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Real Interpolation with Logarithmic Functors and Reiteration

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Abstract. We present "reiteration theorems" with limiting values $\theta = 0$ and $\theta = 1$ for a real interpolation method involving broken-logarithmic functors. The resulting spaces lie outside of the original scale of spaces and to describe them new interpolation functors are introduced. For an ordered couple of (quasi-) Banach spaces similar results were presented without proofs by Doktorskii in [D].

1 Introduction

Let (X_0, X_1) be a compatible couple of (quasi-) Banach spaces. The interpolation space $\bar{X}_{\theta,q;\mathbb{A}} = (X_0, X_1)_{\theta,q;\mathbb{A}}$ (with a "broken logarithmic" functor) is the set of functions $f \in X_0 + X_1$ such that

$$\|f\|_{\theta,q;\mathbb{A}} = \|t^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(t)K(f,t;X_0,X_1)\|_{q,(0,\infty)} < \infty. \tag{1.1}$$

Here $\theta \in [0,1]$, $q \in (0,\infty]$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$,

$$\ell^{\mathbb{A}}(t) = \begin{cases} (1 + |\log t|)^{\alpha_0}, & t \in (0,1] \\ (1 + |\log t|)^{\alpha_\infty}, & t \in (1,\infty), \end{cases}$$

K is the Peetre K-functional, and $\|\cdot\|_{q,(0,\infty)}$ is the usual L^q-(quasi-) norm on the interval $(0,\infty)$. A very important feature of the scale of spaces $(X_0, X_1)_{\theta,q;\mathbb{A}}$, resulting from the logarithmic terms involved in their (quasi-) norms, is that these spaces are well-defined also for $\theta = 0$ and $\theta = 1$ (*cf.* Theorem 2.2 and Corollary 2.3 below).

Assume now that (X_0, X_1) and (Y_0, Y_1) are two compatible couples and that $0 \le \theta_0 < \theta_1 \le 1$, $0 \le \psi_0, \psi_1 \le 1$, $\psi_0 \ne \psi_1$, $\theta \in [0,1]$, $0 < q_0, q_1, q, r_0, r_1 \le \infty$, and $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty})$, $\mathbb{B}_i = (\beta_{i0}, \beta_{i\infty})$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ $(i = 0,1)$. Moreover, let T be a linear or quasilinear (in the sense of [S]) operator such that

$$T\colon \bar{X}_{\theta_0,q_0;\mathbb{A}_0} \to \bar{Y}_{\psi_0,r_0;\mathbb{B}_0},$$
$$T\colon \bar{X}_{\theta_1,q_1;\mathbb{A}_1} \to \bar{Y}_{\psi_1,r_1;\mathbb{B}_1}$$

(here the notation $T\colon X \to Y$ means that T is bounded from X into Y). Then

$$T\colon (\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{\theta,q;\mathbb{A}} \to (\bar{Y}_{\psi_0,r_0;\mathbb{B}_0}, \bar{Y}_{\psi_1,r_1;\mathbb{B}_1})_{\theta,q;\mathbb{A}} \tag{1.2}$$

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(this holds even if spaces involved are trivial; for conditions which ensure that such spaces are nontrivial we refer to Theorem 2.2, Corollary 2.3, Theorem 2.4 and Corollary 2.5 below).

A problem of fundamental importance is the identification of spaces involved in (1.2). In the case when $\theta_0, \theta_1, \theta \in (0,1)$ the identification of interpolation spaces $(\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{\theta,q;\mathbb{A}}$ is known (*cf.* [EOP] and references given therein) and is given by the reiteration formula

$$(1.3) \qquad (\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{\theta,q;\mathbb{A}} = (X_0, X_1)_{\theta',q;\mathbb{A}'},$$

where

$$\theta' = (1-\theta)\theta_0 + \theta\theta_1 \quad \text{and} \quad \mathbb{A}' = (1-\theta)\mathbb{A}_0 + \theta\mathbb{A}_1 + \mathbb{A}.$$

In practice (especially in connection with certain limiting inequalities in analysis—*cf.* [Mi]) it is often important to understand the behaviour of T on spaces which are close to $\bar{X}_{\theta_0,q_0;\mathbb{A}_0}$ or $\bar{X}_{\theta_1,q_1;\mathbb{A}_1}$. This means that we need to identify the spaces

$$(1.4) \qquad (\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_;\mathbb{A}_1})_{\theta,q;\mathbb{A}} \quad \text{with } \theta = 0 \text{ or } \theta = 1.$$

Note that in the case where the space $\bar{X}_{\theta_0,q_0;\mathbb{A}_0}$ or $\bar{X}_{\theta_1;q_1;\mathbb{A}_1}$, respectively, is replaced by X_0 or X_1, such an identification was obtained in [EOP]. In the situation where $X_1 \subseteq X_0$, similar results were presented without proofs in [D]; some particular cases of [D] had been investigated earlier in [GM] and [B].

Although the scale of spaces $(X_0, X_1)_{\theta,q;\mathbb{A}}$ is a particular case of $(X_0, X_1)_{\rho,q}$, where $\rho = \rho(t)$ is the so-called function parameter (*cf.* [Ka], [G], [He], [Me 1, 2], [Per], *etc.*), the results presented in the mentioned literature do not cover the situation described in (1.4). Indeed, to get the reiteration formula involving

$$(\bar{X}_{\rho_0,q_0}, \bar{X}_{\rho_1,q_1})_{\rho,q},$$

where ρ_0, ρ_1 and ρ are function parameters, the authors of these papers impose on ρ assumptions which imply that the Matuszewska-Orlicz indices associated to ρ belong to the interval $(0,1)$ (*cf.* [EOP]). Consequently, the case described in (1.4) is excluded.

The aim of this paper is to fill this gap and identify the spaces which appear in (1.4) provided that $0 < \theta_0 < \theta_1 < 1$ and $\theta = 0$ or $\theta = 1$. In order to do this a new class of (quasi-) Banach spaces is introduced (based on ones appearing in [D] in the case $X_1 \subseteq X_0$), and its relation to the scale of spaces $\bar{X}_{\theta;q,\mathbb{A}}$ is analysed in detail. The identification of the spaces in (1.4) when $X_1 \subseteq X_0$ was stated (without proof) in [D]. In this context we should mention that general reiteration results of [DO], [N] and [BK] imply that

$$(\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{\theta,q;\mathbb{A}} = \bar{X}^K_\Phi,$$

where $\|f\|_{\bar{X}^K_\Phi} = \|K(f,t;X_0,X_1)\|_\Phi$ with

$$\Phi = (L_{p_0,q_0;\mathbb{A}_0}, L_{p_1,q_1;\mathbb{A}_1})_{\theta,q;\mathbb{A}}, \; p_i = (1-\theta_i)^{-1}, \quad i = 0, 1.$$

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This means that the problem of identifying the spaces in (1.4) can be reduced to the special case of couples of Lorentz-Zygmund spaces. However, it will be apparent from our solution below that such a reduction does not make the proofs any easier.

To illustrate our results, for $0 \le \theta \le 1$, $0 < q, r \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, let

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = (X_0, X_1)^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$$

be the family of all functions $f \in X_0 + X_1$ satisfying $\|f\|_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} < \infty$, where

$$\|f\|_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} = \|t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) \|\tau^{-\theta-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) K(f, \tau; X_0, X_1)\|_{q,(0,t)}\|_{r,(0,\infty)}. \tag{1.5}$$

We prove (Theorem 5.3) that

$$(\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = \bar{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \tag{1.6}$$

provided that $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1, q \le \infty$, $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$ $(i = 0, 1)$, and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ is such that

$$\alpha_\infty \le 0 \le \alpha_0 \quad \text{if } q = q_0 = \infty$$

or

$$\alpha_\infty + \frac{1}{q} < 0 < \alpha_0 + \frac{1}{q} \quad \text{if } \max\{q, q_0\} < \infty.$$

If, moreover, $q = q_0$, then (*cf.* Corollary 5.4)

$$\bar{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} = \bar{X}_{\theta_0,q_0;\mathbb{A}+\mathbb{A}_0+\frac{1}{q_0}},$$

which means that the resulting space in (1.6) is a space of the original scale $\bar{X}_{\theta,q;\mathbb{A}}$. In general (*cf.* Theorems 4.7 and 4.10), we have the natural embeddings

$$\bar{X}_{\theta_0;q;\mathbb{A}+\mathbb{A}_0+\frac{1}{\min\{q,q_0\}}} \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \hookrightarrow \bar{X}_{\theta_0,q;\mathbb{A}+\mathbb{A}_0+\frac{1}{\max\{q,q_0\}}} \cap \bar{X}_{\theta_0,\max\{q,q_0\};\mathbb{A}+\mathbb{A}_0+\frac{1}{q}}.$$

The formula (1.6) exhibits a certain kind of stability since the resulting space in (1.6) is independent of the parameters involved in the space $\bar{X}_{\theta_1,q_1;\mathbb{A}_1}$; note also that the same result is obtained if the space $\bar{X}_{\theta_1,q_1;\mathbb{A}_1}$ in (1.6) is replaced by X_1 (Theorem 5.7).

In the case when $X_0 = L^1(\Omega)$, $X_1 = L^\infty(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$, $1 < p < \infty$, $0 < q, r \le \infty$, $\theta = 1 - \frac{1}{p}$, and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$,

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = L^{\mathcal{L}}_{p;r,\mathbb{B},q,\mathbb{A}}(\Omega),$$

where the latter space is the collection of all measurable functions f on Ω satisfying

$$\|t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) \|\tau^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) f^*(\tau)\|_{q,(0,t)}\|_{r,(0,\infty)} < \infty$$

(*cf.* Lemma 8.5); f^* stands for the non-increasing rearrangement of f. The space $L^{\mathcal{L}}_{p;r,\mathbb{B},q,\mathbb{A}}(\Omega)$ is the Lorentz-Zygmund space $L_{p,r;\mathbb{B}+\frac{1}{r}}(\Omega)$ (for the precise definition see Section 8) when $r = q$ and

$$\beta_\infty - \alpha_\infty \le 0 \le \beta_0 - \alpha_0 \quad \text{if } r = \infty$$

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or

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r} \quad \text{if } r < \infty.$$

Similar results are obtained for spaces $(\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{1,q;\mathbb{A}}$ (*cf.* Section 6).

The method of this paper can be used in the situation when the function $\ell^{\mathbb{A}}(t)$ involved in the (quasi-) norm (1.1) is replaced by $\prod_{i=1}^{m} \ell_i^{\mathbb{A}_i}(t)$, where $\ell_1(t) = \ell(t)$, $\ell_{j+1}(t) = 1 + \log \ell_j(t)$ if $j \geq 1$, and $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 1, \dots, m$.

The paper is organized as follows. Section 2 contains the basic definitions and preliminaries, while in Section 3 we present weighted inequalities which are applied in Sections 4–8 to get the desired results. Embedding theorems for spaces $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ are derived in Section 4. Section 5 is devoted to the reiteration formula (1.6) involving spaces of the scale $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$, while spaces $\bar{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ are defined in Section 6, which are of significance in the reiteration (1.4) with $\theta = 1$. In Section 7 we explain how to obtain from our general results the corresponding ones in [D] for an ordered couple of (quasi-) Banach spaces; applications are contained in Section 8.

2 Notation and Preliminaries

Let E be a measurable subset of $\mathbb{R}^n$ (with respect to n-dimensional Lebesgue measure), and denote its measure by $|E|$ and its characteristic function by χ_E. The set of all non-negative measurable functions on E is denoted by $\mathcal{M}^+(E)$; when $E = (a,b) \subseteq \mathbb{R}$ we simply write $\mathcal{M}^+(a,b)$. By $\mathcal{M}^+(a,b;\downarrow)$ we mean the subset of $\mathcal{M}^+(a,b)$ consisting of non-increasing functions on (a,b), and $\mathcal{M}^+(a,b;\uparrow)$ is defined analogously. The set $\mathcal{W}(0,\infty)$ of all weights on $(0,\infty)$ is defined by

$$\mathcal{W}(0,\infty) := \{w \in \mathcal{M}^+(0,\infty) : 0 < w < \infty \text{ a.e. on } (0,\infty)\}.$$

We write $A \lesssim B$ if $A \leq cB$ for some constant c which is independent of significant quantities involved in expressions A and B, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. We use the convention $1/\infty = 0$ and $\infty/a = \infty$ for $0 < a < \infty$, and for $0 < q \leq \infty$ we define q' by $\frac{1}{q'} + \frac{1}{q} = 1$ when $q \neq 1$, and $q' = +\infty$ when $q = 1$; note that $q' < 0$ when $0 < q < 1$.

We understand by $T \colon X \to Y$ that the operator T is bounded from X to Y, where X and Y are (quasi-) normed spaces; hence $\|Tf\|_Y \lesssim \|f\|_X$ for all $f \in X$. The symbol $\hookrightarrow$ in $X \hookrightarrow Y$ means that X is continuously embedded in Y; hence $\|f\|_Y \lesssim \|f\|_X$ for all $f \in X$.

Throughout the paper the spaces X_0, X_1, Y_0, Y_1 are (quasi-) Banach spaces (see [Pi]). We say that (X_0, X_1) is a compatible couple if there is a Hausdorff topological vector space into which each of X_0, X_1 is continuously embedded. The Peetre K-functional $K(f,t;X_0,X_1)$ is defined for $f \in X_0 + X_1$ and $t > 0$ by

$$K(f,t;X_0,X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\}$$

where the infimum is over all representations $f = f_0 + f_1$ with $f_0 \in X_0$, $f_1 \in X_1$. It is known that $K(f,t;X_0,X_1)$ is non-decreasing, and $t^{-1}K(f,t;X_0,X_1)$ non-increasing in t; see [BS]. If there is no danger of confusion, we shall simply write $K(f,t)$ instead of $K(f,t;X_0,X_1)$.

We write $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ for an ordered pair of real numbers, and use the convention $\mathbb{A} + \sigma = (\alpha_0 + \sigma, \alpha_\infty + \sigma)$, $\mathbb{A} + \mathbb{B} = (\alpha_0 + \beta_0, \alpha_\infty + \beta_\infty)$ for $\sigma \in \mathbb{R}$ and

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$\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. If $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and $\alpha_0 = \alpha_\infty = \alpha$, we write the logarithmic function $\ell^{\mathbb{A}}(t)$ appearing in (1.1) as $\ell^{\alpha}(t)$.

The spaces which feature in this paper have already been defined in Section 1: for $0 < r, q \le \infty$, $\theta \in [0,1]$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$,

$$\bar{X}_{\theta,q;\mathbb{A}} \equiv (X_0, X_1)_{\theta,q;\mathbb{A}} = \{f \in X_0 + X_1 : \|f\|_{\theta,q;\mathbb{A}} < \infty\} \tag{2.1}$$

where $\|f\|_{\theta,q;\mathbb{A}}$ is given by (1.1), and

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \equiv (X_0, X_1)^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = \{f \in X_0 + X_1 : \|f\|_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} < \infty\} \tag{2.2}$$

where $\|f\|_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}}$ is defined in (1.5).

It is readily seen that $\bar{X}_{\theta,q;\mathbb{A}}$ and $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ are (quasi-) normed spaces. We establish their completeness in our first theorem.

Theorem 2.1 *Let X_0, X_1 be quasi-Banach spaces. Then, for $\theta \in [0,1]$, $0 < r, q \le \infty$, $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$, the spaces $\bar{X}_{\theta,q;\mathbb{A}}$, $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ are quasi-Banach spaces.*

Proof The proof in the case when X_0, X_1 are Banach spaces and $r, q \in [1,\infty]$ essentially follows that in [BS, Chapter 5, Proposition 1.8]. The modifications necessary in general are based on the following observations.

