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THE ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES OF RIGHT DEFINITE MULTIPARAMETER STURM–LIOUVILLE SYSTEMS

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This paper studies the asymptotic distribution of the multiparameter eigenvalues of a right definite multiparameter Sturm–Liouville eigenvalue problem. A uniform asymptotic analysis of the oscillation number of solutions of a single Sturm–Liouville type equation with potential depending on a general parameter is given; these results are then applied to the system of multiparameter Sturm–Liouville equations to give the asymptotic eigenvalue distribution for the system as a function of a "multi-index" oscillation number.

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1. Introduction

Consider the k-parameter Sturm–Liouville eigenvalue problem

$$u_r''(x_r)+\left(q_r(x_r)+\sum_{s=1}^{k}\lambda_s v_{rs}(x_r)\right)u_r(x_r)=0,\quad 0\leqq x_r\leqq 1, r=1,\ldots,k, \tag{1.1}$$

$$u_r(0)\cos\alpha_r-u_r'(0)\sin\alpha_r=0, u_r(1)\cos\beta_r-u_r'(1)\sin\beta_r=0,\quad r=1,\ldots,k. \tag{1.2}$$

where $q_r, r=1,\ldots,k$, are real valued, continuous functions on the interval $U=[0,1]$, v_{rs}, $r,s=1,\ldots,k$, are real valued, twice continuously differentiable (C^2) function on U and $\alpha_r,\beta_r\in[0,2\pi]$. A k-tuple $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_k)$ of real numbers is called an eigenvalue of (1.1), (1.2) if, for each r, there exists a non-trivial solution u_r of equation (1.1) satisfying the boundary conditions (1.2). We assume that the eigenvalue problem (1.1), (1.2) is uniformly "right definite", i.e.

$$\det_{1\leq r,s\leqq k} v_{rs}(x_r)>0\quad\text{for all } (x_1,\ldots,x_k)\in U^k. \tag{1.3}$$

With this assumption the basic result regarding the existence of eigenvalues of (1.1), (1.2) is Klein's oscillation theorem (see [**8**]):

Theorem 1.1. *For each multi-index* $\mathbf{i}=(i_1,\ldots,i_k)$, *where* $i_1,\ldots,i_k$, *are non-negative integers, there exists a unique eigenvalue* $\boldsymbol{\lambda}^{\mathbf{i}}$ *of* (1.1), (1.2) *such that, for each* r, *a corresponding solution of* (1.1), (1.2) *has precisely* i_r *zeros in the open interval* (0, 1).

In this paper we study the asymptotic behaviour of the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ of (1.1), (1.2), for large $\|\mathbf{i}\|$, and we give a fairly simple characterization of this behaviour in terms of the coefficient functions v_{rs}. For the case $k=2$, this problem has been studied in great detail by Faierman in the papers [**6, 7**]. Faierman's analysis is much more complicated than ours and he obtains more detailed results, but at the cost of more restrictive hypotheses. He also obtains results regarding the eigenfunctions of (1.1), (1.2), which we do not consider. His bounds on the eigenfunctions have been extended to the general k-parameter situation by Schafke and Volkmer in [11].

2. Preliminary results and notation

We begin by discussing the asymptotic behaviour of the number of zeros of the general solution of a second order differential equation similar to (1.1), depending on a single parameter. Our results will be based on the following hypothesis. Suppose that f is a real valued C^2 function defined on U.

Hypothesis F. *Suppose that the set* $\{x \in U : f(x) > 0\}$ *can be decomposed into the union of a finite number of disjoint, open intervals* $I_i = (a_i^1, a_i^2)$, $i = 1, \ldots, n$, *(together with any of the end points* a_i^j, *which are not zeros of* f*) and there exists a constant* $K > 1$ *such that on each interval* I_i *either*

(i)
$$\frac{|(x - a_i^1) f'(x)|}{f(x)} \leqq K, \quad \frac{|(x - a_i^1)^2 f''(x)|}{f(x)} \leqq K, \quad i = 1, \ldots, n; \tag{2.1}$$

(ii) *there is an increasing function* $\tilde{f}_i$ *such that,*

$$K^{-1} \tilde{f}_i(x) \leqq f(x) \leqq K \tilde{f}_i(x); \tag{2.2}$$

or (i) holds with a_i^1 *replaced by* a_i^2, *and (ii) holds with* $\tilde{f}$ *decreasing.*

Now, let $\|f\| = \sup\{|f(x)| : x \in U\}$, and let $[f]_+$ denote the function $x \to \max\{f(x), 0\}$, $x \in U$. The closure of a set A will be denoted by $\bar{A}$.

Lemma 2.1. *Consider the differential equation*

$$w''(x) + p(x)w(x) + \mu f(x)w(x) = 0, \quad x \in U, \tag{2.3}$$

where p *is a real valued, continuous function on* U *and* f *satisfies hypothesis F. Then for all* $\mu > 0$, *the number of zeros* $\nu(w)$ *of any solution* w *of* (2.3) *in the interval* (0, 1) *satisfies*

$$\nu(w) = \pi^{-1} \mu^{1/2} \int_0^1 [f(x)]_+^{1/2} \, dx + O(1), \tag{2.4}$$

where $|O(1)| \leqq 2(n+1)(\pi^{-1}\|p\|^{1/2} + (K^2 + \|p\|)K^2 + 5)$.

