Mathematical Notes.

A Review of Elementary Mathematics and Science.

PUBLISHED BY

THE EDINBURGH MATHEMATICAL SOCIETY

EDITED BY P. PINKERTON, M.A., D.Sc.

No. 11. **October 1912.**

A Puzzle of Dr Whewell's.—

69 NORTHUMBERLAND STREET,
EDINBURGH.

DEAR DR PINKERTON,

Is the following trifle worthy of a place in your *Mathematical Notes?* I came on the puzzle in a volume of the letters of Dr W. Whewell, the well-known Master of Trinity College, Cambridge. At the end of a letter to Professor A. De Morgan, dated 18th January 1859, he says:

"I can express every whole number from 1 to 15 (I think) by means of four nines. Thus $2 = \frac{9}{9} + \frac{9}{9}$. Is it worth while working this out further?"

A note to the letter states that:

"Two more examples from a fragment in Dr Whewell's handwriting will illustrate the meaning of this:

$$6 = \sqrt{9+9+9+9} \qquad 10 = \frac{99-9}{9}\text{"}$$

Here are my solutions of the puzzle for the numbers 0 to 40, excepting 38. Some ingenious young mathematician may be instigated to complement my solutions, and to carry the matter still further.

J. S. MACKAY

$$\begin{aligned} 0 &= 99 - 99 \\ &= (9+9) - (9+9) \\ &= (9-9) + (9-9) \\ &= (9-9) - (9-9) \\ &= 9 \times 9 - 9 \times 9 \end{aligned}$$

$$\begin{aligned} 0 &= \frac{9}{9} - \frac{9}{9} \\ &= 9 - \sqrt{9} - \sqrt{9} - \sqrt{9} \\ &= \sqrt{9} - \cdot\dot{9} - \cdot\dot{9} - \cdot\dot{9} \\ &= (\sqrt{9} + \sqrt{9}) - (\sqrt{9} + \sqrt{9}) \end{aligned}$$

$0 =$ and so on

$= (\cdot\dot{9} + \cdot\dot{9}) - (\cdot\dot{9} + \cdot\dot{9})$

$=$ and so on

$= (9 - 9) + (\sqrt{9} - \sqrt{9})$

$= (9 - 9) - (\sqrt{9} - \sqrt{9})$

$= 9\sqrt{9} - 9\sqrt{9}$

$= \dfrac{9}{\sqrt{9}} - \dfrac{9}{\sqrt{9}}$

$= (9 - 9) + (\cdot\dot{9} - \cdot\dot{9})$

$= (9 - 9) - (\cdot\dot{9} - \cdot\dot{9})$

$= 9 \times \cdot\dot{9} - 9 \times \cdot\dot{9}$

$= \dfrac{9}{\cdot\dot{9}} - \dfrac{9}{\cdot\dot{9}}$

$= (\sqrt{9} - \cdot\dot{9}) - (\sqrt{9} - \cdot\dot{9})$

$=$ and so on

$= \cdot\dot{9}9\dot{9} - \cdot\dot{9}$

$= \cdot\dot{9}\dot{9} - \cdot\dot{9}\dot{9}$

$= \cdot\dot{9} - \cdot\dot{9}9\dot{9}$

$= 9 - 9 \times \cdot\dot{9}\dot{9}$

$= 9 - \dfrac{9}{\cdot\dot{9}\dot{9}}$

$= \dfrac{9 \times \dot{9}}{\sqrt{9}} - \sqrt{9}$

$=$ and so on

$1 = \dfrac{99}{99}$

$= \dfrac{9 + 9 - 9}{9}$

$= \dfrac{9}{9} \times \dfrac{9}{9}$

$= \dfrac{9}{9} \div \dfrac{9}{9}$

$= \dfrac{\sqrt{9 \times 9}}{\sqrt{9 \times 9}}$

$= \dfrac{9}{\sqrt{9}} - \sqrt{9} + \cdot\dot{9}$

$= \sqrt{9} \times \sqrt{9} - 9 + \cdot\dot{9}$

$= \cdot\dot{9}99\dot{9}$

$1 = \cdot\dot{9}9\dot{9} \times \cdot\dot{9}$

$= \cdot\dot{9}9\dot{9} \div \cdot\dot{9}$

$= \cdot\dot{9} \times \cdot\dot{9}9\dot{9}$

$= \cdot\dot{9} \div \cdot\dot{9}9\dot{9}$

$= \cdot\dot{9}\dot{9} \times \cdot\dot{9}\dot{9}$

$= \cdot\dot{9}\dot{9} \div \cdot\dot{9}\dot{9}$

$2 = \dfrac{9}{9} + \dfrac{9}{9}$

$= \dfrac{99}{9} - 9$

$= 9 - 9 + \sqrt{9} - \cdot\dot{9}$

$= \dfrac{9 + \cdot\dot{9}}{\sqrt{9}} - \cdot\dot{9}$

$3 = \dfrac{9 + 9 + 9}{9}$

$= \dfrac{9 + 9}{9} + \cdot\dot{9}$

$= 9 - 9 + \cdot\dot{9} \times \sqrt{9}$

$= \dfrac{9}{9} + \sqrt{9} - \cdot\dot{9}$

$4 = \dfrac{9}{9} + \dfrac{9}{\sqrt{9}}$

$= \cdot\dot{9} + \cdot\dot{9} + \cdot\dot{9} + \cdot\dot{9}$

$= (\cdot\dot{9} + \cdot\dot{9})(\cdot\dot{9} + \cdot\dot{9})$

$= 9 - 9 + \sqrt{9} + \cdot\dot{9}$

$= 9 - \sqrt{9} - \cdot\dot{9} - \cdot\dot{9}$

$= \sqrt{9} + \cdot\dot{9}9\dot{9}$

$= \dfrac{9 \times \cdot\dot{9}}{\sqrt{9}} + \cdot\dot{9}$

$5 = 9 - \dfrac{9}{9} - \sqrt{9}$

$= \dfrac{9}{9} + \sqrt{9} + \cdot\dot{9}$

$= \dfrac{9}{\sqrt{9}} + \sqrt{9} - \cdot\dot{9}$

$= 9 \times \cdot\dot{9} - \sqrt{9} - \cdot\dot{9}$

https://doi.org/10.1017/S1757748900001006 Published online by Cambridge University Press