We first recall that the quasi-norm $\|\cdot\|_X$ on a quasi-normed space X is equivalent to a p-norm for some $p \in (0,1]$, that is a quasi-norm $|||\cdot|||$ say which is such that $|||\cdot|||^p$ satisfies the triangle inequality. It is readily seen that a p-norm $|||\cdot|||$ is also an s-norm for any $s \in (0,p)$. For a space X with a p-norm $|||\cdot|||$, it is known that X is complete if and only if the following implication holds for $f_n \in X$, $n = 1, 2, \dots$:

$$\sum_{n=1}^{\infty} |||f_n|||^p < \infty \Rightarrow \Big|\Big|\Big| \sum_{n=1}^{\infty} f_n \Big|\Big|\Big|^p \le \sum_{n=1}^{\infty} |||f_n|||^p \tag{2.3}$$

(see [Pi, B1.6, page 17]).

Next, let X_i, $i = 1, 2$, be quasi-Banach spaces with quasi-norms $\|\cdot\|_i$, $i = 1, 2$, respectively, which are equivalent to p_i-norms $|||\cdot|||_i$ for some $p_i \in (0,1]$, $i = 1, 2$, and put $p = \min\{p_0, p_1\}$. Then $|||\cdot|||_i$, $i = 1, 2$ are both p-norms, and $X_0 + X_1$ is a p-space with p-norm defined by

$$|||f|||_{X_0+X_1} := \inf_{\substack{f=f_0+f_1\\ f_i\in X_i}} \{|||f_0|||_0 + |||f_1|||_1\} \tag{2.4}$$

which is equivalent to the standard norm $\|\cdot\|_{X_0+X_1}$. The functional

$$\tilde{K}(f,t) := \inf_{\substack{f=f_0+f_1\\ f_i\in X_i}} \{|||f_0|||_0^p + t^p |||f_1|||_1^p\}$$

is readily seen to satisfy the triangle inequality for all $t \in (0,\infty)$, and

$$\tilde{K}(\cdot,t) \approx |||\cdot|||^p_{X_0+X_1}.$$

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Finally, on using the equivalence of $\|\cdot\|_{X_i}$ and $|||\cdot|||_i$, and the fact that for $p \in (0,1]$ and $a_1, a_2 \geq 0$,

$$(a_1 + a_2)^p \leq a_1^p + a_2^p \leq 2^{1-p}(a_1 + a_2)^p,$$

it follows that for all $f \in X_0 + X_1$ and $t \in (0, \infty)$

$$\tilde{K}(f,t) \approx [K(f,t)]^p, \tag{2.5}$$

the constants of equivalence being independent of f and t.

The argument in [BS, Chapter 5, Proposition 1.8] can now be modified as follows. We sketch the proof for $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ (assuming $r, q < \infty$) that for $\bar{X}_{\theta,q;\mathbb{A}}$ being similar. Write

$$\|f\|^p_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} = \|\psi F(f,\cdot)\|_{r/p,(0,\infty)},$$

where

$$\psi(t) = t^{-p/r}\ell^{p(\mathbb{B}-\mathbb{A})}(t), \quad F(f,t) = \|\phi K(f,\cdot)^p\|_{q/p,(0,t)}, \quad \phi(\tau) = \tau^{-p(\theta+\frac{1}{q})}\ell^{p\mathbb{A}}(\tau),$$

and choose $p \in (0, \min\{p_0, p_1, r, q\}]$. The Riesz-Fisher property (2.3) is applied successively in $L^{r/p}\big((0,\infty); \psi^{r/p}\, dt\big)$, $L^{q/p}\big((0,t); \phi^{q/p}\, d\tau\big)$ and $X_0 + X_1$ with the p-norm $|||\cdot|||_{X_0+X_1} \approx K(\cdot, t)$ for each $t \in (0,\infty)$ to give the result. ■

An important question is, when are $\bar{X}_{\theta,q;\mathbb{A}}$ and $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ intermediate spaces between X_0 and X_1, that is

$$X_0 \cap X_1 \hookrightarrow \bar{X}_{\theta,q;\mathbb{A}} \hookrightarrow X_0 + X_1, \tag{2.6}$$

$$X_0 \cap X_1 \hookrightarrow \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow X_0 + X_1. \tag{2.7}$$

The answer for (2.6) is given in [EOP, Theorem 2.2 and Corollary 2.3] and, for convenience, we mention it here.

Theorem 2.2 **([EOP, Theorem 2.2])** *Let $0 \leq \theta \leq 1$, $0 < q \leq \infty$, and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$.*

(i) The space $(X_0, X_1)_{\theta,q;\mathbb{A}}$ is an intermediate space between X_0 and X_1 provided that one of the following conditions is satisfied:

$$\theta \in (0,1); \tag{2.8}$$

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} < 0; \tag{2.9}$$

$$\theta = 0, \quad q = \infty, \quad \alpha_\infty = 0; \tag{2.10}$$

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} < 0; \tag{2.11}$$

$$\theta = 1, \quad q = \infty, \quad \alpha_0 = 0. \tag{2.12}$$

(ii) If none of the conditions (2.8)–(2.12) holds, then $(X_0, X_1)_{\theta,q;\mathbb{A}}$ is a trivial space, that is $(X_0, X_1)_{\theta,q;\mathbb{A}} = \{0\}$.

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Corollary 2.3 (**[EOP, Corollary 2.3]**) *If $X_0 \cap X_1 \neq \{0\}$, then the following statements are equivalent:*

(i) $(X_0, X_1)_{\theta,q;\mathbb{A}}$ *is an intermediate space between X_0 and X_1;*
(ii) $(X_0, X_1)_{\theta,q;\mathbb{A}} \neq \{0\}$*;*
(iii) one of the conditions (2.8)–(2.12) holds.

We now give the answer for (2.7).

Theorem 2.4 *Let $0 \leq \theta \leq 1$, $0 < q, r \leq \infty$, and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$.*

(i) The space $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ is an intermediate space between X_0 and X_1 provided that one of the following conditions is satisfied:
I. $0 < q, r < \infty$ and either

$$0 < \theta < 1, \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0$$

or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} \geq 0, \quad \beta_\infty + \frac{1}{q} + \frac{1}{r} < 0$$

or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} < 0, \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0$$

or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} < 0, \quad \beta_0 + \frac{1}{q} + \frac{1}{r} < 0, \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0;$$

II. $q = \infty$, $0 < r < \infty$ and either

$$0 < \theta < 1, \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0$$

or

$$\theta = 0, \quad \alpha_\infty \geq 0, \quad \beta_\infty + \frac{1}{r} < 0$$

or

$$\theta = 0, \quad \alpha_\infty < 0, \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0$$

or

$$\theta = 1, \quad \alpha_0 \leq 0, \quad \beta_0 + \frac{1}{r} < 0, \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0;$$

III. $0 < q < \infty$, $r = \infty$ and either

$$0 < \theta < 1, \quad \beta_\infty - \alpha_\infty \leq 0$$

or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} > 0, \quad \beta_\infty + \frac{1}{q} \leq 0$$

or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} = 0, \quad \beta_\infty - \alpha_\infty < 0$$

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or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} < 0, \quad \beta_\infty - \alpha_\infty \le 0$$

or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} < 0, \quad \beta_0 + \frac{1}{q} \le 0, \quad \beta_\infty - \alpha_\infty \le 0.$$

IV. $q = \infty$, $r = \infty$ *and either*

$$0 < \theta < 1, \quad \beta_\infty - \alpha_\infty \le 0$$

or

$$\theta = 0, \quad \alpha_\infty \ge 0, \quad \beta_\infty \le 0$$

or

$$\theta = 0, \quad \alpha_\infty < 0, \quad \beta_\infty - \alpha_\infty \le 0$$

or

$$\theta = 1, \quad \alpha_0 \le 0, \quad \beta_0 \le 0, \quad \beta_\infty - \alpha_\infty \le 0.$$

(ii) If none of the conditions I–IV holds, then $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ *is a trivial space, that is* $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = \{0\}$.

Proof The proof is similar to that of [EOP, Theorem 2.2] and thus omitted. ■

Corollary 2.5 *If* $X_0 \cap X_1 \neq \{0\}$, *then the following statements are equivalent:*

(i) $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ *is an intermediate space between* X_0 *and* X_1*;*
(ii) $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \neq \{0\}$*;*
(iii) one of conditions I–IV holds.

Proof It is again omitted since it is similar to that of [EOP, Corollary 2.3]. ■

Other (quasi-) Banach spaces, denoted by $\bar{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$, will be defined in Section 6. They possess similar properties to those of $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$, and, indeed, are closely related to them. The superscripts $\mathcal{L}$, $\mathcal{R}$ are an indication of the fact that the corresponding spaces give a descriptions of the reiterations considered in the paper at the left ($\mathcal{L}$) and right ($\mathcal{R}$) end of the θ-range [0,1].

3 Weighted Inequalities

In this section we collect together weighted inequalities which are needed in the rest of the paper. Some of the results are new, but others are included here (with precise references) for convenience.

Lemma 3.1 **([EOP, Lemma 4.2 (i)])** *Let* $1 \le P \le Q \le \infty$ *and* $\mathbb{D} = (\delta_0, \delta_\infty) \in \mathbb{R}^2$. *Then the inequality*

$$\Big\| t^{-\frac{1}{Q}} \ell^{\mathbb{D}-\frac{1}{Q}}(t) \int_0^t g(\tau)\, d\tau \Big\|_{Q,(0,\infty)} \lesssim \| t^{\frac{1}{P'}} \ell^{\mathbb{D}+\frac{1}{P'}}(t) g(t) \|_{P,(0,\infty)}$$

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holds for every $g \in \mathcal{M}^+(0,\infty)$ *if and only if either*

$$\delta_\infty < 0 < \delta_0,$$

or

$$P = 1, \quad Q = \infty \quad \text{and} \quad \delta_\infty \le 0 \le \delta_0.$$

Theorem 3.2 (**[L, Theorem 2.2]**) *Let* $0 < Q \le P \le 1$, $\Phi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, *and* $v, w \in \mathcal{M}^+(0,\infty)$. *Then the inequality*

$$\Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)^P w(x)\,dx\Big]^{1/P} \le C\Big[\int_0^\infty h^Q(x)v(x)\,dx\Big]^{1/Q} \tag{3.1}$$

holds for every $h \in \mathcal{M}^+(0,\infty;\downarrow)$ *if and only if, for all* $z > 0$,

$$\Big[\int_0^\infty \Big(\int_0^z \Phi(x,y)\,dy\Big)^P w(x)\,dx\Big]^{1/P} \le C\Big[\int_0^z v(x)\,dx\Big]^{1/Q}. \tag{3.2}$$

(The constant C *in both inequalities is the same.)*

Theorem 3.3 (**[L, Theorem 2.1]**) *Let* $1 \le P \le Q < \infty$, $\Phi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, *and* $v, w \in \mathcal{M}^+(0,\infty)$. *Then the inequality*

$$\Big[\int_0^\infty h^Q(x)v(x)\,dx\Big]^{1/Q} \le C\Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)^P w(x)\,dx\Big)^{1/P} \tag{3.3}$$

holds for all $h \in \mathcal{M}^+(0,\infty;\downarrow)$ *if and only if for all* $z > 0$,

$$\Big[\int_0^z v(x)\,dx\Big]^{1/Q} \le C\Big[\int_0^\infty \Big(\int_0^z \Phi(x,y)\,dy\Big)^P w(x)\,dx\Big]^{1/P}. \tag{3.4}$$

(The constant C *in both inequalities is the same.)*

Lemma 3.4 *Let* $s \in (0,1)$, $\varphi, w \in \mathcal{W}(0,\infty)$ *and define* $v \in \mathcal{W}(0,\infty)$ *by*

$$v(y) = w(y)^{1-s}\Big[\varphi(y)\int_y^\infty w(x)\,dx\Big]^s, \quad y \in (0,\infty). \tag{3.5}$$

Then for all $h \in \mathcal{M}^+(0,\infty)$,

$$\int_0^\infty h^s(x)v(x)\,dx \le s^s \int_0^\infty \Big(\int_0^x \varphi(y)h(y)\,dy\Big)^s w(x)\,dx. \tag{3.6}$$

Proof Put

$$(Th)(x) = \int_0^x \varphi(y)h(y)\,dy, \quad h \in \mathcal{M}^+(0,\infty),\ x \in (0,\infty). \tag{3.7}$$

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We can assume that $h \in \mathcal{M}^+(0,\infty)$ satisfies

$$0 < \|(Th)w^{\frac{1}{s}}\|_{s,(0,\infty)} < \infty \tag{3.8}$$

(otherwise (3.6) with such h holds trivially). Using the identity

$$[(Th)(x)]^s = s\int_0^x \Big(\int_0^y \varphi(t)h(t)\,dt\Big)^{s-1}\varphi(y)h(y)\,dy, \quad x \in (0,\infty),$$

Fubini's theorem, and (3.5), we get

$$\begin{aligned} \int_0^\infty [(Th)(x)]^s w(x)\,dx &= s\int_0^\infty \Big(\int_0^x [(Th)(y)]^{s-1}\varphi(y)h(y)\,dy\Big)w(x)\,dx \\ &= s\int_0^\infty [(Th)(y)]^{s-1}h(y)\varphi(y)\Big[\int_y^\infty w(x)\,dx\Big]\,dy \\ &= s\int_0^\infty [(Th)(y)]^{s-1}w^{\frac{1}{s'}}(y)h(y)v^{\frac{1}{s}}(y)\,dy. \end{aligned} \tag{3.9}$$

As $\|(Th)^{s-1}w^{\frac{1}{s'}}\|_{s',(0,\infty)} = \|(Th)w^{\frac{1}{s}}\|^{s-1}_{s,(0,\infty)}$, (3.8) implies that we can estimate the last term in (3.9) by Hölder's inequality (which has the reverse form since $s \in (0,1)$—*cf.* [A., Theorem 2.6]). Thus,

$$\begin{aligned} \int_0^\infty [(Th)(x)]^s w(x)\,dx &\geq s\|(Th)^{s-1}w^{\frac{1}{s'}}\|_{s',(0,\infty)}\|hv^{\frac{1}{s}}\|_{s,(0,\infty)} \\ &= s\|(Th)w^{\frac{1}{s}}\|^{s-1}_{s,(0,\infty)}\|hv^{\frac{1}{s}}\|_{s,(0,\infty)}. \end{aligned}$$

Dividing this inequality by $\|(Th)w^{\frac{1}{s}}\|^{s-1}_{s,(0,\infty)}$ (*cf.* (3.8)) and raising it to the power s, we obtain (3.6). ∎

Lemma 3.5 *Let $\theta < 1$, $0 < q < \infty$, and $\mathbb{A} = (\alpha_0,\alpha_\infty)$, $\mathbb{B} = (\beta_0,\beta_\infty) \in \mathbb{R}^2$. Then for all $h \in \mathcal{M}^+(0,\infty;\downarrow)$,*

$$\|t^{(1-\theta)q}\ell^{\mathbb{B}q}(t)h(t)\|_{\infty,(0,\infty)} \lesssim \Big\|\ell^{(\mathbb{B}-\mathbb{A})q}(t)\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)h(\tau)\,d\tau\Big\|_{\infty,(0,\infty)}. \tag{3.10}$$

Proof Let $t \in (0,\infty)$ and $h \in \mathcal{M}^+(0,\infty;\downarrow)$. Then

$$\begin{aligned} t^{(1-\theta)q}\ell^{\mathbb{B}q}(t)h(t) &= \ell^{(\mathbb{B}-\mathbb{A})q}(t)h(t)t^{(1-\theta)q}\ell^{\mathbb{A}q}(t) \\ &\approx \ell^{(\mathbb{B}-\mathbb{A})q}(t)h(t)\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)\,d\tau \\ &\leq \ell^{(\mathbb{B}-\mathbb{A})q}(t)\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)h(\tau)\,d\tau, \end{aligned}$$

which immediately yields (3.10). ∎

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Remark 3.6 Note that the right-hand side of (3.10) is infinite if $\beta_\infty - \alpha_\infty > 0$ and $0 \neq h \in \mathcal{M}^+(0,\infty;\downarrow)$. Indeed, for such h there is $R \in (0,\infty)$ satisfying $h \geq \chi_{(0,R)}$. Consequently,

$$\Big\|\ell^{(\mathbb{B}-\mathbb{A})q}(t)\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)h(\tau)\,d\tau\Big\|_{\infty,(0,\infty)}$$
$$\geq \Big\|\ell^{(\mathbb{B}-\mathbb{A})q}(t)\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)\chi_{(0,R)}(\tau)\Big\|_{\infty,(R,\infty)}$$
$$= \Big(\int_0^R \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)\,d\tau\Big)\|\ell^{(\mathbb{B}-\mathbb{A})q}(t)\|_{\infty,(R,\infty)}.$$

Assuming that $\beta_\infty - \alpha_\infty > 0$, we have

$$\|\ell^{(\mathbb{B}-\mathbb{A})q}(t)\|_{\infty,(R,\infty)} = \infty,$$

and the result follows.

