# ON THE LATTICE OF TOPOLOGIES

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In many cases Lattice Theory has proven itself to be useful in the study of the totality of mathematical systems of a given type. In this paper we shall continue one of such studies by investigating further the lattice of all topologies on a given set $S$. A considerable amount of research has been done in this field **(1; 2; 3; 5; 6)**. This research, besides satisfying the intrinsic interest in the lattice theoretic properties of this lattice, has aided the study of interconnections of different properties of point set topologies.

We shall show that the lattice of all topologies on a set consisting of more than two elements has only trivial homomorphisms. On the other hand it will be shown that this is not true for the lattice consisting of all $T_1$-topologies on $S$ and the lattice of complete homomorphisms will be constructed in this case. We shall also show that the lattice of all topologies is complemented if $S$ is finite. Finally we shall construct the group of automorphisms for the lattice of all topologies and for the lattice of all $T_1$-topologies on $S$. We shall conclude with a definition of a lattice theoretic property which clarifies the change of properties of the lattice of topologies as we go from the finite to the infinite case.

We shall represent a topology $R$ on the set $S$ by the collection of its closed sets, $R = \{S_\alpha\}$. If $R_1$ and $R_2$ are topologies on $S$ then $R_1 \leqslant R_2$ if and only if every set closed under $R_1$ is also closed under $R_2$. It can be seen that under this ordering the set of all topologies on $S$ forms a complete point lattice. The intersection of two topologies $R_1$ and $R_2$ in the lattice is the topology whose closed sets are the sets closed under $R_1$ and $R_2$. The union of two topologies $R_1$ and $R_2$ is the topology whose closed sets are intersections of finite unions of the closed sets of $R_1$ and $R_2$. Let us denote the lattice of all topologies on $S$ by $LT(S)$ and similarly let $LT_1(S)$ denote the lattice of all $T_1$-topologies on $S$.

We shall now investigate the homomorphisms of $LT(S)$.

LEMMA 1. *If $\theta$ is a nontrivial homomorphism on a point lattice $L$, then there exists a point $p$ of $L$ such that $p \equiv 0(\theta)$.*

*Proof.* Let $\theta$ be a non trivial homomorphism on $L$. Then there exist two elements $a$ and $b$ in $L$, $a > b$, such that $a \equiv b(\theta)$. Since $L$ is a point lattice there exists a point $p$ such that $a \cap p = p$ and $b \cap p = 0$. But then $p = a \cap p \equiv b \cap p = 0$.

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LEMMA 2. *If $\theta$ is a homomorphism on $LT(S)$ which identifies at least two distinct elements, then a topology of the form $\{\phi, p, S\}$, $p \in S$, is identified with the zero element.*

*Proof.* The lattice of topologies on the set $S$ is a point lattice. The points are topologies of the form $\{\phi, D, S\}$, $\phi \neq D \subset S$. Thus by Lemma 1 a topology of this form is congruent to the zero element, say $\{\phi, D, S\} \equiv \{\phi, S\}(\theta)$. If $D = \{p\}$, some $p$ in $S$, then the lemma holds. Otherwise let $p$ be in $D$. Then the quotients $\{\phi, (S - D) \vee p, D, p, S\} : \{\phi, (S - D) \vee p, S\}$ and $\{\phi, D, S\} : \{\phi, S\}$ are perspective. We also observe that the quotient $\{\phi, p, S\}$: $\{\phi, S\}$ is perspective into the first quotient. But then all these quotients are collapsed by the homomorphism $\theta$ since it collapses $\{\phi, D, S\} : \{\phi, S\}$.

THEOREM 1. *There are only trivial homomorphisms on the lattice of topologies on a set consisting of more than two elements.*