Proof. By hypothesis F, there are at most $n+1$ subintervals of U on which f is not strictly positive. On each such subinterval it follows from the Sturm comparison principle (see [4]) that any solution w of (2.3) can have at most $\pi^{-1}\|p\|^{1/2} + 3$ zeros. Now, letting $\nu(w; I)$ denote the number of zeroes of w in any interval I, it follows that

$$\nu(w) = \sum_{i=1}^{n} \nu(w; I_i) + O(1), \tag{2.5}$$

where $|O(1)| \leqq (n+1)(\pi^{-1}\|p\|^{1/2}+3)+2n$ (the term $2n$ is the number of end points of the intervals I_i). We now estimate the number of zeros in an interval I_i. To simplify the notation, and without loss of generality, we will temporarily omit the subscript i, and suppose that $a^1=0$, $a^2=c$, and conditions (i) and (ii) hold as stated in hypothesis F.

By Ex. 2.5 on p. 197 of [**10**], equation (2.3) has the general solution

$$w(x)=A(\mu f(x))^{-1/4}\left(\sin\left\{\mu^{1/2}\int_x^c f(t)^{1/2}\,dt+\delta\right\}+\varepsilon(x)\right), \quad x\in I, \tag{2.6}$$

in which A and δ are arbitrary constants, and

$$|\varepsilon(x)|, (\mu f(x))^{-1/2}|\varepsilon'(x)| \leqq \exp\{V_{x,c}(F)\}-1, \quad x\in I, \tag{2.7}$$

where

$$F(x)=\mu^{-1/2}\int_x^c \{f(t)^{-1/4}(f(t)^{-1/4})''-p(t)f(t)^{-1/2}\}\,dt, \; V_{x,c}(F)=\int_x^c |F'(t)|\,dt.$$

Since f is C^2 on $[0,c]$ and $f(x)\neq 0$ on $(0,c]$,

$$|F'(x)|=\mu^{-1/2}|f(x)^{-1/4}(f(x)^{-1/4})''-p(x)f(x)^{-1/2}|, \quad x\in I,$$

so, by condition (i) in hypothesis F, we find that for all $x\in I$,

$$|f(x)^{-1/4}(f(x)^{-1/4})''| \leqq \tfrac{1}{2}f(x)^{-5/2}|f'(x)|^2+\tfrac{1}{2}f(x)^{-3/2}|f''(x)| \leqq K^2 f(x)^{-1/2}x^{-2},$$

and hence, by (2.2),

$$|F'(x)| \leqq \mu^{-1/2}(K^2+\|p\|)f(x)^{-1/2}x^{-2} \leqq \mu^{-1/2}C\tilde{f}(x)^{-1/2}x^{-2},$$

where $C=(K^2+\|p\|)K$. Thus, since $\tilde{f}$ is increasing we have

$$V_{x,c}(F) \leqq \mu^{-1/2}C\tilde{f}(x)^{-1/2}x^{-1}, \quad x\in I.$$

Now define the number γ_μ to be the solution of the equation

$$\mu^{-1/2}C\tilde{f}(x)^{-1/2}x^{-1}=1/(2\sqrt{2})$$

if a solution exists (since $\tilde{f}$ is increasing, such a solution is unique), otherwise let $\gamma_\mu=c$. By construction,

$$V_{x,c}(F)<1/(2\sqrt{2}), \quad x\in(\gamma_\mu, c),$$

and hence by (2.7)

$$|\varepsilon(x)|,\ (\mu f(x))^{-1/2}|\varepsilon'(x)|<1/\sqrt{2}, \quad x\in(\gamma_\mu, c). \tag{2.8}$$

Now define the strictly decreasing continuous function $g:(\gamma_\mu, c)\to\mathbb{R}$ by

$$g(x)=\mu^{1/2}\int_x^c f(t)^{1/2}\,dt+\delta, \quad x\in(\gamma_\mu c).$$

It follows from (2.6) and (2.8) that any zero x of w in (γ_μ, c) must satisfy

$$g(x)\in(n\pi-\pi/4, n\pi+\pi/4) \tag{2.9}$$

for some integer n, and since, by (2.8),

$$|-\mu^{1/2}f(x)^{1/2}\cos g(x)+\varepsilon'(x)|>0$$

for all x satisfying (2.9), there is at most one zero satisfying (2.9) for each integer n. In addition, by (2.8) and the intermediate value theorem, for any n such that the image of g contains the interval $[n\pi-\pi/4, n\pi+\pi/4]$ there is a zero of w satisfying (2.9). Hence, the number of zeros, $\nu(w;(\gamma_\mu, c))$, of the solution w in the interval (γ_μ, c) satisfies

$$\nu(w;(\gamma_\mu, c))=\pi^{-1}\mu^{1/2}\int_{\gamma_\mu}^c f(x)^{1/2}\,dx+O(1),$$

where $|O(1)|\leqq 3/2$. By hypothesis F,

$$f(x)\leqq K\tilde{f}(x)\leqq K\tilde{f}(\gamma_\mu), \quad x\in[0,\gamma_\mu]. \tag{2.10}$$

Thus, using the definition of γ_μ we have

$$\mu^{1/2}\int_0^{\gamma_\mu} f(x)^{1/2}\,dx\leqq\mu^{1/2}\gamma_\mu K\tilde{f}(\gamma_\mu)^{1/2}\leqq 2\sqrt{2}CK,$$

and hence

$$\nu(w;(\gamma_\mu, c))=\pi^{-1}\mu^{1/2}\int_0^c f(x)^{1/2}\,dx+O(1), \tag{2.11}$$

where $|O(1)|\leqq\pi^{-1}2\sqrt{2}CK+3/2\leqq CK+3/2$.