Lemma 3.7 *Let $\theta \in \mathbb{R}$, $0 < q < \infty$, $s \in (0,1)$, $r = sq$, and let $\mathbb{A} = (\alpha_0,\alpha_\infty)$, $\mathbb{B} = (\beta_0,\beta_\infty) \in \mathbb{R}^2$ be such that*

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r}. \tag{3.11}$$

Then for all $h \in \mathcal{M}^+(0,\infty)$,

$$\int_0^\infty h(x)x^{(1-\theta)q-1}\ell^{\mathbb{B}q+\frac{1}{s}}(x)\,dx \lesssim \Big\{\int_0^\infty \Big[\int_0^x y^{(1-\theta)q-1}\ell^{\mathbb{A}q}(y)h(y)\,dy\Big]^s x^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(x)\,dx\Big\}^{\frac{1}{s}}. \tag{3.12}$$

Proof Let $h \in \mathcal{M}^+(0,\infty)$; put

$$\text{LHS} := \int_0^\infty h(x)x^{(1-\theta)q-1}\ell^{\mathbb{B}q+\frac{1}{s}}(x)\,dx.$$

Using the estimate

$$\ell^{\mathbb{B}q+\frac{1}{s}}(x) \approx \ell^{Aq}(x)\Big(\int_x^\infty y^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(y)\,dy\Big)^{\frac{1}{s}}, \quad x \in (0,\infty),$$

and Minkowski's (integral) inequality, we obtain

$$(\text{LHS})^s \approx \Big\{\int_0^\infty h(x)x^{(1-\theta)q-1}\ell^{\mathbb{A}q}(x)\Big(\int_x^\infty y^{-1}\ell^{(B-A)r}(y)\,dy\Big)^{\frac{1}{s}}\,dx\Big\}^s$$
$$\leq \int_0^\infty \Big[\int_0^y h(x)x^{(1-\theta)q-1}\ell^{\mathbb{A}q}(x)\,dx\Big]^s y^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(y)\,dy$$

and (3.12) follows. ■

Lemma 3.8 **([EOP, Lemma 4.1])** *Let $1 \leq P \leq Q \leq \infty$, $\nu \neq 0$, and $\mathbb{D} = (\delta_0,\delta_\infty) \in \mathbb{R}^2$.*

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(i) *The inequality*

$$\Big\|t^{\nu-\frac{1}{Q}}\ell^{\mathbb{D}}(t)\int_0^t g(s)\,ds\Big\|_{Q,(0,\infty)} \lesssim \|t^{\nu+\frac{1}{P'}}\ell^{\mathbb{B}}(t)g(t)\|_{P,(0,\infty)}$$

holds for every $g \in \mathcal{M}^+(0,\infty)$ *if and only if* $\nu < 0$.

(ii) *The inequality*

$$\Big\|t^{\nu-\frac{1}{Q}}\ell^{\mathbb{D}}(t)\int_t^\infty g(s)\,ds\Big\|_{Q,(0,\infty)} \lesssim \|t^{\nu+\frac{1}{P'}}\ell^{\mathbb{B}}(t)g(t)\|_{P,(0,\infty)}$$

holds for every $g \in \mathcal{M}^+(0,\infty)$ *if and only if* $\nu > 0$.

Theorem 3.9 **([L, Theorem 2.4])** *Let* $0 < Q \le P \le 1$, $\Phi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, *and* $v, w \in \mathcal{M}^+(0,\infty)$. *Then the inequality*

$$\Big[\int_0^\infty\Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)^P w(x)\,dx\Big]^{1/P} \le C\Big[\int_0^\infty h^Q(x)v(x)\,dx\Big]^{1/Q} \tag{3.13}$$

holds for all $h \in \mathcal{M}^+(0,\infty;\uparrow)$ *if and only if, for all* $z > 0$,

$$\Big[\int_0^\infty\Big(\int_z^\infty \Phi(x,y)\,dy\Big)^P w(x)\,dx\Big]^{1/P} \le C\Big[\int_z^\infty v(x)\,dx\Big]^{1/Q}. \tag{3.14}$$

(The constant C in both inequalities is the same.)

Lemma 3.10 *Assume that* $0 < r < \infty$ *and* $\mathbb{A} = (\alpha_0,\alpha_\infty)$, $\mathbb{B} = (\beta_0,\beta_\infty) \in \mathbb{R}^2$ *are such that (3.11) holds. Let either* $\theta \in [0,1)$, *or* $\theta = 1$ *and* $\alpha_0 \le 0 \le \alpha_\infty$. *Then for all* $h \in \mathcal{M}^+(0,\infty;\downarrow)$,

$$\|t^{1-\theta}\ell^{\mathbb{B}+\frac{1}{r}}(t)h(t)\|_{\infty,(0,\infty)} \lesssim \|t^{-\frac{1}{r}}\ell^{\mathbb{B}-\mathbb{A}}(t)\|\tau^{1-\theta}\ell^{\mathbb{A}}(\tau)h(\tau)\|_{\infty,(0,t)}\|_{r,(0,\infty)}.$$

Proof For $h \in \mathcal{M}^+(0,\infty;\downarrow)$ and $T \in [0,\infty)$ put

$$\mathrm{RHS}(T) = \|t^{-\frac{1}{r}}\ell^{\mathbb{B}-\mathbb{A}}(t)\|\tau^{1-\theta}\ell^{\mathbb{A}}(\tau)h(\tau)\|_{\infty,(0,t)}\|_{r,(T,\infty)}.$$

Our assumptions imply that for all $T \in (0,\infty)$,

$$\|\tau^{1-\theta}\ell^{\mathbb{A}}(\tau)h(\tau)\|_{\infty,(0,T)} \approx T^{1-\theta}\ell^{\mathbb{A}}(T)$$

and

$$\|t^{-\frac{1}{r}}\ell^{\mathbb{B}-\mathbb{A}}(t)\|_{r,(T,\infty)} \approx \ell^{\mathbb{B}-\mathbb{A}+\frac{1}{r}}(T).$$

Consequently, for any $T \in (0,\infty)$,

$$\begin{aligned}\mathrm{RHS}(0) \ge \mathrm{RHS}(T) &\ge \|\tau^{1-\theta}\ell^{\mathbb{A}}(\tau)\|_{\infty,(0,T)}h(T)\|t^{-\frac{1}{r}}\ell^{\mathbb{B}-\mathbb{A}}(t)\|_{r,(T,\infty)}\\ &\approx T^{1-\theta}\ell^{\mathbb{B}+\frac{1}{r}}(T)h(T)\end{aligned}$$

and the result follows. ■

Lemma 3.11 **([EOP, Lemma 4.6 (i)])** *Let* $\nu \in (-1,0)$, $0 < P \le 1$, $P \le Q \le \infty$, *and* $\mathbb{D} = (\delta_0,\delta_\infty) \in \mathbb{R}^2$. *Then the inequality*

$$\Big\|t^{\nu-\frac{1}{Q}}\ell^{\mathbb{D}}(t)\int_0^t g(s)\,ds\Big\|_{Q,(0,\infty)} \lesssim \|t^{\nu+\frac{1}{P'}}\ell^{\mathbb{D}}(t)g(t)\|_{P,(0,\infty)}$$

holds every $g \in \mathcal{M}^+(0,\infty;\downarrow)$.

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4 Embeddings

Our objective here is to establish embeddings between spaces of the scales $\bar{X}_{\theta,q;\mathbb{A}}$ and $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$. More precisely, we prove that (under appropriate assumptions)

$$\bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\min\{r,q\}}} \hookrightarrow \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \tag{4.1}$$

and

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}}. \tag{4.2}$$

We shall start with a simple observation.

Lemma 4.1 *Let $\theta \in [0,1]$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ be such that*

$$\beta_\infty - \alpha_\infty \le 0 \le \beta_0 - \alpha_0. \tag{4.3}$$

Then

$$\bar{X}^{\mathcal{L}}_{\theta;\infty,\mathbb{B},\infty,\mathbb{A}} = \bar{X}_{\theta,\infty;\mathbb{B}} \tag{4.4}$$

(and the corresponding norms are equivalent).

Proof Using (4.3), we get

$$\operatorname*{ess\,sup}_{\tau<t<\infty} \ell^{\mathbb{B}-\mathbb{A}}(t) \approx \ell^{\mathbb{B}-\mathbb{A}}(\tau) \quad \text{for all } \tau \in (0,\infty).$$

Hence, for all $f \in \bar{X}^{\mathcal{L}}_{\theta;\infty,\mathbb{B},\infty,\mathbb{A}}$,

$$\begin{aligned}
\|f\|_{\mathcal{L};\theta;\infty,\mathbb{B},\infty,\mathbb{A}} &= \operatorname*{ess\,sup}_{0<t<\infty} \ell^{\mathbb{B}-\mathbb{A}}(t) \operatorname*{ess\,sup}_{0<\tau<t} \tau^{-\theta} \ell^{\mathbb{A}}(\tau) K(f,\tau) \\
&= \operatorname*{ess\,sup}_{0<\tau<\infty} \tau^{-\theta} \ell^{\mathbb{A}}(\tau) K(f,\tau) \operatorname*{ess\,sup}_{\tau<t<\infty} \ell^{\mathbb{B}-\mathbb{A}}(t) \\
&\approx \operatorname*{ess\,sup}_{0<\tau<\infty} \tau^{-\theta} \ell^{\mathbb{B}}(\tau) K(f,\tau) \\
&= \|f\|_{\theta,\infty;\mathbb{B}}.
\end{aligned}$$

■

To deal with other cases, it will be useful to rewrite (1.5). Let $0 < q < \infty$, $0 < r \le \infty$ and set

$$s = r/q \tag{4.5}$$

and

$$g(\tau) = [\tau^{-\theta-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) K(f,\tau)]^q, \quad \tau \in (0,\infty). \tag{4.6}$$

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Then we can rewrite (1.5) as

$$\|f\|^q_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} = \Big\|t^{-\frac{1}{s}}\ell^{(\mathbb{B}-\mathbb{A})q}(t)\int_0^t g(\tau)\,d\tau\Big\|_{s,(0,\infty)}. \tag{4.7}$$

Lemma 4.2 *Let $\theta \in [0,1)$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, and either*

$$0 < q \le r \le \infty \quad \textit{and} \quad q < \infty \tag{4.8}$$

or

$$0 < r < q < \infty. \tag{4.9}$$

If moreover

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r}, \tag{4.10}$$

then

$$\bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\min\{r,q\}}} \hookrightarrow \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}. \tag{4.11}$$

Proof Let s be given by (4.5) and $f \in \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\min\{r,q\}}}$.

(i) Assume first that (4.8) holds. Then $1 \le s \le \infty$. Using (4.7), Lemma 3.1(i) (with $P = Q = s$ and $\mathbb{D} = (\mathbb{B}-\mathbb{A})q + \frac{1}{s}$), (4.10), and (4.6), we obtain

$$\begin{aligned}\|f\|^q_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} &= \Big\|t^{-\frac{1}{s}}\ell^{[(\mathbb{B}-\mathbb{A})q+\frac{1}{s}]-\frac{1}{s}}(t)\int_0^t g(\tau)\,d\tau\Big\|_{s,(0,\infty)} \\ &\lesssim \|t^{\frac{1}{s'}}\ell^{(\mathbb{B}-\mathbb{A})q+1}(t)g(t)\|_{s,(0,\infty)} = \|t^{-\theta q-\frac{1}{s}}\ell^{\mathbb{B}q+1}(t)K(f,t)^q\|_{s,(0,\infty)} \\ &= \|t^{-\theta-\frac{1}{r}}\ell^{\mathbb{B}+\frac{1}{q}}(t)K(f,t)\|^q_{r,(0,\infty)} = \|f\|^q_{\theta,r;\mathbb{B}+\frac{1}{q}}\end{aligned}$$

and (4.11) follows since $\min(r,q) = q$.

(ii) Assume that (4.9) is satisfied. Then $0 < s < 1$. Put $P = Q = s$ and, for $x, y \in (0,\infty)$,

$$\begin{cases}\Phi(x,y) = \chi_{(0,x)}(y)y^{(1-\theta)q-1}\ell^{\mathbb{A}q}(y), \\ w(x) = x^{-1}\ell^{(\mathbb{B}-\mathbb{A})qs}(x), \\ v(x) = x^{(1-\theta)r-1}\ell^{\mathbb{B}r+1}(x), \\ h(y) = [K(f,y)/y]^q;\end{cases} \tag{4.12}$$

note that $h \in \mathcal{M}^+(0,\infty;\downarrow)$.

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Using (4.7), (4.6), (4.12), and the fact that $\min\{r,q\} = r$, we obtain

$$\|f\|^q_{\mathcal{L};\theta,r,\mathbb{B},q,\mathbb{A}} = \Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)^P w(x)\,dx\Big]^{1/P} \tag{4.13}$$

and

$$\|f\|^q_{\theta,r;\mathbb{B}+\frac{1}{\min\{r,q\}}} = \|f\|^q_{\theta,r;B+\frac{1}{r}} = \|x^{1-\theta-\frac{1}{r}}\ell^{\mathbb{B}+\frac{1}{r}}(x)K(f,x)/x\|^q_{r,(0,\infty)} \tag{4.14}$$
$$= \|v^{\frac{1}{r}}(x)h^{\frac{1}{q}}(x)\|^q_{r,(0,\infty)} = \Big[\int_0^\infty h^Q(x)v(x)\,dx\Big]^{1/Q}.$$

Thus, the embedding (4.11) would hold if and only if the inequality (3.1) (with our choice of P, Q, Φ, w, and v) would be satisfied. Consequently, to complete the proof, it is sufficient to verify (3.2) (*cf.* Theorem 3.2).

Let $z \in (0,\infty)$. Since $\theta < 1$,

$$\int_0^z v(x)\,dx = \int_0^z x^{(1-\theta)qs-1}\ell^{\mathbb{B}qs+1}(x)\,dx \approx z^{(1-\theta)qs}\ell^{\mathbb{B}qs+1}(z). \tag{4.15}$$

Moreover, using (4.12), (4.10), and the assumption $\theta < 1$, we obtain

(4.16)
$$\int_0^\infty \Big(\int_0^z \Phi(x,y)\,dy\Big)^P w(x)\,dx$$
$$= \int_0^z \Big(\int_0^z \Phi(x,y)\,dy\Big)^s w(x)\,dx + \int_z^\infty \Big(\int_0^z \Phi(x,y)\,dy\Big)^s w(x)\,dx$$
$$= \int_0^z \Big(\int_0^x y^{(1-\theta)q-1}\ell^{\mathbb{A}q}(y)\,dy\Big)^s w(x)\,dx + \int_z^\infty \Big(\int_0^z y^{(1-\theta)q-1}\ell^{\mathbb{A}q}(y)\,dy\Big)^s w(x)\,dx$$
$$\approx \int_0^z \big(x^{(1-\theta)q}\ell^{\mathbb{A}q}(x)\big)^s w(x)\,dx + \big(z^{(1-\theta)q}\ell^{\mathbb{A}q}(z)\big)^s \int_z^\infty w(x)\,dx$$
$$\approx z^{(1-\theta)qs}\ell^{\mathbb{B}qs+1}(z).$$

The estimates (4.15) and (4.16) show that the condition (3.2) holds. The proof is complete. ■

It remains to prove the embedding (4.1) in the case where $0 < r < q = \infty$. Then the norm in the space $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ *cannot be expressed* by (4.7) and one *cannot directly apply* the results of Section 3. However, we shall show in the proof of the next lemma that there is a way which, ultimately, enables us to make use of results of Section 3 to treat this case.

