity of the generalized Lototsky transform* (to appear).

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