We now estimate the number of zeros of w in the interval $(0,\gamma_\mu]$. By (2.10) and the Sturm comparison principle we have

$$\begin{aligned}\nu(w;(0,\gamma_\mu])&\leqq\gamma_\mu/[\pi(\|p\|+\mu K\tilde{f}(\gamma_\mu))^{-1/2}]+3\leqq\pi^{-1}\gamma_\mu(\|p\|^{1/2}+\mu^{1/2}K\tilde{f}(\gamma_\mu)^{1/2})+3\\&\leqq\pi^{-1}\|p\|^{1/2}+\pi^{-1}2\sqrt{2}CK+3\leqq\pi^{-1}\|p\|^{1/2}+CK+3.\end{aligned}$$

Combining this with (2.11) shows that

$$v(w; I_i) = \pi^{-1}\mu^{1/2} \int_{a_i^1}^{a_i^2} f(x)^{1/2}\,dx + O(1),$$

(where we have reinstated the subscript i and the general end points a_i^j for I_i) with $|O(1)| \leqq CK + 3/2 + \pi^{-1}\|p\|^{1/2} + CK + 3 \leqq \pi^{-1}\|p\|^{1/2} + 2CK + 5$. Since $[f(x)]_+ = f(x)$ on the intervals $\bar{I}_i$, and $[f(x)]_+ = 0$ otherwise, this estimate together with (2.5) yields (2.4).

The asymptotic estimate (2.4) of Lemma 2.1 is, essentially, known; see for example [**5**]. However, the function f is usually regarded as fixed, whereas below it will be essential to regard it as depending on a parameter ξ belonging to some parameter space X (in fact, the parameter will be $\boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|$ and X will be a portion of the unit sphere in $\mathbb{R}^k$). Thus zeros of f may coalesce as ξ varies, and the number of zeros and their orders may change. To ensure that the bound for the error term in (2.4) is independent of ξ we will impose the following hypothesis.

Hypothesis UF. *Suppose that the function $f: U \times X \to \mathbb{R}$ is such that for each $\xi \in X$ the function $f(\cdot, \xi): U \to \mathbb{R}$ satisfies hypothesis F, and let $n(\xi)$, $K(\xi)$ denote the number of intervals and the constant in hypothesis F. Then f is said to satisfy hypothesis UF if $n(\xi)$ and $K(\xi)$ are uniformly bounded for $\xi \in X$, i.e. there exist constants $n > 0$, $K > 0$, such that*

$$n(\xi) \leqq n, \ K(\xi) \leqq K, \quad \xi \in X.$$

Now, if a function f satisfies hypothesis UF then the result of Lemma 2.1 holds for each $\xi \in X$ (with the constants n, K of hypothesis UF). Thus we have the following result.

Lemma 2.2. *Consider the differential equation*

$$w''(x) + p(x)w(x) + \mu f(x, \xi)w(x) = 0, \quad x \in U, \xi \in X, \tag{2.12}$$

where p is a real valued, continuous function on U and f satisfies hypothesis UF. Then for all $\mu > 0$, and $\xi \in X$, the number of zeros $v(w)$ of any solution w of (2.12) *in the interval* (0, 1) *satisfies*

$$v(w) = \pi^{-1}\mu^{1/2} \int_0^1 [f(x, \xi)]_+^{1/2}\,dx + O(1), \tag{2.13}$$

where $|O(1)| \leqq 2(n+1)(\pi^{-1}\|p\|^{1/2} + (K^2 + \|p\|)K^2 + 5)$.

We note that the exact form of the bound for the error term given in Lemma 2.2 is not particularly important and could be improved. Its value lies in the fact that it is given explicitly in terms of $\|p\|$, n and K, and is independent of ξ and μ. The importance of this will become clear below.

We will now show that hypothesis UF holds when X is a compact topological space and the function f depends continuously on (x,ξ) and has an analytic (in x) extension to a complex neighbourhood of U.

Lemma 2.3. *Suppose that X is a compact topological space and A is a complex domain containing U, and suppose that $f:A\times X\to\mathbb{C}$ is continuous on $A\times X$ and for each $\xi\in X$ the function $f(\cdot,\xi)\not\equiv 0$ is analytic on A. Also suppose that $f|_{U\times X}$ (the restriction of f to $U\times X$) is real valued. Then $f|_{U\times X}$ satisfies hypothesis UF.*

Proof. By shrinking A, if necessary, we may suppose that A is bounded and the hypotheses of the theorem also hold on a domain $\tilde{A}$ containing $\bar{A}$. Choose a point $\xi_0\in X$ and let $z_i(\xi_0)$, $i=1,\dots,n_0$ be the zeros of the function $f(\cdot,\xi_0)$ in A, counting multiplicities (the fact that $f(\cdot,\xi_0)$ is analytic on $\tilde{A}\supset\bar{A}$ ensures that the number of zeros in A is finite). By Rouche's theorem there is an open neighbourhood $N_0\subset X$ of ξ_0 such that for each $\xi\in\bar{N}_0$ there are exactly n_0 zeros $z_i(\xi)$, $i=1,\dots,n_0$, of $f(\cdot,\xi)$ in A (if $f(\cdot,\xi_0)$ has a zero on the boundary of A, we shrink A further and choose N_0 sufficiently small to ensure that these zeros do not enter A as ξ varies over $\bar{N}_0$). Also, if the collection of zeros $Z(\xi)=(z_1(\xi)\dots,z_{n_0}(\xi))$ is regarded as an unordered n_0-tuple in the sense described in Section 2.5.2 of [**9**], then the mapping $\xi\to Z(\xi)$ is continuous on $\bar{N}_0$ in the sense of [**9**]. For each $\xi\in\bar{N}_0$, $x\in A$, $x\neq z_i(\xi)$, $i=1,\dots,n_0$, we now define $g(x,\xi)$ by the formula