Lemma 4.3 *Let $\theta \in [0,1)$, $0 < r < q = \infty$, and let $\mathbb{A} = (\alpha_0,\alpha_\infty)$, $\mathbb{B} = (\beta_0,\beta_\infty) \in \mathbb{R}^2$ satisfy (4.10). Then the embedding (4.11) holds.*

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Proof On putting

$$\bar{X} = \bar{X}_{\theta,r;B+\frac{1}{\min\{r,q\}}} = \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{r}}, \quad \bar{X}^{\mathcal{L}} = \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}},$$

and

$$h(t) = [K(f,t)/t]^r, \quad t \in (0,\infty),$$

we have, for any $f \in X_0 + X_1$,

$$\|f\|_{\bar{X}} = \|t^{-\theta-\frac{1}{r}}\ell^{\mathbb{B}+\frac{1}{r}}(t)K(f,t)\|_{r,(0,\infty)} = \|t^{(1-\theta)r-1}\ell^{\mathbb{B}r+1}(t)h(t)\|^{\frac{1}{r}}_{1,(0,\infty)} \tag{4.17}$$

and

$$\|f\|_{\bar{X}^{\mathcal{L}}} = \|t^{-\frac{1}{r}}\ell^{\mathbb{B}-\mathbb{A}}(t)N(t)\|_{r,(0,\infty)},$$

where

$$N(t) = \operatorname*{esssup}_{0<\tau<t} \tau^{-\theta}\ell^{\mathbb{A}}(\tau)K(f,\tau) = \operatorname*{esssup}_{0<\tau<t} h^{\frac{1}{r}}(\tau)\tau^{1-\theta}\ell^{\mathbb{A}}(\tau).$$

Since $\tau^{1-\theta}\ell^{\mathbb{A}}(\tau) \approx \|\sigma^{1-\theta-\frac{1}{r}}\ell^{\mathbb{A}}(\sigma)\|_{r,(0,\tau)}$ and $h \in \mathcal{M}^+(0,\infty;\downarrow)$,

$$N(t) \lesssim \operatorname*{esssup}_{0<\tau<t} \|\sigma^{1-\theta-\frac{1}{r}}\ell^{\mathbb{A}}(\sigma)h^{\frac{1}{r}}(\sigma)\|_{r,(0,\tau)} = \|\sigma^{1-\theta-\frac{1}{r}}\ell^{\mathbb{A}}(\sigma)h^{\frac{1}{r}}(\sigma)\|_{r,(0,t)},$$

which in turn yields

$$\begin{aligned} \|f\|_{\bar{X}^{\mathcal{L}}} &\lesssim \Big\| t^{-\frac{1}{r}}\ell^{\mathbb{B}-\mathbb{A}}(t)\Big(\int_0^t \sigma^{(1-\theta)r-1}\ell^{\mathbb{A}r}(\sigma)h(\sigma)\,d\sigma\Big)^{\frac{1}{r}}\Big\|_{r,(0,\infty)} \\ &= \Big\| t^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(t)\int_0^t \sigma^{(1-\theta)r-1}\ell^{\mathbb{A}r}(\sigma)h(\sigma)\,d\sigma\Big\|^{\frac{1}{r}}_{1,(0,\infty)}. \end{aligned} \tag{4.18}$$

On putting, for $x, y \in (0,\infty)$,

$$\Phi(x,y) = \chi_{(0,x)}(y)y^{(1-\theta)r-1}\ell^{\mathbb{A}r}(y),$$
$$w(x) = x^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(x),$$
$$v(x) = x^{(1-\theta)r-1}\ell^{\mathbb{B}r+1}(x),$$

we have from (4.18) and (4.17),

$$\|f\|^r_{\bar{X}^{\mathcal{L}}} = \int_0^\infty \Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)w(x)\,dx, \quad \|f\|^r_{\bar{X}} = \int_0^\infty h(x)v(x)\,dx.$$

Together with Theorem 3.2 this implies that the embedding (4.11) holds if and only if the condition (3.2) (with $P = Q = 1$) is satisfied for all $z \in (0,\infty)$. Since this is the case, the proof is complete. ■

Now, we turn our attention to the embedding (4.2).

Lemma 4.4 *Let* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Suppose that either*

$$\theta \in [0,1) \quad \textit{and} \quad 0 < q \le r < \infty \tag{4.19}$$

or

$$\theta \in [0,1] \quad \textit{and} \quad 0 < r < q < \infty. \tag{4.20}$$

If moreover

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r}, \tag{4.21}$$

then

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}}. \tag{4.22}$$

Proof Let s be given by (4.5) and $f \in \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$.

(i) Assume first that (4.19) holds. Then $1 \le s < \infty$. Put $P = Q = s$ and define Φ, w, v, and h by (4.12). On using (4.7), (4.6), (4.12), and the fact that $\max\{r,q\} = r$, we obtain (4.13) and

$$\|f\|^q_{\theta,r;B+\frac{1}{\max\{r,q\}}} = \|f\|^q_{\theta,r;\mathbb{B}+\frac{1}{r}} = \Big[\int_0^\infty h^Q(x)v(x)\,dx\Big]^{1/Q} \quad (\textit{cf.}\ (4.14)).$$

Thus, the embedding (4.22) holds if and only if the inequality (3.3) (with our choice of P, Q, Φ, w, and v) is satisfied. By Theorem 3.3, this is the case if we verify (3.4). However, this follows from (4.15) and (4.16).

(ii) Assume now that (4.20) is satisfied. Then $0 < s < 1$. For $x, y \in (0,\infty)$ put

$$\begin{cases} \varphi(y) = y^{(1-\theta)q-1}\ell^{\mathbb{A}q}(y), & h(y) = [K(f,y)/y]^q, \\ w(x) = x^{-1}\ell^{(\mathbb{B}-\mathbb{A})qs}(x), & v(x) = x^{(1-\theta)r-1}\ell^{\mathbb{B}r+s}(x). \end{cases} \tag{4.23}$$

Then (*cf.* (4.21))

$$\int_y^\infty w(x)\,dx \approx \ell^{(\mathbb{B}-\mathbb{A})qs+1}(y), \quad y \in (0,\infty),$$

which implies that

$$\begin{aligned} w^{1-s}(y)\Big[\varphi(y)\int_y^\infty w(x)\,dx\Big]^s &= y^{s-1}\ell^{(\mathbb{B}-A)qs(1-s)}(y)y^{[(1-\theta)q-1]s}\ell^{[\mathbb{A}q+(\mathbb{B}-\mathbb{A})qs+1]s}(y) \\ &= v(y), \quad y \in (0,\infty). \end{aligned}$$

Thus, by Lemma 3.4, the inequality (3.6) holds on $\mathcal{M}^+(0,\infty)$. However, on using (4.7), (4.6), and (4.23), we see that

$$\|f\|^q_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} = \Big[\int_0^\infty \Big(\int_0^x \varphi(y)h(y)\,dy\Big)^s w(x)\,dx\Big]^{1/s}.$$

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Moreover,

$$\|f\|^q_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}} = \|f\|^q_{\theta,r;\mathbb{B}+\frac{1}{q}} = \|x^{1-\theta-\frac{1}{r}}\ell^{\mathbb{B}+\frac{1}{q}}(x)K(f,x)/x\|^q_{r,(0,\infty)}$$
$$= \|v^{\frac{1}{r}}(x)h^{\frac{1}{q}}(x)\|^q_{r,(0,\infty)} = \Big[\int_0^\infty h^s(x)v(x)\,dx\Big]^{1/s}.$$

Consequently, (4.22) follows from (3.6). ■

The next two assertions show that Lemma 4.4 continues to hold when the condition (4.19) is replaced by $0 < q < r = \infty$ and the condition (4.20) by $0 < r < q = \infty$.

Lemma 4.5 *Let $\theta \in [0,1)$, $\theta < q < r = \infty$, and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. Then the embedding (4.22) holds.*

Proof Let $f \in \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ and define s by (4.5) (*i.e.* $s = \infty$). Put

$$h(t) = [K(f,t)/t]^q, \quad t \in (0,\infty). \tag{4.24}$$

Then, on using (4.7), (4.6) and the fact that $\max\{r,q\} = \infty$ we obtain

$$\|f\|^q_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} = \|\ell^{(\mathbb{B}-\mathbb{A})q}(t)\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)h(\tau)\,d\tau\|_{\infty,(0,\infty)}$$

and

$$\|f\|^q_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}} = \|f\|^q_{\theta,\infty;\mathbb{B}} = \|t^{1-\theta}\ell^{\mathbb{B}}(t)h^{\frac{1}{q}}(t)\|^q_{\infty,(0,\infty)} = \|t^{(1-\theta)q}\ell^{\mathbb{B}q}(t)h(t)\|_{\infty,(0,\infty)}.$$

Consequently, the result follows from Lemma 3.5. ■

Lemma 4.6 *Let $\theta \in [0,1]$, $0 < r < q = \infty$, and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. Then the embedding (4.22) holds.*

Proof Let $f \in \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$. Since for every $t \in (0,\infty)$,

$$\|\tau^{-\theta}\ell^{\mathbb{A}}(\tau)K(f,\tau)\|_{\infty,(0,t)} \geq t^{-\theta}\ell^{\mathbb{A}}(t)K(f,t),$$

we get

$$\|f\|_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} \geq \|t^{-\theta-\frac{1}{r}}\ell^{\mathbb{B}}(t)K(f,t)\|_{r,(0,\infty)} = \|f\|_{\theta,r;\mathbb{B}}$$

and the result follows for $\max(r,q) = \infty$. ■

Summarizing our results with $\theta \in [0,1)$, we have the following assertion.

Theorem 4.7 *Let $\theta \in [0,1)$, $0 < r, q \leq \infty$, and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$.*

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(i) *Let*

$$\beta_\infty - \alpha_\infty \le 0 \le \beta_0 - \alpha_0 \quad \text{if } r = q = \infty$$

and

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r} \quad \text{if } \min\{r, q\} < \infty.$$

Then

$$\bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\min\{r,q\}}} \hookrightarrow \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}. \tag{4.25}$$

(ii) *Let*

$$\beta_\infty - \alpha_\infty \le 0 \le \beta_0 - \alpha_0 \quad \text{if } r = q = \infty$$

and

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r} \quad \text{if } \max\{r, q\} < \infty.$$

Then

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}}. \tag{4.26}$$

Corollary 4.8 *Let* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Let either*

$$\theta \in [0,1),\ 0 < r < \infty, \quad \text{and} \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r} \tag{4.27}$$

or

$$\theta \in [0,1],\ r = \infty, \quad \text{and} \quad \beta_\infty - \alpha_\infty \le 0 \le \beta_0 - \alpha_0. \tag{4.28}$$

Then

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},r,\mathbb{A}} = \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{r}}. \tag{4.29}$$

Proof The result follows from Theorem 4.7 and Lemma 4.1. ■

Note that the direct proof of (4.29) shows that the restriction $\theta < 1$ in (4.27) can be relaxed (*cf.* our next result).

Lemma 4.9 *Let* $\theta \in [0,1]$, *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Suppose either*

$$0 < r < \infty \quad \text{and} \quad \beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r} \tag{4.30}$$

or

$$r = \infty \quad \text{and} \quad \beta_\infty - \alpha_\infty \le 0 \le \beta_0 - \alpha_0.$$

Then

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},r,\mathbb{A}} = \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{r}}.$$

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Proof If $r = \infty$, then Lemma 4.1 yields the result. If $r \in (0, \infty)$, then (4.7), (4.5), Fubini's theorem, (4.30), and (4.6) imply that

$$\begin{aligned}\|f\|^r_{\mathcal{L};\theta;r,\mathbb{B},r,\mathbb{A}} &= \int_0^\infty t^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(t)\Big(\int_0^t g(\tau)\,d\tau\Big)\,dt \\ &= \int_0^\infty g(\tau)\Big(\int_\tau^\infty t^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(t)\,dt\Big)\,d\tau \approx \int_0^\infty g(\tau)\ell^{(\mathbb{B}-\mathbb{A})r+1}(\tau)\,d\tau \\ &= \int_0^\infty [\tau^{-\theta-\frac{1}{r}}\ell^{\mathbb{B}+\frac{1}{r}}(\tau)K(f,\tau)]^r\,d\tau = \|f\|^r_{\theta,r;\mathbb{B}+\frac{1}{r}},\end{aligned}$$

which completes the proof. ∎

Now, we are going to show that the target space in (4.26) can be improved. In fact, we shall prove that

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow \bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}} \cap \bar{X}_{\theta,\max\{r,q\};\mathbb{B}+\frac{1}{r}} \tag{4.31}$$

provided that $\theta \in [0,1]$, $0 < r < q < \infty$, and $\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r}$.

Note that the target space in (4.31), in general, does not coincide with either of the spaces $\bar{X}_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}}$, $\bar{X}_{\theta,\max\{r,q\};\mathbb{B}+\frac{1}{r}}$ (*cf.* [EOP, Theorems 3.1 and 3.11]). Observe also that the target spaces on the right hand side of (4.31) coincide if $0 < q \le r \le \infty$.

Theorem 4.10 *Let $\theta \in [0,1]$, $0 < r < q < \infty$, and let $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ be such that*

$$\beta_\infty - \alpha_\infty + \frac{1}{r} < 0 < \beta_0 - \alpha_0 + \frac{1}{r}. \tag{4.32}$$

Then (4.31) holds.

Proof By Lemma 4.4, the embedding (4.26) holds. Thus, it remains to prove that

$$\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow \bar{X}_{\theta,\max\{r,q\};\mathbb{B}+\frac{1}{r}}. \tag{4.33}$$

Let s be given by (4.5) and $f \in \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$. Define $h \in \mathcal{M}^+(0,\infty)$ by (4.24). Then, using (4.7), (4.6), (4.24), and the fact that $\max\{r,q\} = q$, we obtain

$$\|f\|^q_{\mathcal{L};\theta;r,\mathbb{B},q,\mathbb{A}} = \Big\{\int_0^\infty \Big[\int_0^t \tau^{(1-\theta)q-1}\ell^{\mathbb{A}q}(\tau)h(\tau)\,d\tau\Big]^s t^{-1}\ell^{(\mathbb{B}-\mathbb{A})r}(t)\,dt\Big\}^{1/s}$$

and

$$\begin{aligned}\|f\|^q_{\theta,\max\{r,q\};\mathbb{B}+\frac{1}{r}} &= \|f\|^q_{\theta,q;\mathbb{B}+\frac{1}{r}} \\ &= \|t^{1-\theta-\frac{1}{q}}\ell^{\mathbb{B}+\frac{1}{r}}(t)h^{\frac{1}{q}}(t)\|^q_{q,(0,\infty)} = \int_0^\infty h(t)t^{(1-\theta)q-1}\ell^{\mathbb{B}q+\frac{1}{s}}(t)\,dt.\end{aligned}$$

Thus, (4.33) is a consequence of Lemma 3.7. ∎

The next theorem shows that (4.31) may hold even if $0 < r < q = \infty$.

Theorem 4.11 *Assume that $0 < r < q = \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$ are such that (4.32) is satisfied. Let either $\theta \in [0,1)$, or $\theta = 1$ and $\alpha_\infty \ge 0$. Then (4.31) holds.*

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Proof Since, by Lemma 4.6, the embedding (4.22) holds, it remains to prove (4.33).

