A NORM INEQUALITY FOR LINEAR TRANSFORMATIONS

B.N. Moyls[1] and N.A. Khan[2]

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In 1949 Ky Fan [1] proved the following result: Let $\lambda_1 \leq \dots \leq \lambda_n$ be the eigenvalues of an Hermitian operator H on an n-dimensional vector space V_n. If $x_1, \dots, x_q$ is an orthonormal set in V_n, and q is a positive integer such that $1 \leq q \leq n$, then

$$(1) \qquad \Sigma_{j=1}^{q} \lambda_j \leq \Sigma_{j=1}^{q} (Hx_j, x_j) \leq \Sigma_{j=1}^{q} \lambda_{n-j+1} .$$

We shall use this result to prove the following

THEOREM. Let A be a normal operator and let B be any operator on V_n. Let the eigenvalues $\alpha_j = \lambda_j + i\mu_j$, $j = 1, \dots, n$, of A be ordered so that $\lambda_1 \leq \dots \leq \lambda_n$ and $\mu_{\sigma(1)} \leq \dots \leq \mu_{\sigma(n)}$ for some permutation σ. Let $\nu_j = \mu_{\sigma(j)}$. Let $\beta_1 \leq \dots \leq \beta_n$ be the eigenvalues of $B + B^*$, and $\gamma_1 \leq \dots \leq \gamma_n$ the eigenvalues of $i(B - B^*)$. Then

$$(2) \qquad \Sigma_{j=1}^{n} \lambda_j(\lambda_j - \beta_j) + \Sigma_{j=1}^{n} \nu_j(\nu_j + \gamma_{n-j+1})$$

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$$\le ||B - A||^2 - ||B||^2$$

$$\le \Sigma_{j=1}^{n} \lambda_j(\lambda_j - \beta_{n-j+1}) + \Sigma_{j=1}^{n} \nu_j(\nu_j + \gamma_j),$$

where $||B||^2$ is the trace of BB^*.

Proof. By the linearity of the trace,

(3) $$||B - A||^2 - ||B||^2 = tr(B-A)(B-A)^* - tr\, BB^* = tr(AA^*) - tr(AB^* + BA^*).$$

Choose $x_1, \ldots, x_n$ to be an orthonormal set of eigenvectors for the normal matrix A, corresponding to $\alpha_1, \ldots, \alpha_n$. Then

(4) $$tr(AA^*) = \Sigma_{j=1}^{n} (AA^* x_j, x_j) = \Sigma_{j=1}^{n} |\alpha_j|^2 = \Sigma_{j=1}^{n} \lambda_j^2 + \Sigma_{j=1}^{n} \nu_j^2.$$

Similarly,

$$tr(AB^* + BA^*) = \Sigma_{j=1}^{n} ([AB^* + BA^*]x_j, x_j)$$

$$= \Sigma_{j=1}^{n} (B^* x_j, A^* x_j) + \Sigma_{j=1}^{n} (A^* x_j, B^* x_j)$$

$$= \Sigma_{j=1}^{n} \alpha_j (B^* x_j, x_j) + \Sigma_{j=1}^{n} \overline{\alpha}_j (Bx_j, x_j).$$

Thus

(5) $$tr(AB^* + BA^*) = \Sigma_{j=1}^{n} \lambda_j (Hx_j, x_j) - \Sigma_{j=1}^{n} \nu_j (Kx_j, x_j),$$

where $H = B + B^*$ and $K = i(B - B^*)$. If we set $\lambda_0 = 0$, then

(6) $$\Sigma_{j=1}^{n} \lambda_j (Hx_j, x_j) = \Sigma_{s=1}^{n} (\lambda_s - \lambda_{s-1}) \Sigma_{j=s}^{n} (Hx_j, x_j).$$

By (1) we have

(7) $$\Sigma_{j=s}^{n} \beta_{n-j+1} \le \Sigma_{j=s}^{n} (Hx_j, x_j) \le \Sigma_{j=s}^{n} \beta_j.$$

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Since $\lambda_s - \lambda_{s-1} \geq 0$, it follows from (6) and (7) that

$$\Sigma_{s=1}^{n} (\lambda_s - \lambda_{s-1}) \Sigma_{j=s}^{n} \beta_{n-j+1} \leq \Sigma_{j=1}^{n} \lambda_j (Hx_j, x_j)$$

$$\leq \Sigma_{s=1}^{n} (\lambda_s - \lambda_{s-1}) \Sigma_{j=s}^{n} \beta_j,$$

and, by the identity implied in (6),

(8) $$\Sigma_{j=1}^{n} \lambda_j \beta_{n-j+1} \leq \Sigma_{j=1}^{n} \lambda_j (Hx_j, x_j) \leq \Sigma_{j=1}^{n} \lambda_j \beta_j.$$

A similar argument gives

(9) $$\Sigma_{j=1}^{n} \nu_j \gamma_{n-j+1} \leq \Sigma_{j=1}^{n} \nu_j (Kx_j, x_j) \leq \Sigma_{j=1}^{n} \nu_j \gamma_j.$$

Combining (5), (8) and (9), we have

(10) $$\Sigma_{j=1}^{n} \lambda_j \beta_{n-j+1} - \Sigma_{j=1}^{n} \nu_j \gamma_j \leq \mathrm{tr}(AB^* + BA^*)$$

$$\leq \Sigma_{j=1}^{n} \lambda_j \beta_j - \Sigma_{j=1}^{n} \nu_j \gamma_{n-j+1}.$$

The result (2) follows immediately from (3), (4) and (10).

Two corollaries arise when A is Hermitian or skew Hermitian.

COROLLARY 1. Let A be an Hermitian operator on V_n with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$; and let B be an arbitrary operator, where $B + B^*$ has eigenvalues $\beta_1 \leq \ldots \leq \beta_n$. Then

$$\Sigma_{j=1}^{n} \lambda_j (\lambda_j - \beta_j) \leq ||B - A||^2 - ||B||^2$$

$$\leq \Sigma_{j=1}^{n} \lambda_j (\lambda_j - \beta_{n-j+1}).$$

COROLLARY 2. Let A be a skew Hermitian operator on

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V_n with eigenvalues $i\mu_j$, where $\mu_1 \leq \ldots \leq \mu_n$; and let B be an arbitrary operator, where $i(B - B^*)$ has eigenvalues $\gamma_1 \leq \ldots \leq \gamma_n$. Then

$$\Sigma_{j=1}^{n} \mu_j(\mu_j + \gamma_{n-j+1}) \leq ||B - A||^2 - ||B||^2 \leq \Sigma_{j=1}^{n} \mu_j(\mu_j + \gamma_j).$$

REMARK. The theorem is not true for arbitrary A. For example, consider $A = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$. Here $\alpha_1 = \alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 8$, and $\gamma_1 = \gamma_2 = 0$. Then

$$||B - A||^2 - ||B||^2 = 12 - 16$$

$$< \Sigma_{j=1}^{2} \lambda_j(\lambda_j - \beta_j) + \Sigma_{j=1}^{2} \mu_j(\mu_j - \gamma_{n-j+1}) = 0.$$

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REFERENCE

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University of British Columbia
and
Muslim University, Aligarh, India