*Proof.* Let $\theta$ be a homomorphism which identifies two distinct elements of $LT(S)$. Then by Lemma 1 a topology of the form $\{\phi, D, S\}$ is congruent to the zero element. Assume that $D$ contains at least two distinct elements, say $p$ and $q$. Then by Lemma 2 we have that $\{\phi, p, S\} \equiv \{\phi, q, S\} \equiv \{\phi, S\}$. Let us denote by $R(-p)$ the topology on $S$ whose closed sets are the set $S$ and the subsets of $S$ which do not contain the element $p$. Let $R(p)$ denote the topology whose closed sets are the void set and all the subsets of $S$ which contain the element $p$. Then we observe that $R(-p) \cap R(p) = \{\phi, S\}$ and $R(-p) \cup \{\phi, p, S\} = I$. This implies that $R(-p) \equiv I(\theta)$. Thus $R(-p) \cap R(p) \equiv I \cap R(p) \equiv R(p) \equiv \{\phi, S\}$. Similarly, it follows that $R(q) \equiv \{\phi, S\}$. Since $R(p) \cup R(q) = I$ we conclude that $I \equiv \{\phi, S\}$ which shows that $\theta$ is a trivial homomorphism. In the case $D$ consists of a single element, $D = \{p\}$, we obtain similarly that $R(p) \equiv \{\phi, S\}$. From this it follows, if we recall that the set $S$ contains more than two elements, that there exists an element $q$ in $S$, $q \neq p$, so that $\{\phi, S\} < \{\phi, q \vee p, S\} < R(p)$. This implies that $\{\phi, q \vee p, S\} \equiv \{\phi, S\}$. Now we can complete the proof as in the previous case if we set $D = p \vee q$.

We shall now show that the result previously derived does not hold if we consider only $T_1$ − topologies on a set $S$.

The set of all $T_1$-topologies on $S$ forms a complete sublattice of the lattice of all topologies on $S$. We shall assume that $S$ is infinite since otherwise the lattice of $T_1$-topologies consists of a single element. It can be seen that $LT_1(S)$ is not a point lattice and that the join irreducible elements are topologies of the form $\{F_\alpha, D \vee F_\alpha, S\}$, where $\{F_\alpha\}$ is the set of finite subsets of $S$ and $D$ is a proper infinite subset of $S$. From this we conclude that if $J$ is a join irreducible element and $R$ is any element of $LT_1(S)$ then $J \cap R$ is a join irreducible element, provided that $J \cap R \neq 0$. We also observe that two join irreducible elements are comparable if and only if there exists a finite set $A$ such that they induce the same topology on $S - A$. From this it follows that the only join irreducible elements which contain a point of the lattice are topologies of the form $\{F_\alpha, (S - A) \vee F_\alpha\}$, where $A$ is a finite subset of $S$. Note that the points of this lattice are topologies of the form $\{F_\alpha, S - p, S\}$, $p \in S$.

THEOREM 2. *The lattice of the complete homomorphisms of $LT_1(S)$ is isomorphic to the lattice consisting of finite subsets of S and the set S ordered under set inclusion.*

*Proof.* First we shall show that to each finite subset $A$ of $S$ there corresponds a complete homomorphism $\theta(A)$. To see this let $R_1$ be congruent to $R_2$ mod $\theta(A)$ if and only if $R_1$ and $R_2$ induce the same topology on $S - A$. It can be seen that this does define a complete homomorphism on $LT_1(S)$ and that to two distinct subsets there correspond distinct homomorphisms.