If $\bar{X}^{\mathcal{L}} := \bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = \{0\}$, then (4.33) is plain. Let $\bar{X}^{\mathcal{L}} \neq \{0\}$. If $\theta = 1$, then Theorem 2.4 and the second inequality in (4.32) imply that $\alpha_0 < 0$. Hence, $\theta, r, \mathbb{A}$ and $\mathbb{B}$ satisfy the assumption of Lemma 3.10. Therefore, Lemma 3.10 applied to $h \in \mathcal{M}^+(0,\infty;\downarrow)$, given by $h(t) = K(f,t)/t$, $t \in (0,\infty)$, with $f \in \bar{X}^{\mathcal{L}}$, yields the result. ■

5 Reiteration

The aim of this section is to prove a limiting form of reiteration formula involving spaces $\bar{X}_{\theta,q;\mathbb{A}} = (X_0, X_1)_{\theta,q;\mathbb{A}}$, namely, the formula for the space

$$(\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}}.$$

To this end we shall use the following notation:

$$\bar{K}(f,t) = K(f,t;\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1}).$$

Lemma 5.1 *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1, q \le \infty$, *and* $\mathbb{A}_0 = (\alpha_{00}, \alpha_{0\infty})$, $\mathbb{A}_1 = (\alpha_{10}, \alpha_{1\infty})$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *Then*

$$(5.1) \qquad (\bar{X}_{\theta_0,q_0;\mathbb{A}_0}, \bar{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} \hookrightarrow \bar{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}.$$

Proof Put

$$(5.2) \qquad \overline{X} = (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}}$$

and

$$(5.3) \qquad \overline{X}^{\mathcal{L}} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}.$$

By [EOP, Theorem 6.3], for all $f \in \overline{X}$,

$$(5.4) \qquad \overline{K}\big(f,\rho(t)\big) \approx \|s^{-\theta_0-\frac{1}{q_0}}\ell^{\mathbb{A}_0}(s)K(f,s;X_0,X_1)\|_{q_0,(0,t)} + \rho(t)\|s^{-\theta_1-\frac{1}{q_1}}\ell^{\mathbb{A}_1}(s)K(f,s;X_0,X_1)\|_{q_1,(t,\infty)},$$

where

$$(5.5) \qquad \rho(t) = t^{\theta_1-\theta_0}\ell^{\mathbb{A}_0-\mathbb{A}_1}(t), \quad t \in (0,\infty).$$

Note that the function ρ is continuous on $(0,\infty)$, it maps $(0,\infty)$ onto $(0,\infty)$, and there exist numbers $\delta, M \in (0,\infty)$, $\delta < 1 < M$, such that ρ is increasing on $(0,\delta) \cup (M,\infty)$. Since $\rho(t) \to 0$ as $t \to 0_+$ and $\rho(t) \to \infty$ as $t \to \infty$, we may assume that δ and M are chosen in such a way that also $\rho(\delta) < 1 < \rho(M)$. One can verify that

$$(5.6) \qquad \rho'(t) \approx \rho(t)/t, \quad t \in (0,\infty),$$

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and

$$\rho^{-1}(\tau) \approx [\tau \ell^{\mathbb{A}_1-\mathbb{A}_0}(\tau)]^{1/(\theta_1-\theta_0)}, \tag{5.7}$$

provided $\tau \in \big(0,\rho(\delta)\big) \cup \big(\rho(M),\infty\big)$.

On using (5.4), we obtain for all $f \in \overline{X}$,

$$\begin{aligned} \|f\|_{\overline{X}^{\mathcal{L}}} &= \|f\|_{\mathcal{L};\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \\ &= \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\|\tau^{-\theta_0-\frac{1}{q_0}}\ell^{\mathbb{A}_0}(\tau)K(f,\tau)\|_{q_0,(0,t)}\|_{q,(0,\infty)} \\ &\lesssim \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}\big(f,\rho(t)\big)\|_{q,(0,\infty)} \lesssim N_1+N_2+N_3, \end{aligned} \tag{5.8}$$

where

$$N_i = \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}\big(f,\rho(t)\big)\|_{q,I_i}, \quad i=1,2,3,$$

and

$$I_1=(0,\delta), \quad I_2=(\delta,M), \quad I_3=(M,\infty). \tag{5.9}$$

Changing the variable to $\tau=\rho(t)$, $t\in I_1\cup I_3$, and using the relations $(dt/t)\approx(d\tau/\tau)$ and $\log\rho^{-1}(\tau)\approx\log\tau$, we get for $i=1,3$,

$$N_i \approx \|\tau^{-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)\overline{K}(f,\tau)\|_{q,\rho(I_i)} \le \|f\|_{\overline{X}}, \tag{5.10}$$

where $\rho(I_i)=\{\tau;\tau=\rho(t),t\in I_i\}$. Since $t\approx 1$ and $\ell(t)\approx 1$ on I_2, and

$$c_1\overline{K}(f,t) \le \overline{K}\big(f,\rho(t)\big) \le c_2\overline{K}(f,t), \quad t\in I_2,$$

where

$$c_1=\min_{t\in I_2}\{1,t^{-1}\rho(t)\}, \quad c_2=\max_{t\in I_2}\{1,t^{-1}\rho(t)\},$$

we have

$$N_2 \approx \|\overline{K}\big(f,\rho(t)\big)\|_{q,I_2} \approx \|\overline{K}(f,t)\|_{q,I_2} \approx \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}(f,t)\|_{q,I_2} \le \|f\|_{\overline{X}}. \tag{5.11}$$

Thus, on using estimates (5.8), (5.10), and (5.11), we obtain (5.1). ∎

Lemma 5.2 *Let* $0<\theta_0<\theta_1<1$, $0<q_0,q_1,q\le\infty$, *and* $\mathbb{A}_i=(\alpha_{i0},\alpha_{i\infty})\in\mathbb{R}^2$, $i=0,1$. *Suppose that* $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$ *is such that*

$$\alpha_\infty \le 0 \le \alpha_0 \quad \text{if } q=q_0=\infty \tag{5.12}$$

and

$$\alpha_\infty+\frac{1}{q}<0\le\alpha_0+\frac{1}{q} \quad \text{if } \max\{q,q_0\}<\infty. \tag{5.13}$$

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Then

$$\overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \hookrightarrow (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}}.$$

Proof We shall use notation (5.2) and (5.3). Let $f \in \overline{X}^{\mathcal{L}}$. Then

$$\|f\|_{\overline{X}} = \|\tau^{-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)\overline{K}(f,\tau)\|_{q,(0,\infty)} \lesssim \overline{N}_1 + \overline{N}_2 + \overline{N}_3,$$

where

$$\overline{N}_i = \|\tau^{-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)\overline{K}(f,\tau)\|_{q,\rho(I_i)}, \quad i = 1,2,3,$$

ρ is from (5.5), and I_1, I_2, I_3 are given by (5.9). On substituting $\tau = \rho(t)$, $t \in I_1 \cup I_3$, we obtain for $i = 1,3$,

$$\overline{N}_i \approx \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}\big(f,\rho(t)\big)\|_{q,I_i} \le \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}\big(f,\rho(t)\big)\|_{q,(0,\infty)}.$$

Since, moreover,

$$\begin{aligned}\overline{N}_2 &\approx \|\overline{K}(f,\tau)\|_{q,\rho(I_2)} \approx \|\overline{K}\big(f,\rho(\tau)\big)\|_{q,\rho(I_2)}\\ &\approx \|\tau^{-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)\overline{K}\big(f,\rho(\tau)\big)\|_{q,\rho(I_2)} \le \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}\big(f,\rho(t)\big)\|_{q,(0,\infty)},\end{aligned}$$

we have

$$\|f\|_{\overline{X}} \lesssim \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\overline{K}\big(f,\rho(t)\big)\|_{q,(0,\infty)} \tag{5.14}$$

and, on making use of (5.4) and (5.5),

$$\|f\|_{\bar{X}} \lesssim I + J,$$

where

$$I = \|t^{-\frac{1}{q}}\ell^{\mathbb{A}}(t)\|s^{-\theta_0-\frac{1}{q_0}}\ell^{\mathbb{A}_0}(s)K(f,s)\|_{q_0,(0,t)}\|_{q,(0,\infty)} = \|f\|_{\overline{X}^{\mathcal{L}}}$$

and

$$J = \|t^{\theta_1-\theta_0-\frac{1}{q}}\ell^{\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1}(t)\|s^{-\theta_1-\frac{1}{q_1}}\ell^{\mathbb{A}_1}(s)K(f,s)\|_{q_1,(t,\infty)}\|_{q,(0,\infty)}.$$

Thus, to conclude the proof, it is sufficient to verify that

$$J \lesssim \|f\|_{\overline{X}^{\mathcal{L}}} \quad \text{for all } f \in \overline{X}^{\mathcal{L}}. \tag{5.15}$$

To this end, we shall distinguish several cases:

(i) Assume first that $q_1 = \infty$, $q = \infty$. Since

$$\begin{aligned}J &= \operatorname*{esssup}_{0<t<\infty} t^{\theta_1-\theta_0}\ell^{\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1}(t) \operatorname*{esssup}_{t<s<\infty} s^{-\theta_1}\ell^{\mathbb{A}_1}(s)K(f,s)\\ &= \operatorname*{esssup}_{0<s<\infty} s^{-\theta_1}\ell^{\mathbb{A}_1}(s)K(f,s) \operatorname*{esssup}_{0<t<s} t^{\theta_1-\theta_0}\ell^{\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1}(t)\\ &\approx \operatorname*{esssup}_{0<s<\infty} s^{-\theta_1}\ell^{\mathbb{A}_1}(s)K(f,s)s^{\theta_1-\theta_0}\ell^{\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1}(s)\\ &= \operatorname*{esssup}_{0<s<\infty} s^{-\theta_0}\ell^{\mathbb{A}+\mathbb{A}_0}(s)K(f,s) = \|f\|_{\theta_0,\infty;\mathbb{A}+\mathbb{A}_0}\end{aligned}$$

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and, by Lemma 4.5 (if $q_0 < \infty$) or by Lemma 4.1 (if $q_0 = \infty$; note that then (5.12) is used),

$$\|f\|_{\overline{X}^{\mathcal{L}}} = \|f\|_{\mathcal{L};\theta_0;\infty,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \gtrsim \|f\|_{\theta_0,\infty;\mathbb{A}+\mathbb{A}_0},$$

the estimate (5.15) follows.

(ii) Let $0 < q_1 < \infty$ and $0 < q \le \infty$. On putting

$$\sigma = q/q_1 \tag{5.16}$$

and

$$g(s) = [s^{-\theta_1 - \frac{1}{q_1}} \ell^{\mathbb{A}_1}(s) K(f,s)]^{q_1}, \tag{5.17}$$

we can write

$$J^{q_1} = \left\| t^{(\theta_1-\theta_0)q_1 - \frac{1}{\sigma}} \ell^{(\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1)q_1}(t) \int_t^\infty g(s)\, ds \right\|_{\sigma,(0,\infty)}. \tag{5.18}$$

(ii$_1$) Assume additionally that $q_1 \le q$ (consequently, $0 < q_1 \le q \le \infty$ and $q_1 < \infty$). Then $1 \le \sigma \le \infty$. Using (5.18), Lemma 3.8(ii) (with $P = Q = \sigma$, $\nu = (\theta_1 - \theta_0)q_1$, and $\mathbb{D} = (\mathbb{A} + \mathbb{A}_0 - \mathbb{A}_1)q_1$), (5.17), and (5.16), we obtain

$$\begin{aligned} J^{q_1} &\lesssim \|t^{(\theta_1-\theta_0)q_1 + \frac{1}{\sigma'}} \ell^{(\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1)q_1}(t) g(t)\|_{\sigma,(0,\infty)} \\ &= \|t^{-\theta_0 q_1 - \frac{1}{\sigma}} \ell^{(\mathbb{A}+\mathbb{A}_0)q_1}(t) K(f,t)^{q_1}\|_{\sigma,(0,\infty)} \\ &= \|t^{-\theta_0 - \frac{1}{q}} \ell^{(\mathbb{A}+\mathbb{A}_0)}(t) K(f,t)\|_{q,(0,\infty)}^{q_1} \\ &= \|f\|_{\theta_0,q;\mathbb{A}+\mathbb{A}_0}^{q_1} \le \|f\|_{\theta_0,q;\mathcal{A}}^{q_1} \end{aligned} \tag{5.19}$$

where

$$\mathcal{A} = \mathbb{A} + \mathbb{A}_0 + \frac{1}{\max\{q, q_0\}}. \tag{5.20}$$

Since, by Theorem 4.7(ii) (*cf.* (5.12) and (5.13)),

$$\overline{X}^{\mathcal{L}} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \hookrightarrow \overline{X}_{\theta_0,q;\mathcal{A}}, \tag{5.21}$$

the estimate (5.15) follows from (5.19).

(ii$_2$) Assume additionally that $q < q_1$ (consequently, $0 < q < q_1 < \infty$). Then $0 < \sigma < 1$. Let $\mathcal{A}$ be given by (5.20). Put $P = Q = \sigma$ and, for $x, y \in (0,\infty)$,

$$\begin{cases} \Phi(x,y) = \chi_{(x,\infty)}(y) y^{-\theta_1 q_1 - 1} \ell^{\mathbb{A}_1 q_1}(y), \\ w(x) = x^{(\theta_1-\theta_0)q-1} \ell^{(\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1)q}(x), \\ v(x) = x^{-\theta_0 q - 1} \ell^{\mathcal{A}q}(x), \\ h(y) = [K(f,y)]^{q_1}; \end{cases} \tag{5.22}$$

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note that $h \in \mathcal{M}^+(0,\infty;\uparrow)$. Using (5.18), (5.17), and (5.22), we obtain

$$J^{q_1} = \Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)^P w(x)\,dx\Big]^{1/P}$$

and

$$\Big[\int_0^\infty h^Q(x)v(x)\,dx\Big]^{1/Q} = \|h^{\frac{1}{q_1}}(x)v(x)^{\frac{1}{q}}\|_{q,(0,\infty)}^{q_1}$$
$$= \|x^{-\theta_0-\frac{1}{q}}\ell^{\mathcal{A}}(x)K(f,x)\|_{q,(0,\infty)}^{q_1} = \|f\|_{\theta_0,q;\mathcal{A}}^{q_1}.$$

Since, by Theorem 4.7(ii), the embedding (5.21) holds, the estimate (5.15) holds if and only if the inequality (3.13) (with our choice of P, Q, Φ, w, and v) is satisfied. By Theorem 3.9, this is the case if we verify (3.14).

Let $z \in (0,\infty)$. Then

$$\int_z^\infty v(x)\,dx = \int_z^\infty x^{-\theta_0 q-1}\ell^{\mathcal{A}q}(x)\,dx \approx z^{-\theta_0 q}\ell^{\mathcal{A}q}(z). \tag{5.23}$$

Moreover,

$$\begin{aligned}\int_0^\infty \Big(\int_z^\infty \Phi(x,y)\,dy\Big)^P w(x)\,dx &\\ = \int_0^z \Big(\int_z^\infty \Phi(x,y)\,dy\Big)^P w(x)\,dx + \int_z^\infty \Big(\int_z^\infty \Phi(x,y)\,dy\Big)^P w(x)\,dx &\\ = \int_0^z \Big(\int_z^\infty y^{-\theta_1 q_1-1}\ell^{\mathbb{A}_1 q_1}(y)\,dy\Big)^\sigma w(x)\,dx &\\ + \int_z^\infty \Big(\int_x^\infty y^{-\theta_1 q_1-1}\ell^{\mathbb{A}_1 q_1}(y)\,dy\Big)^\sigma w(x)\,dx &\\ \approx z^{-\theta_0 q}\ell^{(\mathbb{A}+\mathbb{A}_0)q}(z).&\end{aligned} \tag{5.24}$$

The estimates (5.23) and (5.24) imply that the condition (3.14) holds.

(iii) Finally, let $0 < q < q_1 = \infty$. It is sufficient to prove that

$$J \le \|f\|_{\theta_0,q;\mathcal{A}} \quad \text{for all} \quad f \in \overline{X}^{\mathcal{L}} \tag{5.25}$$

with $\mathcal{A}$ from (5.20) since then the estimate (5.15) follows on applying the embedding (5.21).