$$f(x,\xi)=g(x,\xi)\prod_{i=1}^{n_0}(x-z_i(\xi)). \tag{2.14}$$

It follows from Taylor's theorem that the points $x=z_i(\xi)$ are removable singularities of the function $g(\cdot,\xi)$ (see Ch. 4, Section 3.1 of [**1**]), thus we can extend the definition of $g(\cdot,\xi)$ to the whole of A, to yield an analytic function on A (which we continue to denote by $g(\cdot,\xi)$). In addition, $g(x,\xi_0)$ is a continuous function of (x,ξ) in $A\times\bar{N}_0$. This follows from (2.14) and the continuity of the n_0-tuple $Z(\xi)$, except at points $(x,\xi)=(z_i(\xi),\xi)$, $i=1,\dots,n_0$. To prove the continuity of g at such a point, say $(z_j(\xi_1),\xi_1)$, we choose a contour $C\subset A$ surrounding $z_j(\xi_1)$ and not passing through any zero $z_i(\xi_1)$ and let C^- denote the open subset of $\mathbb{C}$ surrounded by the contour C. Then Cauchy's integral formula

$$g(x,\xi)=\frac{1}{2\pi i}\int_C\frac{g(\zeta,\xi)}{\zeta-x}\,d\zeta,\quad x\in C^-,$$

holds for all ξ in a neighbourhood N_1 of ξ_1 sufficiently small that the zeros $z_i(\xi)$ are bounded away from C for $\xi\in N_1$. Since $g(x,\xi)$ is continuous on $C\times N_1$, $g(x,\xi)$ is continuous on $C^-\times N_1$.

Now let A' be a domain containing U such that $\bar{A}'\subset A$. Then g is a continuous, non-zero function on the compact set $S=\bar{N}_0\times\bar{A}'$ (g is non-zero on S since any zero of g

would contribute another zero of f other than the zeros $z_i(\xi)$). Similarly it can be shown that the functions $g'(x,\xi)$, $g''(x,\xi)$, are continuous on S (where $'$ denotes derivative with respect to x). Thus there exist constants $M_0, m_0 > 0$ such that

$$|g^{(r)}(x,\xi)| \leqq M_0\,,\ r=0,1,2,\ |g(x,\xi)| \geqq m_0, \quad (x,\xi)\in S. \tag{2.15}$$

Now choose an arbitrary point $\xi \in N_0$ and for each $i=1,\dots,n_0$, let $e_i(\xi)=\operatorname{Re} z_i(\xi)$. We suppose, without loss of generality, that the numbering of the zeros $z_i(\xi)$ is such that $e_1(\xi)\leqq e_2(\xi)\leqq\cdots\leqq e_{n_0}(\xi)$. Now, for $i=1,\dots,n_0$, let

$$I_i^-(\xi)=\{x\in U: f(x,\xi)>0 \text{ and } \tfrac{1}{2}(e_{i-1}(\xi)+e_i(\xi))<x<e_i(\xi)\},$$

$$I_i^+(\xi)=\{x\in U: f(x,\xi)>0 \quad\text{and}\quad e_i(\xi)<x<\tfrac{1}{2}(e_i(\xi)+e_{i+1}(\xi))\},$$

taking $e_0(\xi)=-e_1(\xi)$, $e_{n_0+1}(\xi)=2-e_{n_0}(\xi)$. Note that some of these sets may be empty. Considering only the non-empty sets $I_i^{\pm}(\xi)$, we obtain a collection of disjoint, open intervals which cover the set $\{x\in U: f(x,\xi)>0\}$ (except for any end points e at which $f(e,\xi)>0$).

Now suppose that $I_i^+(\xi)$ is non-empty. For any $x\in I_i^+(\xi)$,

$$\frac{|(x-e_i(\xi))f'(x,\xi)|}{f(x,\xi)}=\frac{(x-e_i(\xi))\left|g'(x,\xi)\prod_{j=1}^{n_0}(x-z_j(\xi))+g(x,\xi)\sum_{l=1}^{n_0}\prod_{\substack{j=1\\ j\neq l}}^{n_0}(x-z_j(\xi)\right|}{g(x,\xi)\prod_{j=1}^{n_0}(x-z_j(\xi))}$$

$$\leqq (x-e_i(\xi))\frac{|g'(x,\xi)|}{|g(x,\xi)|}+\sum_{l=1}^{n_0}\frac{|x-e_i(\xi)|}{|x-z_l(\xi)|}\leqq\frac{M_0}{m_0}+n_0. \tag{2.16}$$

Similarly it can be shown that

$$\frac{|(x-e_i(\xi))^2 f''(x,\xi)|}{f(x,\xi)}\leqq\frac{M_0}{m_0}+2\,\frac{M_0}{m_0}\,n_0+n_0^2. \tag{2.17}$$

Next suppose that $i<n_0$ (the case $i=n_0$ is similar), and define the function

$$\tilde{f}(x,\xi)=\prod_{j\leqq i}|x-z_j(\xi)|\prod_{j>i}|e_i(\xi)-z_j(\xi)|,\quad x\in I_i^+(\xi).$$

Clearly $\tilde{f}(\cdot,\xi)$ is increasing on $I_i^+(\xi)$, and

$$\frac{\tilde{f}(x,\xi)}{f(x,\xi)}=\frac{1}{|g(x,\xi)|}\prod_{j>i}\frac{|e_i(\xi)-z_j(\xi)|}{|x-z_j(\xi)|}. \tag{2.18}$$