To complete the proof we shall show that every complete homomorphism is of this type. First we shall show that if a complete homomorphism $\theta$ maps a join irreducible element $J = \{F_\alpha, D \vee F_\alpha, S\}$ which does not contain a point of the lattice into the zero element then the homomorphism is trivial. Let $D_1$ be a subset of $S$ such that $D - D_1$ and $D_1 - D$ are infinite. Then $J \cup \{F_\alpha, D_1 \vee F_\alpha, S\} \equiv \{F_\alpha, D_1 \vee F_\alpha, D \vee F_\alpha, (D_1 \wedge D) \vee F_\alpha, D_1 \vee D \vee F_\alpha, S\}$ from which it follows that $\{F_\alpha, (D_1 \vee D) \vee F_\alpha, S\} \equiv 0$ and $\{F_\alpha, (D_1 \wedge D) \vee F_\alpha, S\} \equiv 0$. Proceeding this way we can show that all the join irreducible elements are mapped into the zero element. Thus because of the completeness of the homomorphism $\theta$ we conclude that it identifies all the elements of $LT_1(S)$. Assume now that $\theta$ is a non-trivial complete homomorphism which identifies two topologies $R_1$ and $R_2$, $R_1 < R_2$. Then there exists a join irreducible element $J_1$ such that $R_2 \cap J_1 = J_1$ and $R_1 \cap J_1 = J_2 \neq J_1$. Then $J_2$ is a join irreducible element or the zero element. If we let $J_1 = \{F_\alpha, D \vee F_\alpha, S\}$ then $J_2 = \{F_\alpha, D \vee A \vee F_\alpha, S\}$, where $A$ is a finite subset of $S$ and $D \wedge A = \phi$. From this we shall show that $\theta$ identifies all topologies which agree on $S - A$. To see this let $C$ be an infinite subset of $D$ such that $D - C$ is also infinite. Then the quotients $\{F_\alpha, D \vee F_\alpha, S\} : \{F_\alpha, D \vee A \vee F_\alpha, S\}$ and $\{F_\alpha, C \vee F_\alpha, D \vee F_\alpha, S\} : \{F_\alpha, C \vee A \vee F_\alpha, D \vee A \vee F_\alpha, S\}$ are perspective, and the quotient $\{F_\alpha, C \vee F_\alpha, S\} : \{F_\alpha, C \vee A \vee F_\alpha, S\}$ is perspective into the second quotient. From this, since the first quotient is collapsed by $\theta$, we obtain that the last quotient is also collapsed. Repeating the same argument for $\{F_\alpha, C \vee F_\alpha, S\} : \{F_\alpha, C \vee A \vee F_\alpha, S\}$, $\{F_\alpha, C \vee F_\alpha, C_1 \vee F_\alpha, (S - A) \vee F_\alpha\} : \{F_\alpha, C \vee A \vee F_\alpha, C_1 \vee F_\alpha, S\}$ and $\{F_\alpha, (S - A) \vee F_\alpha\} : \{F_\alpha, S\}$, where $C_1 = S - (C \vee A)$, we obtain that $\{F_\alpha, (S - A) \vee F_\alpha\} \equiv 0$. But then if $M_1$ and $M_2$ are any two topologies which induce the same topology on $S - A$ we conclude that $M_1 \cup \{F_\alpha, (S - A) \vee F_\alpha\} = M_2 \cup \{F_\alpha, (S - A) \vee F_\alpha\}$ and thus $M_1 \equiv M_2$. It can now be easily seen that if for some other finite set $B$ we have that $\{F_\alpha, (S - B) \vee F_\alpha\} \equiv 0$ then all the topologies which agree on $S - (A \vee B)$ are identified by $\theta$. Because of the completeness of the homomorphism this holds for the union of any number of such subsets. But for the homomorphism to be non-trivial there must exist a finite set $V$ such that two topologies are congruent if and only if they agree on $S - V$. Note that otherwise there would exist a join irreducible element which does not contain a point of $LT_1(S)$ but which would be mapped into the zero element. But this would force the homomorphism to be trivial. This completes the proof by showing that there is a one-to-one order preserving correspondence between the non-trivial complete homomorphisms on $LT_1(S)$ and the proper finite subsets of $S$.

We shall now investigate the problem of complementation in the lattice of all topologies on $S$.

THEOREM 3. *$LT(S)$ is complemented if $S$ is finite.*

*Proof.* Assume that $S$ is finite. We shall call a closed non-void set $C$ of a topology $R$ minimal if no proper subset of $C$ is a closed set of $R$. To construct a complement $R'$ of a topology $R$ we pick a point from each minimal set and denote this collection of points by $A$. Let the union of all minimal sets be denoted by $U$. If $U = S$ then let $R'$ be the topology on $S$ which is generated by $A$ and the subsets of $S - A$. If $U \subset S$ then let $R'$ be the topology generated by $A \vee (S - U)$ and the subsets of $S - A$. It can be seen that in either case $R \cup R' = I$ and $R \cap R' = 0$.