Let $f \in \overline{X}^{\mathcal{L}}$. For $x, y \in (0,\infty)$ put

$$\begin{cases} \Phi(x,y) = \chi_{(x,\infty)}(y)y^{-\theta_1 q-1}\ell^{\mathbb{A}_1 q}(y), \\ w(x) = x^{(\theta_1-\theta_0)q-1}\ell^{(\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1)q}(x), \\ v(x) = x^{-\theta_0 q-1}\ell^{\mathcal{A}q}(x), \\ h(y) = [K(f,y)]^q; \end{cases} \tag{5.26}$$

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again $h \in \mathcal{M}^+(0,\infty;\uparrow)$. As

$$\|s^{-\theta_1}\ell^{\mathbb{A}_1}(s)K(f,s)\|_{\infty,(t,\infty)} \approx \operatorname*{esssup}_{t<s<\infty} K(f,s)\|\sigma^{-\theta_1-\frac{1}{q}}\ell^{\mathbb{A}_1}(\sigma)\|_{q,(s,\infty)} \leq \|\sigma^{-\theta_1-\frac{1}{q}}\ell^{\mathbb{A}_1}(\sigma)K(f,\sigma)\|_{q,(t,\infty)},$$

we obtain

$$J^q \lesssim \|t^{\theta_1-\theta_0-\frac{1}{q}}\ell^{\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1}(t)\Big(\int_t^\infty \sigma^{-\theta_1 q-1}\ell^{\mathbb{A}_1 q}(\sigma)h(\sigma)\,d\sigma\Big)^{1/q}\|_{q,(0,\infty)}^q$$
$$= \Big\|t^{(\theta_1-\theta_0)q-1}\ell^{(\mathbb{A}+\mathbb{A}_0-\mathbb{A}_1)q}(t)\int_t^\infty \sigma^{-\theta_1 q-1}\ell^{\mathbb{A}_1 q}(\sigma)h(\sigma)\,d\sigma\Big\|_{1,(0,\infty)}$$
$$= \int_0^\infty\Big(\int_0^\infty \Phi(x,y)h(y)\,dy\Big)w(x)\,dx. \tag{5.27}$$

Moreover,

$$\|f\|_{\theta_0,q;\mathcal{A}}^q = \|t^{-\theta_0-\frac{1}{q}}\ell^{\mathcal{A}}(t)K(f,t)\|_{q,(0,\infty)}^q$$
$$= \|t^{-\theta_0 q-1}\ell^{\mathcal{A}q}(t)h(t)\|_{1,(0,\infty)} = \int_0^\infty h(x)v(x)\,dx. \tag{5.28}$$

Since the condition (3.14) with $P = Q = 1$ (and Φ, w, and v given by (5.26)) is satisfied, by Theorem 3.9 the inequality (3.13) holds on $\mathcal{M}^+(0,\infty;\uparrow)$. Together with (5.27) and (5.28) this yields (5.25), which completes the proof. ■

Combining Lemmas 5.1 and 5.2, we obtain the desired reiteration formula.

Theorem 5.3 *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1, q \leq \infty$, *and* $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 0, 1$. *Suppose that* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *is such that*

$$\alpha_\infty \leq 0 \leq \alpha_0 \quad \text{if } q = q_0 = \infty \tag{5.29}$$

and

$$\alpha_\infty + \frac{1}{q} < 0 < \alpha_0 + \frac{1}{q} \quad \text{if } \max\{q, q_0\} < \infty. \tag{5.30}$$

Then

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}. \tag{5.31}$$

Corollary 5.4 *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1 \leq \infty$, *and* $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 0, 1$. *Suppose that* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *is such that*

$$\alpha_\infty \leq 0 \leq \alpha_0 \quad \text{if } q_0 = \infty$$

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and

$$\alpha_\infty + \frac{1}{q_0} < 0 < \alpha_0 + \frac{1}{q_0} \quad \text{if } q_0 < \infty.$$

Then

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q_0;\mathbb{A}} = \overline{X}_{\theta_0,q_0;\mathbb{A}+\mathbb{A}_0+\frac{1}{q_0}}. \tag{5.32}$$

Proof The assertion is a consequence of Theorem 5.3 and Lemma 4.9. ■

One can see from Theorem 5.3 that the resulting space in (5.31), that is the space $\overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}$, is independent of parameters θ_1, q_1, and $\mathbb{A}_1$. That means there is a certain kind of *stability* in the reiteration formula (5.31) with respect to the space $\overline{X}_{\theta_1,q_1;\mathbb{A}_1}$. Now, we are going to show that (5.31) continues to hold even if the space $\overline{X}_{\theta_1,q_1;\mathbb{A}_1}$ is replaced by any intermediate space $\overline{X}_1$ of class 1, that is, by any intermediate space $\bar{X}_1$ (between X_0 and X_1) satisfying (*cf.* [BS, Chapter 5, Definition 2.2])

$$X_1 \hookrightarrow \bar{X}_1 \hookrightarrow X_1 + \infty X_0 := (X_0, X_1)_{1,\infty;(0,0)}.$$

Lemma 5.5 *Let* $0 < \theta_0 < 1$, $0 < q_0, q \le \infty$, *and* $\mathbb{A}_0 = (\alpha_{00}, \alpha_{0\infty})$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *If* $\overline{X}_1$ *is an intermediate space of class 1, then*

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q;\mathbb{A}} \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}. \tag{5.33}$$

The proof uses the same arguments as that of Lemma 5.1, except that the estimate (*cf.* [EOP, Theorem 6.6])

$$K(f, \rho(t); \overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1) \approx \| s^{-\theta_0-\frac{1}{q_0}} \ell^{\mathbb{A}_0}(s) K(f,s) \|_{q_0,(0,t)} \tag{5.34}$$

with

$$\rho(t) = t^{1-\theta_0} \ell^{\mathbb{A}_0}(t), \quad t \in (0,\infty), \tag{5.35}$$

is applied instead of (5.4). ■

The next lemma deals with the opposite embedding to (5.33).

Lemma 5.6 *Let* $0 < \theta_0 < 1$, $0 < q_0, q \le \infty$, *and* $\mathbb{A}_0 = (\alpha_{00}, \alpha_{0\infty})$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *If* $\bar{X}_1$ *is an intermediate space of class 1, then*

$$\overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \hookrightarrow (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q;\mathbb{A}}. \tag{5.36}$$

Proof Let $\overline{X}^{\mathcal{L}}$ be given by (5.3); put $\overline{X} = (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q;\mathbb{A}}$ and $\overline{K}(f,t) = K(f,t;\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)$. Using the arguments of the proof of Lemma 5.2 (except that ρ is now given by (5.35)), one obtains (5.14), and, on applying (5.34),

$$\|f\|_{\overline{X}} \lesssim \| t^{-\frac{1}{q}} \ell^{\mathbb{A}}(t) \| s^{-\theta_0-\frac{1}{q_0}} \ell^{\mathbb{A}_0}(s) K(f,s) \|_{q_0,(0,t)} \|_{q,(0,\infty)} = \|f\|_{\overline{X}^{\mathcal{L}}}.$$ ■

Combining Lemmas 5.5 and 5.6, we get the desired result.

Theorem 5.7 *Let* $0 < \theta_0 < 1$, $0 < q_0, q \le \infty$, *and* $\mathbb{A}_0 = (\alpha_{00}, \alpha_{0\infty})$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *If* $\bar{X}_1$ *is an intermediate space of class 1, then*

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q;\mathbb{A}} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}. \tag{5.37}$$

The next assertion is a consequence of Theorem 5.7 and Lemma 4.9.

Corollary 5.8 *Let* $0 < \theta_0 < 1$, $0 < q_0 \le \infty$, *and* $\mathbb{A}_0 = (\alpha_{00}, \alpha_{0\infty})$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *Suppose that*

$$\alpha_\infty \le 0 \le \alpha_0 \quad \text{if } q_0 = \infty$$

or

$$\alpha_\infty + \frac{1}{q_0} < 0 < \alpha_0 + \frac{1}{q_0} \quad \text{if } q_0 < \infty.$$

If $\overline{X}_1$ *is an intermediate space of class 1, then*

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q_0;\mathbb{A}} = \overline{X}_{\theta_0,q_0;\mathbb{A}+\mathbb{A}_0+\frac{1}{q_0}}. \tag{5.38}$$

Combining Theorems 5.3 and 5.7 (or Corollaries 5.4 and 5.8), we obtain the following result on stability of the reiteration formula (5.31).

Theorem 5.9 *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1, q \le \infty$, *and* $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 0, 1$. *Suppose that* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *is such that*

$$\alpha_\infty \le 0 \le \alpha_0 \quad \text{if } q = q_0 = \infty$$

and

$$\alpha_\infty + \frac{1}{q} < 0 < \alpha_0 + \frac{1}{q} \quad \text{if } \max\{q, q_0\} < \infty.$$

If, moreover, $\overline{X}_1$ *is an intermediate space of class 1, then*

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q;\mathbb{A}} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}.$$

Corollary 5.10 *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1 \le \infty$, *and* $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 0, 1$. *Suppose that* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ *is such that*

$$\alpha_\infty \le 0 \le \alpha_0 \quad \text{if } q_0 = \infty$$

and

$$\alpha_\infty + \frac{1}{q_0} < 0 < \alpha_0 + \frac{1}{q_0} \quad \text{if } q_0 < \infty.$$

If, moreover, $\overline{X}_1$ *is an intermediate space of class 1, then*

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q_0;\mathbb{A}} = (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_1)_{0,q_0;\mathbb{A}} = \overline{X}_{\theta_0,q_0;\mathbb{A}+\mathbb{A}_0+\frac{1}{q_0}}.$$

Remark 5.11 Note that a similar stability property to that of Theorem 5.9 has the reiteration formula

$$(\overline{X}_0, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = (\overline{X}_0, \overline{X}_1)_{0,q;\mathbb{A}} = \overline{X}_{0,q;\mathbb{A}}$$

provided that $\alpha_\infty + \frac{1}{q} < 0$ or $\alpha_\infty \le 0$ when $q = \infty$, and $\overline{X}_i$ $(i = 0, 1)$ is an intermediate space of class i (see [EOP, Theorem 7.1 (iii)]; *cf.* also [GM, Theorem 4.1]).

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6 The Spaces $\bar{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$

For $0 < r, q \le \infty$, $\theta \in [0,1]$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$, we define

$$\bar{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \equiv (X_0, X_1)^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}} := \{ f \in X_0 + X_1 : \|f\|_{\mathcal{R};\theta;r,\mathbb{B},q,\mathbb{A}} < \infty \}, \tag{6.1}$$

where

$$\|f\|_{\mathcal{R};\theta;r,\mathbb{B},q,\mathbb{A}} := \| t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) \| \tau^{-\theta-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) K(f,\tau;X_0,X_1)\|_{q,(t,\infty)} \|_{r,(0,\infty)}. \tag{6.2}$$

These spaces are needed to obtain analogous results to the preceding ones at the right end (θ=1) of the θ-range $[0,1]$. They are in fact related to the spaces $\bar{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ by the identity

$$(X_0, X_1)^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = (X_1, X_0)^{\mathcal{L}}_{1-\theta;r,\tilde{\mathbb{B}},q,\tilde{\mathbb{A}}}, \tag{6.3}$$

where, if $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty)$, we have $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$ and $\tilde{\mathbb{B}} = (\beta_\infty, \beta_0)$. This follows from a change of variables $t \to 1/t$ and the fact that (see [BS, Chapter 5, Proposition 1.2])

$$tK(f, t^{-1}; X_0, X_1) = K(f, t; X_1, X_0). \tag{6.4}$$

Similarly, on using (6.4) we have

$$(X_0, X_1)_{\theta,q;\mathbb{A}} = (X_1, X_0)_{1-\theta,q;\tilde{\mathbb{A}}} \tag{6.5}$$

and hence

$$\begin{aligned} &\big((X_0, X_1)_{\theta_0,q_0;\mathbb{A}_0}, (X_0, X_1)_{\theta_1,q_1;\mathbb{A}_1} \big)_{1,q;\mathbb{A}} \\ &\qquad = \big((X_1, X_0)_{1-\theta_1,q_1;\tilde{\mathbb{A}}_1}, (X_1, X_0)_{1-\theta_0,q_0;\tilde{A}_0} \big)_{0,q;\tilde{A}}. \end{aligned} \tag{6.6}$$

These remarks lead to the following consequences of Theorems 2.4.

Theorem 2.4*** *Let $0 \le \theta \le 1$, $0 < q, r \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$.*

(i) The space $\bar{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ is a (quasi-) Banach space which is intermediate between X_0 and X_1, provided that one of the following conditions is satisfied:
I. $0 < q, r < \infty$ and either

$$0 < \theta < 1, \quad \beta_0 - \alpha_0 + \frac{1}{r} < 0$$

or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} < 0, \quad \beta_\infty + \frac{1}{q} + \frac{1}{r} < 0, \quad \beta_0 - \alpha_0 + \frac{1}{r} < 0$$

or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} < 0, \quad \beta_0 - \alpha_0 + \frac{1}{r} < 0$$

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or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} \geq 0, \quad \beta_0 + \frac{1}{q} + \frac{1}{r} < 0;$$

II. $q = \infty$, $0 < r < \infty$ *and either*

$$0 < \theta < 1, \quad \beta_0 - \alpha_0 + \frac{1}{r} < 0$$

or

$$\theta = 0, \quad \alpha_\infty \leq 0, \quad \beta_\infty + \frac{1}{r} < 0, \quad \beta_0 - \alpha_0 + \frac{1}{r} < 0$$

or

$$\theta = 1, \quad \alpha_0 < 0, \quad \beta_0 - \alpha_0 + \frac{1}{r} < 0$$

or

$$\theta = 1, \quad \alpha_0 \geq 0, \quad \beta_0 + \frac{1}{r} < 0;$$

III. $0 < q < \infty$, $r = \infty$ *and either*

$$0 < \theta < 1, \quad \beta_0 - \alpha_0 \leq 0$$

or

$$\theta = 0, \quad \alpha_\infty + \frac{1}{q} < 0, \quad \beta_\infty + \frac{1}{q} \leq 0, \quad \beta_0 - \alpha_0 \leq 0$$

or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} < 0, \quad \beta_0 - \alpha_0 \leq 0$$

or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} = 0, \quad \beta_0 - \alpha_0 < 0$$

or

$$\theta = 1, \quad \alpha_0 + \frac{1}{q} > 0, \quad \beta_0 + \frac{1}{q} \leq 0;$$

IV. $q = \infty$, $r = \infty$ *and either*

$$0 < \theta < 1, \quad \beta_0 - \alpha_0 \leq 0$$

or

$$\theta = 0, \quad \alpha_\infty \leq 0, \quad \beta_\infty \leq 0, \quad \beta_0 - \alpha_0 \leq 0$$

or

$$\theta = 1, \quad \alpha_0 < 0, \quad \beta_0 - \alpha_0 \leq 0$$

or

$$\theta = 1, \quad \alpha_0 \geq 0, \quad \beta_0 \leq 0.$$

(ii) If none of the conditions I–IV holds, then $\overline{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ *is a trivial space.*

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Similarly, one can obtain the symmetric results to those of Sections 4 and 5 involving spaces $\overline{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$. For example, the symmetric counterparts of Theorems 4.7 and 5.9 read as follows.