Since $I_i^+(\xi)$ is non-empty we must have $e_j(\xi) > e_i(\xi)$, for $j > i$, so $|x - z_j(\xi)|$ is decreasing on $I_i^+(\xi)$. Hence

$$1 \leqq \frac{|e_i(\xi) - z_j(\xi)|^2}{|x - z_j(\xi)|^2} \leqq \frac{|e_i(\xi) - e_j(\xi)|^2 + |\operatorname{Im} z_j(\xi)|^2}{|\frac{1}{2}(e_i(\xi) + e_{i+1}(\xi)) - e_j(\xi)|^2 + |\operatorname{Im} z_j(\xi)|^2}$$

$$\leqq \frac{|e_i(\xi) - e_j(\xi)|^2 + |\operatorname{Im} z_j(\xi)|^2}{\frac{1}{4}|e_i(\xi) - e_j(\xi)|^2 + |\operatorname{Im} z_j(\xi)|^2} \leqq 4, \quad x \in I_i^+(\xi),$$

and so

$$\frac{1}{M_0} \leqq \frac{\tilde{f}(x,\xi)}{f(x,\xi)} \leqq \frac{1}{m_0} 2^{n_0}, \quad x \in I_i^+(\xi). \tag{2.19}$$

The inequalities (2.16), (2.17) and (2.19) show that $f|_{U \times X}$ satisfies conditions (i) and (ii) as stated in hypothesis F on the non-empty intervals $I_i^+(\xi)$; similarly it can be shown that the alternative form of conditions (i) and (ii) hold on the non-empty intervals $I_i^-(\xi)$. The constants in (2.16), (2.17) and (2.19) are clearly uniform with respect to ξ in N_0, as is the number of zeros of $f(\cdot,\xi)$ in U, so hypothesis UF holds for ξ in N_0. Since the set X is assumed to be compact it can be covered by a finite collection of such neighbourhoods, and choosing the maximum values of the corresponding finite collection of constants proves that hypothesis UF holds for all $\xi \in X$. This completes the proof of the lemma.

The condition $f(\cdot,\xi) \not\equiv 0$, $\xi \in X$, above is not necessary, e.g. $f(x,\xi) = \xi \sin x$, $\xi \in [0,1]$ satisfies hypothesis UF. However, $f(x,\xi) = \xi \sin(x/\xi)$, $\xi \in (0,1]$, $f(x,0) = 0$, does not.

In the case of general C^r functions hypothesis F is satisfied by functions f which behave 'roughly' like powers of $(x-a)$ near a zero a; e.g. x^n, $n>0$, or $x^n \log x$ satisfy it, whereas $\exp(-1/x)$ does not. However, general conditions which ensure that hypothesis UF holds are not known.

We now introduce some further notation. For any point $\mathbf{a} \in \mathbb{R}^k$, we write $\mathbf{a} > \mathbf{0}$ (respectively $\mathbf{a} \geqq \mathbf{0}$) if, and only if, $a_r > 0$ (respectively $a_r \geqq 0$), for all $r = 1, \ldots, k$ and we let $\mathbb{R}^k_+$ (respectively $\bar{\mathbb{R}}^k_+$) denote the set of points $\mathbf{a} \in \mathbb{R}^k$ for which $\mathbf{a} > \mathbf{0}$ (respectively $\mathbf{a} \geqq \mathbf{0}$). For each $r = 1, \ldots, k$, we define the functions

$$\mathbf{v}_r(x) := (v_{r1}(x), \ldots, v_{rk}(x)),\ \boldsymbol{\lambda} \cdot \mathbf{v}_r(x) := \sum_{s=1}^{k} \lambda_s v_{rs}(x), \quad x \in U,\ \boldsymbol{\lambda} \in \mathbb{R}^k.$$

$$\phi_r(\boldsymbol{\lambda}) := \int_U [\boldsymbol{\lambda} \cdot \mathbf{v}_r(x)]_+^{1/2}\, dx,\ \boldsymbol{\phi}(\boldsymbol{\lambda}) := (\phi_1(\boldsymbol{\lambda}), \ldots, \phi_k(\boldsymbol{\lambda})), \quad \boldsymbol{\lambda} \in \mathbb{R}^k.$$

Clearly, the function $\boldsymbol{\phi}: \mathbb{R}^k \to \mathbb{R}^k$ is continuous and for all $\boldsymbol{\lambda} \in \mathbb{R}^k$, $\boldsymbol{\phi}(\boldsymbol{\lambda}) \geqq \mathbf{0}$. Also, for any real $c \geqq 0$,

$$\phi(c\lambda) = c^{1/2}\phi(\lambda). \tag{2.20}$$

Let Q denote the set of points λ for which $\phi(\lambda) > \mathbf{0}$, and let $\bar{Q}$ be the closure of Q. It follows from (2.20) that the sets Q and $\bar{Q}$ are cones (a set $A \subset \mathbb{R}^k$ is said to be a cone if, for any $\mathbf{a} \in A$, $c\mathbf{a} \in A$ for all $c > 0$). We will now prove some simple preliminary lemmas concerning ϕ and Q. The Euclidean norm on $\mathbb{R}^k$ will be denoted by $\|\cdot\|$. In the following, constants c_i, $i = 1, 2, \ldots$, will be strictly positive and may depend on the functions p_r, v_{rs}, but will not depend on λ, or any other variables.