COROLLARY 1. *Let $S$ be a finite set which consists of more than two elements. Then $R$ in $LT(S)$ has a unique complement if and only if $R = 0$ or $R = I$.*

*Proof.* From the proof of Theorem 3 we see that if in $R$, $R \neq 0, I$, a minimal closed set consists of more than one element then we have more than one way to construct the set $A$. Thus the complements are not unique. If all the minimal closed sets of $R$ consist of a single element than either $R = I$ or the union of the minimal sets $U \subset S$. In the second case we can construct one of the complements $R'$ as it was done in the previous proof. To construct a different complement $R''$ for $R$ we shall proceed as follows. Assume that $S - U$ contains at least two distinct elements. If $R$ contains a closed set $C$, $C \subset S$, such that $p$ in $C$, $q$ not in $C$, and $p$ and $q$ in $S - U$, then a complement $R''$ can be chosen to be the topology generated by all subsets of $S - (U \vee p)$ and $p \vee q$. If $R$ does not contain a closed set $C$ as described above then we can let $R''$ be the topology generated by all subsets of $S - U$ and $p \vee r$, where $p$ in $S - U$ and $r$ in $U$. If $S - U$ consists of a single element $p$ and there is a closed set $C$ in $R$ such that $\{p\} \subset C \subset S$ then let $R'' = \{\phi, (S - C) \vee p, S\}$. In the case if there is no such set $C$ in $R$ we let $R'' = \{\phi, p, q \vee p, S\}$. It is seen that in all cases $R'' \neq R'$ and $R'' \cup R = I$, $R'' \cap R = 0$. This completes the proof.

The corresponding questions about complements of $LT(S)$ and $LT_1(S)$ when $S$ is an infinite set are interesting problems and have not been answered in this paper.

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We shall now investigate the group of automorphisms of the lattice of topologies and the lattice of $T_1$-topologies on $S$.

We shall say that the points $p$ and $q$ of a point lattice $L$ form a union of type $n$ if $p \cup q$ contains $n$ distinct points.

THEOREM 4. *The group of automorphisms of $LT(S)$ is isomorphic to the symmetric group on $S$ if $S$ consists of one or two elements or is infinite; otherwise the group of automorphisms is isomorphic to the direct product of the symmetric group on $S$ with the two element group.*

*Proof.* Since the lattice of topologies on $S$ is a complete point lattice any automorphism is characterized by the permutation it induces on the set of points of $LT(S)$. If $S$ consists of one or two elements then it can be seen that the group of automorphisms of $LT(S)$ is the symmetric group on $S$. Let us now assume that $S$ consists of three or more elements. We shall denote the collection consisting of topologies of the form $\{\phi, p, S\}$, $p$ in $S$, by $\mathfrak{n}$, and the collection consisting of topologies of the form $\{\phi, S - p, S\}$, $p$ in $S$, by $\mathfrak{m}$. Then any element from $\mathfrak{n} \vee \mathfrak{m}$ forms a union of type at most three with any other point of $LT(S)$. On the other hand, for every point $P$ of $LT(S)$, $P$ not in $\mathfrak{n} \vee \mathfrak{m}$ there is a point $Q$ such that $Q \cup P$ is of type four. From this it follows that every automorphism maps the set $\mathfrak{n} \vee \mathfrak{m}$ onto itself. Furthermore, any two distinct elements from $\mathfrak{n}$ or $\mathfrak{m}$ form a union of type three, and an element from $\mathfrak{n}$ always forms a union of type two with an element from $\mathfrak{m}$. This implies that every automorphism has to map either $\mathfrak{n}$ onto $\mathfrak{n}$ and $\mathfrak{m}$ onto $\mathfrak{m}$, or $\mathfrak{n}$ onto $\mathfrak{m}$ and $\mathfrak{m}$ onto $\mathfrak{n}$. We shall now show that an automorphism which maps $\mathfrak{n}$ onto $\mathfrak{n}$ and $\mathfrak{m}$ onto $\mathfrak{m}$ corresponds to a permutation of the set $S$ and we know that to every permutation of $S$ there corresponds such an automorphism. First let us show that if $\{\phi, a, S\} \to \{\phi, b, S\}$ then $\{\phi, S - a, S\} \to \{\phi, S - b, S\}$. To see this observe that the third point $\{\phi, a \vee x, S\}$, $x \neq a$, $x$ in $S$, contained in the union of $\{\phi, a, S\}$ and $\{\phi, x, S\}$ forms unions of type three with $\{\phi, S - a, S\}$. Thus its image $\{\phi, a' \vee x', S\}$ has to form unions of type three with the image of $\{\phi, S - a, S\}$. Let this image be $\{\phi, S - a'', S\}$. But this can hold for all possible $x$ in $S$ only if $a' = a''$. Furthermore, if $\{\phi, D, S\}$ not in $\mathfrak{n} \vee \mathfrak{m}$ then $p$ in $D$ if and only if $\{\phi, p, S\} \cup \{\phi, D, S\}$ is a union of type two. Thus $p$ in $D$ if and only if $p'$ in $D'$, which shows that the mapping corresponds to a permutation on $S$. It can be seen that the lattice operations are preserved under this mapping and that to distinct automorphisms of this type there correspond distinct permutations of $S$ and vice versa. By a similar argument one can show that if there exists an automorphism which maps $\mathfrak{n}$ onto $\mathfrak{m}$ and $\mathfrak{m}$ onto $\mathfrak{n}$ then this automorphism corresponds to a permutation on $S$ followed by a complementation:

$$\{\phi, D, S\} \to \{\phi, D', S\} \to \{\phi, S - D', S\}.$$

Such a mapping preserves the lattice operations if $S$ is finite. Thus for every permutation on $S$ we have an automorphism which maps the permuted elements of their complements. Thus if $S$ is finite and contains three or more elements the group of automorphisms is isomorphic to the group which is the direct product of the symmetric group on $S$ and the two-element group. If $S$ is infinite then there can be no automorphism which maps $\mathfrak{n}$ onto $\mathfrak{m}$ and $\mathfrak{m}$. onto $\mathfrak{n}$ because $\bigcup \mathfrak{n} < \bigcup \mathfrak{m}$ and thus the lattice operations are not preserved. In this case the group of automorphisms on $LT(S)$ is isomorphic to the symmetric group on $S$.

THEOREM 5. *The group of automorphisms of $LT_1(S)$ is isomorphic to the symmetric group on $S$ if $S$ is infinite.*

*Proof.* We recall that the points of $LT_1(S)$ are topologies of the form $\{F_\alpha, S - p, S\}$. Any automorphism on $LT_1(S)$ has to map the set of points onto itself. Similarly every automorphism has to map the set $\mathfrak{L}$ of join irreducible elements of $LT_1(S)$, that is, the set consisting of topologies of the form $\{\phi, D \vee F_\alpha, S\}$, $D$ proper infinite subset of $S$, onto itself. But since any topology of $LT_1(S)$ can be written as a union of topologies of $\mathfrak{L}$ we see that every automorphism is defined by the permutation it induces on the set $\mathfrak{L}$. On the other hand, we note that $p$ is in $D$, $S - D$ infinite, if and only if $\{F_\alpha, D \vee F_\alpha, S\} \cup \{F_\alpha, S - p, S\}$ does not cover $\{F_\alpha, D \vee F_\alpha, S\}$. Similarly $p$ is in $D$, $S - D$ finite, if and only if $\{F_\alpha, S - p, S\}$ is not contained in $\{F_\alpha, D \vee F_\alpha\}$. Thus every automorphism is characterized by the permutation it induces on the set of points of $LT_1(S)$ and every permutation of the set of points defines an automorphism. Thus the group of automorphisms is isomorphic to the symmetric group on $S$.

We shall conclude by giving a definition of a lattice theoretic property which clarifies the difference between the lattice of topologies on a finite set and an infinite set, and which is in general useful in the study of point lattices. Let $L$ be a complete point lattice with the set of points $P = \{p, q, r, s, \ldots, \}$. If $A \subseteq P$ then let

$$\bar{A} = \wedge \{B | A \subseteq B \subseteq P; p, q \text{ in } B \text{ and } r \leqslant p \cup q \text{ implies } r \text{ in } B\}.$$

DEFINITION. Let $L$ be a complete point lattice with the set of points $P$. Then $L$ is said to be *tall* if for every $A \subseteq P$, $\bigcup A = a$, we have that $\bar{A} = \{p \in P | p \leqslant a\}$.

This means that if $L$ is a tall lattice then $L$ is completely determined if we know the unions of pairs of points in $L$ and it is the largest possible lattice which can be constructed with these given unions of pairs of points. Note that if we consider a lattice of subspaces of a geometry **(4)** then $\bar{A}$ is the smallest subspace which contains the set of points $A$. The following result can now be obtained.

THEOREM 6. *$LT(S)$ is a tall lattice if and only if $S$ is finite.*

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