Theorem 4.7* *Let $\theta \in (0,1]$, $0 < r, q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$.*

(i) Let

$$\beta_0 - \alpha_0 \le 0 \le \beta_\infty - \alpha_\infty \quad \textit{if } r = q = \infty$$

and

$$\beta_0 - \alpha_0 + \frac{1}{r} < 0 < \beta_\infty - \alpha_\infty + \frac{1}{r} \quad \textit{if } \min\{r,q\} < \infty.$$

Then

$$\overline{X}_{\theta,r;\mathbb{B}+\frac{1}{\min\{r,q\}}} \hookrightarrow \overline{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}. \tag{4.25*}$$

(ii) Let

$$\beta_0 - \alpha_0 \le 0 \le \beta_\infty - \alpha_\infty \quad \textit{if } r = q = \infty$$

and

$$\beta_0 - \alpha_0 + \frac{1}{r} < 0 < \beta_\infty - \alpha_\infty + \frac{1}{r} \quad \textit{if } \max\{r,q\} < \infty.$$

Then

$$\overline{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}} \hookrightarrow \overline{X}_{\theta,r;\mathbb{B}+\frac{1}{\max\{r,q\}}}. \tag{4.26*}$$

Theorem 5.9* *Let $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1, q \le \infty$ and $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 0, 1$. Suppose that $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ is such that*

$$\alpha_0 \le 0 \le \alpha_\infty \quad \textit{if } q = q_1 = \infty$$

and

$$\alpha_0 + \frac{1}{q} < 0 < \alpha_\infty + \frac{1}{q} \quad \textit{if } \max\{q, q_1\} < \infty.$$

Then, if $\bar{X}_0$ is an intermediate space of class 0,

$$(\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{1,q;\mathbb{A}} = (\overline{X}_0, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{1,q;\mathbb{A}} = \overline{X}^{\mathcal{R}}_{\theta_1;q,\mathbb{A}+\mathbb{A}_1,q_1,\mathbb{A}_1}.$$

7 Ordered Couples

We now indicate the changes which result from the assumptions that the compatible couple (X_0, X_1) of (quasi-) Banach spaces is ordered in the sense that $X_1 \subseteq X_0$ algebraically and topologically, *i.e.* the identification map is continuous, and hence there exists $K > 0$ such that $\|f\|_{X_0} \le K\|f\|_{X_1}$ for all $f \in X_1$. These are the spaces considered in [D]. An example is $X_0 = L^1(\Omega)$, $X_1 = L^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$ such that $|\Omega| < \infty$. When $X_1 \subseteq X_0$ and X_0 is a

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Banach space, then it is readily seen that $K(f,t;X_0,X_1) = \|f\|_{X_0}$ for all $f \in X_0$ and $t \ge K$ (*cf.* [BL]), whereas if X_0 is a quasi-Banach space, there exists $c > 1$ such that

$$\frac{1}{c}\|f\|_{X_0} \le K(f,t;X_0,X_1) \le \|f\|_{X_0}, \quad f \in X_0$$

for $t \ge K$; in fact, in this case the functional $\tilde{K}$ in the proof of Theorem 2.1 satisfies $\tilde{K}(f,t;X_0,X_1) = \|f\|_{X_0}^p$ for $f \in X_0$, where $p \in (0,1]$ is such that $\|\cdot\|_{X_0}$ is equivalent to a p-norm.

The spaces $(X_0,X_1)_{\theta,q;\mathbb{A}}$ and $(X_0,X_1)^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$ are now defined (*cf.* [D] and [EOP]) with $\mathbb{A} = (\alpha,\alpha)$ and $\mathbb{B} = (\beta,\beta)$, and (quasi-) norms

$$\|f\|_{\theta,q;\alpha} := \|t^{-\theta-\frac{1}{q}}\ell^{\alpha}(t)K(f,t,X_0,X_1)\|_{q,(0,1)},$$

$$\|f\|_{\mathcal{L};\theta;r,\beta,q,\alpha} := \|t^{-1/r}\ell^{\beta-\alpha}(t)\|\tau^{-\theta-\frac{1}{q}}\ell^{\alpha}(\tau)K(f,\tau;X_0,X_1)\|_{q,(0,t)}\|_{r,(0,1)}$$

respectively: they are denoted by $\overline{X}_{\theta,q;\alpha}$ and $\overline{X}^{\mathcal{L}}_{\theta;r,\beta,q,\alpha}$.

The proofs of our general results in this paper are easily adapted to give analogous results for the ordered couple case: these essentially involve replacing the intervals $(0,\infty)$, (t,∞) by $(0,1)$, $(t,1)$ respectively, the vector exponents $\mathbb{A}$, $\mathbb{B}$ by their first components, and by omitting all the assumptions on their second components. For example, the results which correspond to Theorems 2.4, 4.7, 4.10, 5.7 and 5.9 are as follows.

Theorem 2.4^{+} *Let* $0 \le \theta \le 1$, $0 < q, r \le \infty$ *and* $\alpha, \beta \in \mathbb{R}$.

(i) *The space* $(X_0,X_1)^{\mathcal{L}}_{\theta;r,\beta,q,\alpha}$ *is an intermediate space between* X_0 *and* X_1 *provided that one of the following conditions is satisfied:*

(a) $0 \le \theta < 1$;

(b) $\theta = 1$ *and either*

$$0 < q, r < \infty, \quad \alpha + \frac{1}{q} < 0, \quad \beta + \frac{1}{q} + \frac{1}{r} < 0$$

or

$$q = \infty, \quad 0 < r < \infty, \quad \alpha \le 0, \quad \beta + \frac{1}{r} < 0$$

or

$$0 < q < \infty, \quad r = \infty, \quad \alpha + \frac{1}{q} < 0, \quad \beta + \frac{1}{q} \le 0$$

or

$$q = \infty, \quad r = \infty, \quad \alpha \le 0, \quad \beta \le 0.$$

(ii) *If none of the above conditions holds, then* $(X_0,X_1)^{\mathcal{L}}_{\theta;r,\beta,q,\alpha} = \{0\}$.

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Theorem 4.7$^+$ *Let* $\theta \in [0,1)$, $0 < r, q \le \infty$ *and* $\alpha, \beta \in \mathbb{R}$.

(i) Let

$$\beta - \alpha \ge 0 \quad \text{if } r = q = \infty,$$
$$\beta - \alpha + \frac{1}{r} > 0 \quad \text{if } \min\{r,q\} < \infty.$$

Then

$$\overline{X}_{\theta,r;\beta+\frac{1}{\min\{r,q\}}} \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta;r,\beta,q,\alpha}. \tag{4.25$^+$}$$

(ii) Let

$$\beta - \alpha \ge 0 \quad \text{if } r = q = \infty,$$
$$\beta - \alpha + \frac{1}{r} > 0 \quad \text{if } \max\{r,q\} < \infty.$$

Then

$$\overline{X}^{\mathcal{L}}_{\theta;r,\beta,q,\alpha} \hookrightarrow \overline{X}_{\theta,r;\beta+\frac{1}{\max\{r,q\}}}. \tag{4.26$^+$}$$

Theorem 4.10$^+$ *Let* $\theta \in [0,1]$, $0 < r < q < \infty$, *and* $\alpha, \beta \in \mathbb{R}$ *with* $\beta - \alpha + \frac{1}{r} > 0$. *Then*

$$\overline{X}^{\mathcal{L}}_{\theta;r,\beta,q,\alpha} \hookrightarrow \overline{X}_{\theta,r;\beta+\frac{1}{q}} \cap \overline{X}_{\theta,q;\beta+\frac{1}{r}}. \tag{4.31$^+$}$$

Theorem 5.9$^+$ *Let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0, q_1, q \le \infty$ *and* $\alpha_0, \alpha_1 \in \mathbb{R}$. *Let* $\alpha \in \mathbb{R}$ *be such that*

$$\alpha \ge 0 \quad \text{if } q = q_0 = \infty,$$
$$\alpha + \frac{1}{q} > 0 \quad \text{if } \max\{q, q_0\} < \infty.$$

Then

$$(\overline{X}_{\theta_0,q_0;\alpha_0}, \overline{X}_{\theta_1,q_1;\alpha_1})_{0,q;\alpha} = (\overline{X}_{\theta_0,q_0;\alpha_0}, \overline{X}_1)_{0,q;\alpha} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\alpha+\alpha_0,q_0,\alpha_0}$$

where $\overline{X}_1$ *is an intermediate space of class 1.*

Results for an ordered couple (X_0, X_1) (still with $X_1 \subseteq X_0$) corresponding to $\theta = 1$ in the reiteration formulae are obtained in terms of the spaces $\overline{X}^{\mathcal{R}}_{\theta;r,\beta,q,\alpha} = (X_0, X_1)^{\mathcal{R}}_{\theta;r,\beta,q,\alpha}$ with (quasi-) norms

$$\|f\|_{\mathcal{R};\theta;r,\beta,q,\alpha} := \|t^{-\frac{1}{r}}\ell^{\beta-\alpha}(t)\|\tau^{-\theta-\frac{1}{q}}\ell^{\alpha}(\tau)K(f,\tau,X_0,X_1)\|_{q,(t,1)}\|_{r,(0,1)}.$$

Such results can be easily obtained from general results for the spaces $\overline{X}^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$, replacing the intervals $(0,\infty)$, (t,∞) by $(0,1)$, $(t,1)$ respectively, the vector exponents $\mathbb{A}$, $\mathbb{B}$ by their

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first components, and by omitting all the assumptions on their second components. (The formula (6.3) offers another way of getting such results from our general results for the spaces $\overline{X}^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$: these essentially involve replacing θ by $1-\theta$, the vector exponents $\mathbb{A}$, $\mathbb{B}$ by their second components, and by omitting all the assumptions on their first components).

For example, the result corresponding to Theorem 4.7* reads as follows (compare also with Theorem 4.7):

Theorem 4.7^{*+} *Let* $\theta \in (0,1]$, $0 < r, q \le \infty$ *and* $\alpha, \beta \in \mathbb{R}$.

(i) Let

$$\beta - \alpha \le 0 \quad \text{if } r = q = \infty,$$

$$\beta - \alpha + \frac{1}{r} < 0 \quad \text{if } \min\{r,q\} < \infty.$$

Then

$$\overline{X}_{\theta,r;\beta+\frac{1}{\min\{r,q\}}} \hookrightarrow \overline{X}^{\mathcal{R}}_{\theta;r,\beta,q,\alpha}. \tag{4.25^{*+}}$$

(ii) Let

$$\beta - \alpha \le 0 \quad \text{if } r = q = \infty,$$

$$\beta - \alpha + \frac{1}{r} < 0 \quad \text{if } \max\{r,q\} < \infty.$$

Then

$$\overline{X}^{\mathcal{R}}_{\theta;r,\beta,q,\alpha} \hookrightarrow \overline{X}_{\theta,r;\beta+\frac{1}{\max\{r,q\}}}. \tag{4.26^{*+}}$$

8 Applications

Let $(\mathcal{R},\mu)$ denote a totally σ-finite measure space with a non-atomic measure μ. By $\mathcal{M}(\mathcal{R},\mu)$ we mean the set of all μ-measurable functions on $\mathcal{R}$. If $f \in \mathcal{M}(\mathcal{R},\mu)$, f^* denotes the non-increasing rearrangement of f with respect to μ, and $f^{**}(t) := t^{-1}\int_0^t f^*(s)\,ds$. We adopt the convention:

$$(+\infty)^s = +\infty \quad \text{for all } s \in (0,\infty). \tag{$*$}$$

Definition 8.1 Let $0 < p, q \le \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. The *Lorentz-Zygmund space* $L_{p,q;\mathbb{A}} = L_{p,q;\mathbb{A}}(\mathcal{R},\mu)$ is the set of all $f \in \mathcal{M}(\mathcal{R},\mu)$ such that

$$\|f; L_{p,q;\mathbb{A}}\| := \|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t) f^*(t)\|_{q,(0,\infty)} < \infty. \tag{8.1}$$

(For detail study of spaces $L_{p,q;\mathbb{A}}$ we refer to [OP].)

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J. Peetre [Pe] has shown that

$$K(f,t;L^1,L^\infty)=\int_0^t f^*(\tau)\,d\tau. \tag{8.2}$$

This result has been generalized by P. Krée [Kr] who has proved that

$$K(f,t;L^s,L^\infty)\approx\Big(\int_0^{t^s} f^*(\tau)^s\,d\tau\Big)^{1/s},\quad 0<s<\infty. \tag{8.3}$$

When the space L^s is replaced by $L^{s,\infty}$, then (*cf.*, *e.g.* , [Ho])

$$K(f,t;L^{s,\infty},L^\infty)\approx\sup_{0<\tau<t^s}\tau^{\frac{1}{s}}f^*(\tau),\quad 0<s<\infty. \tag{8.4}$$

We shall need the following two lemmas (where we make use of the convention (∗)).

8.2. Lemma *Let* $\theta\in(0,1)$, $0<q\le\infty$, $0<s<\infty$, *and* $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$. *Then for all* $t\in(0,\infty]$ *and every* $f\in L^s+L^\infty$,

$$\|\tau^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)K(f,\tau;L^s,L^\infty)\|_{q,(0,t)}\approx\|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)f^*(y)\|_{q,(0,t^s)}. \tag{8.5}$$

Proof Put $N(t)=\|\tau^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)K(f,\tau;L^s,L^\infty)\|_{q,(0,t)}$. On using (8.3) and the change of variables $\tau^s=y$, we get

$$\begin{aligned}N(t)&\approx\Big\|\tau^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)\Big(\int_0^{\tau^s}f^*(z)^s\,dz\Big)^{1/s}\Big\|_{q,(0,t)}\\&\approx\Big\|y^{-\frac{\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)\Big(\int_0^{y}f^*(z)^s\,dz\Big)^{1/s}\Big\|_{q,(0,t^s)}.\end{aligned} \tag{8.6}$$

The estimate

$$\int_0^y f^*(z)^s\,dz\ge y f^*(y)^s,\quad y\in(0,\infty),$$

implies that

$$N(t)\ge\|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)f^*(y)\|_{q,(0,t^s)}. \tag{8.7}$$

On the other hand, by (8.6),

$$N(t)\approx\Big\|y^{-\theta-\frac{s}{q}}\ell^{\mathbb{A}s}(y)\int_0^y f^*(z)^s\,dz\Big\|^{\frac{1}{s}}_{\frac{q}{s},(0,t^s)}.$$

Consequently, putting $g=f^*\chi_{(0,t^s)}$ (note that $g\in\mathcal{M}^+(0,\infty;\downarrow)$) and applying Lemma 3.8(i) and Lemma 3.11 if $\frac{q}{s}\ge 1$ and $\frac{q}{s}\in(0,1)$, respectively, we obtain

$$\begin{aligned}N(t)&\le\Big\|y^{-\theta-\frac{s}{q}}\ell^{\mathbb{A}s}(y)\int_0^y g(z)^s\,dz\Big\|^{\frac{1}{s}}_{\frac{q}{s},(0,\infty)}\lesssim\|y^{-\theta+1/(\frac{q}{s})'}\ell^{\mathbb{A}s}(y)g(y)^s\|^{\frac{1}{s}}_{\frac{q}{s},(0,\infty)}\\&=\|y^{1-\theta-\frac{s}{q}}\ell^{\mathbb{A}s}(y)f^*(y)^s\|^{\frac{1}{s}}_{\frac{q}{s},(0,t^s)}=\|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)f^*(y)\|_{q,(0,t^s)}.\end{aligned}$$

This estimate and (8.7) yield the result. ■

Lemma 8.3 *Let* $\theta \in (0,1)$, $0 < q \le \infty$, $0 < s < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *Then for all* $t \in (0,\infty]$ *and every* $f \in L^{s,\infty} + L^\infty$,

$$\|\tau^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)K(f,\tau;L^{s,\infty},L^\infty)\|_{q,(0,t)} \approx \|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)f^*(y)\|_{q,(0,t^s)}. \tag{8.8}$$

Proof Put $N(t) = \|\tau^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(y)K(f,\tau;L^{s,\infty},L^\infty)\|_{q,(0,t)}$. On using (8.4) and the change of variables $\tau^s = y$, we get

$$\begin{aligned} N(t) &\approx \|\tau^{-\theta-\frac{1}{q}}\ell^{\mathbb{A}}(\tau)\sup_{0<z<\tau^s} z^{\frac{1}{s}}f^*(z)\|_{q,(0,t)} \\ &\approx \|y^{-\frac{\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)\sup_{0<z<y} z^{\frac{1}{s}}f^*(z)\|_{q,(0,t^s)}. \end{aligned} \tag{8.9}$$

The estimate

$$\sup_{0<z<y} z^{\frac{1}{s}}f^*(z) \ge y^{\frac{1}{s}}f^*(y), \quad y \in (0,\infty),$$

implies that

$$N(t) \gtrsim \|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)f^*(y)\|_{q,(0,t^s)}. \tag{8.10}$$