Lemma 2.4. *There is a constant $c_1 > 0$ such that for any $\lambda \in \mathbb{R}^k$, there is an r, $1 \leqq r \leqq k$, such that*

$$|\lambda \cdot \mathbf{v}_r(x)| \geqq c_1 \|\lambda\|, \quad x \in U.$$

Proof. Suppose that the lemma is false. Then there must exist a $\lambda \neq \mathbf{0}$ such that for each $r = 1, \ldots, k$, there is a point $x_r^0 \in U$ with

$$\lambda \cdot \mathbf{v}_r(x_r^0) = 0.$$

However, this contradicts the right definiteness assumption (1.3), which proves the lemma.

Lemma 2.5. *There are constants $c_2, c_3 > 0$ such that for all $\lambda \in \bar{Q}$,*

$$c_2 \|\lambda\| \leqq \|\phi(\lambda)\|^2 \leqq c_3 \|\lambda\|. \tag{2.21}$$

Proof. If $\lambda \in Q$, then $\phi(\lambda) > \mathbf{0}$ and so by Lemma 2.4 there must be an r such that

$$c_1 \|\lambda\| \leqq \lambda \cdot \mathbf{v}_r(x), \quad x \in U.$$

Thus the first inequality in (2.21) holds for all $\lambda \in Q$, and hence, by continuity of ϕ, it holds for all $\lambda \in \bar{Q}$. The second inequality in (2.21) follows immediately from the definition of ϕ and the boundedness of the functions v_{rs}.

Lemma 2.6. *Suppose that $\lambda^1, \lambda^2 \in \bar{Q}$ and $\|\lambda^1\|, \|\lambda^2\| \leqq 1$. Then:*

(i)
$$\|\phi(\lambda^2) - \phi(\lambda^1)\| \geqq c_4 \|\lambda^2 - \lambda^1\|^{3/2}; \tag{2.22}$$

(ii) *if $\phi(\lambda^j) \geqq (\varepsilon, \ldots, \varepsilon)$, $j = 1, 2$, for some ε, $0 < \varepsilon \leqq 1$, then*

$$\|\phi(\lambda^2) - \phi(\lambda^1)\| \geqq \varepsilon c_5 \|\lambda^2 - \lambda^1\|; \tag{2.23}$$

(iii) *suppose that for all $\lambda \in \bar{Q}$ with $\|\lambda\| \leqq 1$, all $r = 1, \ldots, k$, and all $\gamma \in [0, 1]$, the set $\{x \in U : \lambda \cdot \mathbf{v}_r(x) \geqq -\gamma\}$ has Lebesgue measure at least $c_6 \gamma^{1/2}$. Then,*

$$\|\boldsymbol{\phi}(\boldsymbol{\lambda}^2|-\boldsymbol{\phi}(\boldsymbol{\lambda}^1)\| \geqq c_7\|\boldsymbol{\lambda}^2-\boldsymbol{\lambda}^1\|. \tag{2.24}$$

Proof. Suppose that $\boldsymbol{\mu}:=\boldsymbol{\lambda}^2-\boldsymbol{\lambda}^1\neq\mathbf{0}$. Then by Lemma 2.4 there is an r such that

$$\boldsymbol{\mu}\cdot\mathbf{v}_r(x)\geqq c_1\|\boldsymbol{\mu}\|, \quad x\in U \tag{2.25}$$

(or $\boldsymbol{\mu}\cdot\mathbf{v}_r(x)\leqq -c_1\|\boldsymbol{\mu}\|$, $x\in U$, in which case we interchange $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ to obtain (2.25)). By definition,

$$\phi_r(\boldsymbol{\lambda}^2)-\phi_r(\boldsymbol{\lambda}^1)=\int_U \delta(x)\,dx. \tag{2.26}$$

where $\delta(x):=[\boldsymbol{\lambda}^1\cdot\mathbf{v}_r(x)+\boldsymbol{\mu}\cdot\mathbf{v}_r(x)]_+^{1/2}-[\boldsymbol{\lambda}^1\cdot\mathbf{v}_r(x)]_+^{1/2}$. Now let a be the maximum of the function $\boldsymbol{\lambda}^1\cdot\mathbf{v}_r$ in U. Since $\boldsymbol{\lambda}^1\in\bar{Q}$, it follows from continuity that $a\geqq 0$. Using (2.25) and the simple inequality

$$(\alpha+\beta)^{1/2}-\alpha^{1/2}=\beta/((\alpha+\beta)^{1/2}+\alpha^{1/2})\geqq\tfrac{1}{3}\beta(\max\{\alpha,\beta\})^{-1/2}, \quad \alpha,\beta\geqq 0, \max\{\alpha,\beta\}>0,$$

it can be seen that

$$\delta(x)\geqq c_8\|\boldsymbol{\mu}\|(\max\{a,\|\boldsymbol{\mu}\|\})^{-1/2}, \quad x\in P, \tag{2.27}$$

$$\delta(x)\geqq\tfrac{1}{4}c_1^{1/2}\|\boldsymbol{\mu}\|^{1/2}, \quad x\in N, \tag{2.28}$$

where

$$P=\{x\in U:\boldsymbol{\lambda}^1\cdot\mathbf{v}_r(x)\geqq 0\},\ N=\{x\in U:0>\boldsymbol{\lambda}^1.\mathbf{v}_r(x)\geqq -c_9\|\boldsymbol{\mu}\|\},$$

where $c_9=\frac{1}{2}\min\{1,c_1\}$. Since the functions v_{rs} are C^2 on U it follows that $|P|\geqq c_{10}a$, and there is a sufficiently small constant c_{11} such that