On the other hand,

$$\begin{aligned} \sup_{0<z<y} z^{\frac{1}{s}}f^*(z) &\approx \sup_{0<z<y} f^*(z)\int_0^z \sigma^{\frac{1}{s}-1}\,d\sigma \\ &\le \sup_{0<z<y}\int_0^z \sigma^{\frac{1}{s}-1}f^*(\sigma)\,d\sigma = \int_0^y \sigma^{\frac{1}{s}-1}f^*(\sigma)\,d\sigma, \end{aligned}$$

which, together with (8.9), yields

$$N(t) \lesssim \Big\|y^{-\frac{\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)\int_0^y \sigma^{\frac{1}{s}-1}f^*(\sigma)\,d\sigma\Big\|_{q,(0,t^s)}.$$

Consequently, putting $g = f^*\chi_{(0,t^s)}$ (note that $g \in \mathcal{M}^+(0,\infty;\downarrow)$) and applying Lemma 3.8(i) and Theorem 3.2, respectively, if $q \ge 1$ and $q \in (0,1)$, we obtain

$$\begin{aligned} N(t) &\le \Big\|y^{-\frac{\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)\int_0^y \sigma^{\frac{1}{s}-1}g(\sigma)\,d\sigma\Big\|_{q,(0,\infty)} \\ &\lesssim \|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)g(y)\|_{q,(0,\infty)} = \|y^{\frac{1-\theta}{s}-\frac{1}{q}}\ell^{\mathbb{A}}(y)f^*(y)\|_{q,(0,t^s)}. \end{aligned}$$

Together with (8.10), this gives the result. ■

Corollary 8.4 *Let* $0 < s < p < \infty$, $\theta = 1 - \frac{s}{p}$, $0 < q \le \infty$, *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *Then*

$$(L^s, L^\infty)_{\theta,q;\mathbb{A}} = L_{p,q;\mathbb{A}}, \tag{8.11}$$

$$(L^{s,\infty}, L^\infty)_{\theta,q;\mathbb{A}} = L_{p,q;\mathbb{A}}. \tag{8.12}$$

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In particular:

$$(L^s, L^\infty)_{\theta,p} = L^p \quad (\text{cf. [Ho, Lemma 4.1]}), \tag{8.13}$$

$$(L^{s,\infty}, L^\infty)_{\theta,p} = L^p \quad (\text{cf. [Ho, Lemma 4.2]}), \tag{8.14}$$

$$(L^1, L^\infty)_{\frac{1}{p'},q;\mathbb{A}} = L_{p,q;\mathbb{A}} \quad \text{if } 1 < p < \infty, \tag{8.15}$$

$$(L^{1,\infty}, L^\infty)_{\frac{1}{p'},p;\mathbb{A}} = L_{p,q;\mathbb{A}} \quad \text{if } 1 < p < \infty. \tag{8.16}$$

Proof Put $\overline{X}_{\theta,q;\mathbb{A}} = (X_0, X_1)_{\theta,q;\mathbb{A}}$, where $X_0 = L^s$ or $X_0 = L^{s,\infty}$, and $X_1 = L^\infty$. Then for all $f \in X_0 + X_1$, $\|f; \overline{X}_{\theta,q;\mathbb{A}}\| = \|\tau^{-\theta-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) K(f, \tau; X_0, X_1)\|_{q,(0,\infty)}$. Applying Lemma 8.2 or 8.3 (with $t = \infty$), respectively, and using the identity $(1-\theta)/s = 1/p$ and (8.1), we get

$$\|f; \overline{X}_{\theta,q;\mathbb{A}}\| \approx \|y^{\frac{1-\theta}{s}-\frac{1}{q}} \ell^{\mathbb{A}}(y) f^*(y)\|_{q,(0,\infty)} = \|f; L_{p,q;\mathbb{A}}\|$$

which proves (8.11) and (8.12). ■

Lemma 8.5 *Let* $0 < s < p < \infty$, $\theta = 1 - \frac{s}{p}$, $0 < q, r \le \infty$, *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Then*

$$(L^s, L^\infty)^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = (L^{s,\infty}, L^\infty)^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = L^{\mathcal{L}}_{p;r,\mathbb{B},q,\mathbb{A}} = L^{\mathcal{L}}, \tag{8.17}$$

where

$$L^{\mathcal{L}}_{p;r,\mathbb{B},q,\mathbb{A}} = \{f \in \mathcal{M}(\mathcal{R}, \mu); \|f; L^{\mathcal{L}}\| < \infty\} \tag{8.18}$$

and

$$\|f; L^{\mathcal{L}}\| = \|f; L^{\mathcal{L}}_{p;r,\mathbb{B},q,\mathbb{A}}\| = \|t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) \|\tau^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) f^*(\tau)\|_{q,(0,t)}\|_{r,(0,\infty)}. \tag{8.19}$$

Proof Put $\overline{X}^{\mathcal{L}} = (X_0, X_1)^{\mathcal{L}}_{\theta;r,\mathbb{B},q,\mathbb{A}}$, where $X_0 = L^s$ or $X_0 = L^{s,\infty}$, respectively, and $X_1 = L^\infty$. Then, by (2.2)

$$\|f; \overline{X}^{\mathcal{L}}\| = \|t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) N(t)\|_{r,(0,\infty)},$$

where

$$N(t) = \|\tau^{-\theta-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) K(f, \tau; X_0, X_1)\|_{q,(0,t)}.$$

Using Lemma 8.2 or 8.3, and the identity $(1-\theta)/s = 1/p$, we obtain

$$\|f; L^{\mathcal{L}}\| \approx \|t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) \|y^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(y) f^*(y)\|_{q,(0,t^s)}\|_{r,(0,\infty)}$$

and, on changing the variable to $x = t^s$,

$$\|f; \overline{X}^{\mathcal{L}}\| \approx \|x^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(x) \|y^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(y) f^*(y)\|_{q,(0,x)}\|_{r,(0,\infty)} = \|f; L^{\mathcal{L}}\|.$$ ■

Lemma 8.6 *Let* $1 < p < \infty$, $\theta = \frac{1}{p'}$, $0 < q, r \le \infty$, *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Then*

$$(8.20)\qquad (L^1, L^\infty)^{\mathcal{R}}_{\theta;r,\mathbb{B},q,\mathbb{A}} = L^{(\mathcal{R})}_{p;r,\mathbb{B},q,\mathbb{A}} = L^{(\mathcal{R})},$$

where

$$(8.21)\qquad L^{(\mathcal{R})}_{p;r,\mathbb{B},q,\mathbb{A}} = \{f \in \mathcal{M}(\mathcal{R}, \mu); \|f; L^{(\mathcal{R})}\| < \infty\}$$

and

$$(8.22)\qquad \|f; L^{(\mathcal{R})}\| = \|f; L^{(\mathcal{R})}_{p;r,\mathbb{B},q,\mathbb{A}}\| = \|t^{-\frac{1}{r}} \ell^{\mathbb{B}-\mathbb{A}}(t) \|\tau^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(\tau) f^{**}(\tau)\|_{q,(t,\infty)}\|_{r,(0,\infty)}.$$

The *proof* is trivial and is hence omitted. ■

In [EOP, Section 8] we have used the spaces

$$(8.23)\qquad \mathfrak{M}_{p,\mathbb{A}}(\Omega) := \big(L^1(\Omega), L^{p,1}(\Omega)\big)_{1,\infty;\mathbb{A}}, \quad 1 < p < \infty,$$

and

$$(8.24)\qquad S_{p,\mathbb{A}}(\Omega) := \big(L^{p,\infty}(\Omega), L^\infty(\Omega)\big)_{0,1;\mathbb{A}}, \quad 1 \le p < \infty,$$

with $\Omega \subseteq \mathbb{R}^n$ to describe mapping properties of certain (quasi-) linear operators. The following theorem shows that these spaces are particular cases of $L^{(\mathcal{R})}_{p;r,\mathbb{B},q,\mathbb{A}}$ and $L^{\mathcal{L}}_{p;r,\mathbb{B},q,\mathbb{A}}$ (with $(\mathcal{R}, \mu) = (\Omega, dx)$).

Theorem 8.7

(i) Let $1 < p < \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *Then*

$$\mathfrak{M}_{p,\mathbb{A}}(\Omega) = L^{(\mathcal{R})}_{p;\infty,\mathbb{A},1,(0,0)} = \big(L^1(\Omega), L^\infty(\Omega)\big)^{\mathcal{R}}_{\frac{1}{p'};\infty,\mathbb{A},1,(0,0)}.$$

(ii) Let $1 \le p < \infty$, $0 < s < p$, $\theta = 1 - \frac{s}{p}$, *and* $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$. *Then*

$$S_{p,\mathbb{A}} = L^{\mathcal{L}}_{p;1,\mathbb{A},\infty,(0,0)} = \big(L^{s,\infty}(\Omega), L^\infty(\Omega)\big)^{\mathcal{L}}_{\theta;1,\mathbb{A},\infty,(0,0)}.$$

Proof (i) Put $X_0 = L^1 = L^1(\Omega)$, $X_1 = L^\infty = L^\infty(\Omega)$. Since, by (8.15),

$$L^{p,1} = (L^1, L^\infty)_{\frac{1}{p'},1}, \quad 1 < p < \infty,$$

we have by (8.23), a symmetric analogue (dealing with the right end of the interpolation scale) of Theorem 5.7, and Lemma 8.6,

$$\begin{aligned}\mathfrak{M}_{p,\mathbb{A}}(\Omega) &= \big(X_0, (X_0, X_1)_{\frac{1}{p'},1}\big)_{1,\infty;\mathbb{A}} = \overline{X}^{\mathcal{R}}_{\frac{1}{p'};\infty,\mathbb{A},1,(0,0)}\\ &= \big(L^1(\Omega), L^\infty(\Omega)\big)^{\mathcal{R}}_{\frac{1}{p'};\infty,\mathbb{A},1,(0,0)} = L^{(\mathcal{R})}_{p;\infty,\mathbb{A},1,(0,0)}.\end{aligned}$$

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(ii) Put $Y_0 = L^{s,\infty} = L^{s,\infty}(\Omega)$, $Y_1 = L^\infty = L^\infty(\Omega)$. Since, by (8.12),

$$L^{p,\infty} = (L^{s,\infty}, L^\infty)_{\theta,\infty},$$

we have by (8.24), Theorem 5.7, and Lemma 8.5,

$$\begin{aligned} S_{p,\mathbb{A}}(\Omega) &= \big((Y_0, Y_1)_{\theta,\infty}, Y_1\big)_{0,1;\mathbb{A}} = \overline{Y}^{\mathcal{L}}_{\theta;1,\mathbb{A},\infty,(0,0)} \\ &= \big(L^{s,\infty}(\Omega), L^\infty(\Omega)\big)^{\mathcal{L}}_{\theta;1,\mathbb{A},\infty,(0,0)} = L^{\mathcal{L}}_{p;1,\mathbb{A},\infty,(0,0)}. \end{aligned}$$

■

Next, recall the result of [Me 1, Example 3]:
Let $0 < p_0, p_1 < \infty$, $p_0 \neq p_1$, $0 < q_0, q_1, q \leq \infty$, $\alpha_0, \alpha_1, \alpha \in \mathbb{R}$, $0 < \theta < 1$, and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1.$$

Then

$$(L_{p_0,q_0;(\alpha_0,\alpha_0)}, L_{p_1,q_1;(\alpha_1,\alpha_1)})_{\theta,q;(\alpha,\alpha)} = L_{p_\theta,q;(\alpha_\theta,\alpha_\theta)}.$$

In the next theorem we determine the corresponding space with $\theta = 0$.

Theorem 8.9 *Let $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \leq \infty$, $\mathbb{A}_i = (\alpha_{i0}, \alpha_{i\infty}) \in \mathbb{R}^2$, $i = 0, 1$. Assume that $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ is such that*

$$\alpha_\infty \leq 0 \leq \alpha_0 \quad \textit{if } q = q_0 = \infty$$

and

$$\alpha_\infty + \frac{1}{q} < 0 < \alpha_0 + \frac{1}{q} \quad \textit{if } \max\{q, q_0\} < \infty.$$

Then

$$(L_{p_0,q_0;\mathbb{A}_0}, L_{p_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = L^{\mathcal{L}}_{p_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0}. \tag{8.25}$$

In particular,

$$(L_{p_0,q_0;\mathbb{A}_0}, L_{p_1,q_1;\mathbb{A}_1})_{0,q_0;\mathbb{A}} = L_{p_0,q_0;\mathbb{A}+\mathbb{A}_0+\frac{1}{q_0}} \tag{8.26}$$

provided that

$$\alpha_\infty \leq 0 \leq \alpha_0 \quad \textit{and} \quad q_0 = \infty$$

and

$$\alpha_\infty + \frac{1}{q_0} < 0 < \alpha_0 + \frac{1}{q_0} \quad \textit{and} \quad q_0 < \infty.$$

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Proof Let $0 < s < \min\{p_0, p_1\}$, $\theta_i = 1 - \frac{s}{p_i}$, $i = 0, 1$, and $X_0 = L^s$, $X_1 = L^\infty$. Then by (8.11), Theorem 5.9, and Lemma 8.5,

$$\begin{aligned}(L_{p_0,q_0;\mathbb{A}_0}, L_{p_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} &= \big((L^s, L^\infty)_{\theta_0,q_0;\mathbb{A}_0}, (L^s, L^\infty)_{\theta_1,q_1;\mathbb{A}_1}\big)_{0,q;\mathbb{A}} \\ &= (\overline{X}_{\theta_0,q_0;\mathbb{A}_0}, \overline{X}_{\theta_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = \overline{X}^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \\ &= (L^s, L^\infty)^{\mathcal{L}}_{\theta_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0} \\ &= L^{\mathcal{L}}_{p_0;q,\mathbb{A}+\mathbb{A}_0,q_0,\mathbb{A}_0},\end{aligned}$$

which proves (8.25). Taking $q = q_0$ in (8.25) and applying Lemma 4.9, we obtain (8.26). ∎

Remarks 8.10 (i) By Theorem 5.9, the assertion of Theorem 8.9 remains true if the space $L_{p_1,q_1;\mathbb{A}_1}$ is replaced in (8.25) or (8.26) by L^∞.

(ii) Suppose that the assumption $0 < p_0 < p_1 < \infty$ of Theorem 8.9 is replaced by $1 < p_1 < p_0 < \infty$. Then

(8.27) $$(L_{p_0,q_0;\mathbb{A}_0}, L_{p_1,q_1;\mathbb{A}_1})_{0,q;\mathbb{A}} = (L_{p_1,q_1;\mathbb{A}_1}, L_{p_0,q_0;\mathbb{A}_0})_{1,q;\tilde{\mathbb{A}}} = L^{(\mathcal{R})}_{p_0;q,\tilde{\mathbb{A}}+\mathbb{A}_0,q_0,\mathbb{A}_0},$$

where $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$ (the last equality in (8.27) follows from Theorem 5.9* and Lemma 8.6).

(iii) Note that in the case when $\mu(\mathcal{R}) < \infty$ the result corresponding to (8.26) (with $\mathbb{A}_0 = (\alpha_0, \alpha_0)$, $\mathbb{A}_1 = (\alpha_1, \alpha_1)$, and $\mathbb{A} = (\alpha, \alpha)$) was mentioned without proof in [D].

(iv) Let $\mu(\mathcal{R}) < \infty$. Taking $\mathbb{A}_0 = (0, 0)$, $\mathbb{A}_1 = (0, 0)$ and $\mathbb{A} = (\alpha, \alpha)$ in (8.26), we get

$$L_{p_0,q_0;\alpha+\frac{1}{q_0}} = (L^{p_0,q_0}, L^{p_1,q_1})_{0,q_0;\alpha}$$

provided that

$$0 \le \alpha \quad \text{and} \quad q_0 = \infty$$

or

$$0 < \alpha + \frac{1}{q_0} \quad \text{and} \quad q_0 < \infty.$$

(Compare with extrapolation results in [ET, Section 2.6.2, Theorem 2] and [Mi, (2.26)].)

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