$$\max\{|P|,|N|\}>c_{11}\|\boldsymbol{\mu}\|, \tag{2.29}$$

where $|\cdot|$ denotes Lebesgue measure (we use the assumption that $\|\boldsymbol{\lambda}^1\|, \|\boldsymbol{\lambda}^2\|\leqq 1$ here). Also, when the hypothesis of part (iii) of the lemma holds, (2.29) can be replaced with the stronger inequality

$$\max\{|P|,|N|\}\geqq c_{12}\|\boldsymbol{\mu}\|^{1/2}. \tag{2.30}$$

Suppose that $a\geqq\|\boldsymbol{\mu}\|$. Then by (2.26) and (2.27),

$$\phi_r(\boldsymbol{\lambda}^2)-\phi_r(\boldsymbol{\lambda}^1)\geqq c_8\|\boldsymbol{\mu}\|a^{-1/2}c_{10}a=c_8c_{10}\|\boldsymbol{\mu}\|a^{1/2}. \tag{2.31}$$

Since $a\geqq\|\boldsymbol{\mu}\|$, (2.22) follows immediately from this inequality. Also, we easily see that

$$a^{1/2} \geqq \phi_r(\boldsymbol{\lambda}^1), \tag{2.32}$$

and (2.23) now follows from (2.31), (2.32) and the assumption that $\phi_r(\boldsymbol{\lambda}^1) \geqq \varepsilon$. Assuming the hypothesis of (iii), (2.24) follows from (2.26), (2.27), (2.28), and (2.30).

Now suppose that $a < \|\boldsymbol{\mu}\|$. Then (2.22), (2.23) and (2.24) again follow from the above inequalities in a similar manner. This completes the proof of the lemma.

We remark that the measure condition (iii) of Lemma 2.6 holds if, for each $\boldsymbol{\lambda} \in \bar{Q}$, $\|\boldsymbol{\lambda}\| \leqq 1$, $r = 1, \ldots, k$, the maximum value of the function $x \to \boldsymbol{\lambda} . \mathbf{v}_r(x)$ is attained in the interior of U, i.e. where the derivative of the function is zero. However, this is not a necessary condition for (iii) to hold.

3. The main result

We can now prove our main result. For any $\varepsilon > 0$, let $\bar{\mathbb{R}}^k_\varepsilon$ denote the set $\{\mathbf{a} \in \mathbb{R}^k_+ : \|\mathbf{a}\| \geqq (\varepsilon, \ldots, \varepsilon)\}$.

Theorem 3.1. *Suppose that the functions* $(x, \boldsymbol{\lambda}) \to \boldsymbol{\lambda} . \mathbf{v}_r(x)$, $(x, \boldsymbol{\lambda}) \in U \times \bar{Q}$, $r = 1, \ldots, k$, *satisfy hypothesis UF (with bounds* n_r *and* K_r*). Then the mapping* $\boldsymbol{\phi}: \bar{Q} \to \bar{\mathbb{R}}^k_+$ *is a homeomorphism. Letting* $\boldsymbol{\phi}^{-1}: \bar{\mathbb{R}}^k_+ \to \bar{Q}$ *denote the inverse of this homeomorphism, the eigenvalues of the multiparameter problem satisfy*

$$\boldsymbol{\lambda}^{\mathbf{i}} = \pi^2 \boldsymbol{\phi}^{-1}(\mathbf{i}) + O(\|\mathbf{i}\|^{4/3}) = \pi^2 \|\mathbf{i}\|^2 \boldsymbol{\phi}^{-1}(\mathbf{i}/\|\mathbf{i}\|) + O(\|\mathbf{i}\|^{4/3}), \tag{3.1}$$

for all multi-indices $\mathbf{i} \neq \mathbf{0}$. *If* $\varepsilon > 0$, *then for all* $\mathbf{i} \in \bar{\mathbb{R}}^k_\varepsilon$,

$$\boldsymbol{\lambda}^{\mathbf{i}} = \pi^2 \boldsymbol{\phi}^{-1}(\mathbf{i}) + \varepsilon^{-1} O(\|\mathbf{i}\|) = \pi^2 \|\mathbf{i}\|^2 \boldsymbol{\phi}^{-1}(\mathbf{i}/\|\mathbf{i}\|) + \varepsilon^{-1} O(\|\mathbf{i}\|). \tag{3.2}$$

If the hypothesis of part (iii) *of Lemma* 2.6 *holds then*

$$\boldsymbol{\lambda}^{\mathbf{i}} = \pi^2 \boldsymbol{\phi}^{-1}(\mathbf{i}) + O(\|\mathbf{i}\|) = \pi^2 \|\mathbf{i}\|^2 \boldsymbol{\phi}^{-1}(\mathbf{i}/\|\mathbf{i}\|) + O(\|\mathbf{i}\|), \tag{3.3}$$

for all $\mathbf{i} \neq \mathbf{0}$.

Proof. For each $r = 1, \ldots, k$ let T_r denote the self-adjoint differential operator associated with the differential expression on the left-hand side of (1.1), with $\boldsymbol{\lambda} = \mathbf{0}$, together with the boundary conditions (1.2). We will assume that the greatest eigenvalue of each operator T_r is strictly negative and the functions q_r are negative. This assumption entails no loss of generality since we can ensure that this condition holds by making a translation of the eigenvalue parameter $\boldsymbol{\lambda}$, see Lemma 2.1 of [**2**], and this translation only affects the values of the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ by $O(1)$, which does not affect the statement of the theorem. With this assumption it follows immediately that any eigenvalue $\boldsymbol{\lambda}^{\mathbf{i}}$ of (1.1), (1.2) must belong to Q.

Consider the eigenvalue $\lambda^{\mathbf{i}}$, for any multi-index $\mathbf{i} \neq \mathbf{0}$. By definition, for each r there is a solution of the differential equation (1.1) with i_r zeros in $(0, 1)$. Thus, by Lemma 2.2,

$$\mathbf{i} = \pi^{-1}\boldsymbol{\phi}(\boldsymbol{\lambda}^{\mathbf{i}}) + O(1), \tag{3.4}$$

where the term $O(1)$ is bounded by a constant (given in Lemma 2.2) depending on the constants n_r and K_r, but not on $\boldsymbol{\lambda}$. By rearranging (3.4) and using (2.20), we find that

$$\boldsymbol{\phi}(\boldsymbol{\lambda}^{\mathbf{i}}/\|\mathbf{i}\|^2) = \pi\mathbf{i}/\|\mathbf{i}\| + O(\|\mathbf{i}\|^{-1}). \tag{3.5}$$

Now choose any $\mathbf{a} \in \bar{\mathbb{R}}^k_+$ such that $\|\mathbf{a}\| = 1$ and choose a sequence of multi-indices $\mathbf{i}^n$, $n = 1, 2, \ldots$, such that $\|\mathbf{i}^n\| \to \infty$ and $\pi\mathbf{i}^n/\|\mathbf{i}^n\| \to \mathbf{a}$ as $n \to \infty$. By Theorem 1.1 the eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}^n}$ exist for all $n = 1, 2, \ldots$, and by Lemma 2.5 and (3.5), the sequence $\boldsymbol{\lambda}^{\mathbf{i}^n}/\|\mathbf{i}^n\|^2$ is bounded, so it has an accumulation point, $\boldsymbol{\lambda}^*$, say. It follows from (3.5) and the continuity of $\boldsymbol{\phi}$ that

$$\boldsymbol{\phi}(\boldsymbol{\lambda}^*) = \mathbf{a},$$

and hence the range of $\boldsymbol{\phi}$ contains the set $\{\mathbf{a} \in \bar{\mathbb{R}}^k_+ : \|\mathbf{a}\| = 1\}$. Combining this with (2.20) shows that the range of $\boldsymbol{\phi}$ is $\bar{\mathbb{R}}^k_+$.

This result, together with Lemma 2.6, shows that the equation

$$\boldsymbol{\phi}(\boldsymbol{\lambda}) = \mathbf{a} \tag{3.6}$$

has a unique solution $\boldsymbol{\lambda} \in \bar{Q}$ for all $\mathbf{a} \in \bar{\mathbb{R}}^k_+$. Thus we can define the bijective function $\boldsymbol{\phi}^{-1} : \bar{\mathbb{R}}^k_+ \to \bar{Q}$. Now, for $j = 1, 2$, choose any $\mathbf{c}^j \in \bar{\mathbb{R}}^k_+$ with $\|\mathbf{c}^j\| \leqq c_2^{-1/2}$. Then by Lemma 2.5, $\boldsymbol{\phi}^{-1}(\mathbf{c}^j) \leqq 1$, and by Lemma 2.6 (with $\boldsymbol{\lambda}^j = \boldsymbol{\phi}^{-1}(\mathbf{c}^j) \in \bar{Q}$, $j = 1, 2$),

$$\|\mathbf{c}^2 - \mathbf{c}^1\| \geqq c_4 \|\boldsymbol{\phi}^{-1}(\mathbf{c}^2) - \boldsymbol{\phi}^{-1}(\mathbf{c}^1)\|^{3/2}. \tag{3.7}$$

This, together with (2.20), shows that the function $\boldsymbol{\phi}^{-1}$ is continuous and so, by definition, the mapping $\boldsymbol{\phi} : \bar{Q} \to \bar{\mathbb{R}}^k_+$ is a homeomorphism. Also, since $\boldsymbol{\lambda}^{\mathbf{i}} \in Q$, for all $\mathbf{i}$, (3.1) follows immediately from (2.22). Similarly, (3.2) and (3.3) follow from (3.5) together with (2.23) and (2.24) respectively. This completes the proof of the theorem.

Combining Lemma 2.3 and Theorem 3.1 yields the following result.

Theorem 3.2. *If the functions* v_{rs}, $r, s = 1, \ldots, k$, *are analytic and* $\boldsymbol{\lambda} . \mathbf{v}_r \equiv 0$, $\boldsymbol{\lambda} \neq \mathbf{0}$, *then the results of Theorem* 3.1 *hold.*

Finally we remark that it can be shown that $Q = -\bar{C}$, where C is the "comparison cone" of Binding and Browne [3], although $Q = -C$ need not be true (the minus sign here is not significant, it occurs because of the way in which we have written the multiparameter system, compared with that in [3]). The cone C is used in [3] to "compare" the multiparameter eigenvalues, e.g. it is shown in [3] that

$$\boldsymbol{\lambda}^{\mathbf{j}} \in \boldsymbol{\lambda}^{\mathbf{i}} - C, \text{ for all } \mathbf{j} - \mathbf{i} \geqq \mathbf{0}, \tag{3.8}$$

(again the minus sign arises from the different definitions of the multiparameter systems). Theorems 3.1 and 3.2 show that, in these cases, the cone C is, essentially, the best possible comparison cone in the sense that (3.8) cannot hold for any cone $C' \subset Q$ with $Q \backslash C'$ non-empty and open.

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