$$5=\frac{9}{\cdot\dot{9}}-\sqrt{9}-\cdot\dot{9}$$

$$\begin{aligned}
6&=\sqrt{9+9+9+9}\\
&=9-9+9-\sqrt{9}\\
&=9-\sqrt{9}+\cdot\dot{9}-\cdot\dot{9}\\
&=9-9+\sqrt{9}+\sqrt{9}\\
&=9-\cdot\dot{9}-\cdot\dot{9}-\cdot\dot{9}\\
&=\frac{9}{\sqrt{9}}+\frac{9}{\sqrt{9}}\\
&=\frac{9}{\cdot\dot{9}}-\frac{\sqrt{9}}{\cdot\dot{9}}\\
&=9-\frac{9\times\cdot\dot{9}}{\sqrt{9}}\\
&=\sqrt{9}+\frac{9\times\cdot\dot{9}}{\sqrt{9}}
\end{aligned}$$

$$\begin{aligned}
7&=9-\frac{9}{9}-\cdot\dot{9}\\
&=\frac{9}{\sqrt{9}}+\sqrt{9}+\cdot\dot{9}\\
&=9\times\cdot\dot{9}-\sqrt{9}+\cdot\dot{9}\\
&=\frac{9}{\cdot\dot{9}}-\sqrt{9}+\cdot\dot{9}\\
&=\cdot\dot{9}\times\cdot\dot{9}+9-\sqrt{9}\\
&=\frac{\cdot\dot{9}}{\cdot\dot{9}}+9-\sqrt{9}
\end{aligned}$$

$$\begin{aligned}
8&=\frac{9\times 9-9}{9}\\
&=\frac{99}{9}-\sqrt{9}\\
&=9-9+9-\cdot\dot{9}\\
&=9-\sqrt{9}+\cdot\dot{9}+\cdot\dot{9}
\end{aligned}$$

$$\begin{aligned}
9&=\frac{9+9+9}{\sqrt{9}}\\
&=\frac{9\sqrt{9\times 9}}{9}\\
&=9\cdot\dot{9}-\frac{9}{9}
\end{aligned}$$

$$\begin{aligned}
9&=9\cdot\dot{9}-\cdot\dot{9}\dot{9}\\
&=9+\frac{\sqrt{9}}{\sqrt{9}}-\cdot\dot{9}
\end{aligned}$$

$$\begin{aligned}
10&=\frac{9\times 9+9}{9}\\
&=\frac{99-9}{9}\\
&=\sqrt{9\times 9}+\frac{9}{9}\\
&=\sqrt{9\times 9}+\cdot\dot{9}\dot{9}\\
&=9-9+9+\cdot\dot{9}\\
&=9+\sqrt{9}-\cdot\dot{9}-\cdot\dot{9}
\end{aligned}$$

$$\begin{aligned}
11&=\frac{9+9}{9}+9\\
&=9+\sqrt{9}-\frac{9}{9}\\
&=9+\sqrt{9}-\cdot\dot{9}\dot{9}\\
&=\sqrt{9\times 9}+\cdot\dot{9}+\cdot\dot{9}\\
&=9\times\cdot\dot{9}+\sqrt{9}-\cdot\dot{9}\\
&=\frac{9}{\cdot\dot{9}}+\sqrt{9}-\cdot\dot{9}\\
&=9+\frac{\sqrt{9}}{\sqrt{9}}+\cdot\dot{9}\\
&=9+\cdot\dot{9}\times\cdot\dot{9}+\cdot\dot{9}\\
&=9+\frac{\cdot\dot{9}}{\cdot\dot{9}}+\cdot\dot{9}
\end{aligned}$$

$$\begin{aligned}
12&=\frac{99+9}{9}\\
&=9+9-9+\sqrt{9}\\
&=9+\sqrt{9}+\cdot\dot{9}-\cdot\dot{9}\\
&=\frac{9\times\cdot\dot{9}}{\sqrt{9}}+9\\
&=9\times\cdot\dot{9}+\sqrt{9}\times\cdot\dot{9}\\
&=\sqrt{9}+\sqrt{9}+\sqrt{9}+\sqrt{9}
\end{aligned}$$

$$13=9+\frac{9}{9}+\sqrt{9}$$

https://doi.org/10.1017/S1757748900001006 Published online by Cambridge University Press

$$13 = 9 \times \cdot\dot{9} + \sqrt{9} + \cdot\dot{9}$$
$$= \frac{9}{\cdot\dot{9}} + \sqrt{9} + \cdot\dot{9}$$
$$= 9 + \sqrt{9} + \cdot\dot{9} \times \cdot\dot{9}$$
$$= 9 + \sqrt{9} + \frac{\cdot\dot{9}}{\cdot\dot{9}}$$
$$= 9 + \sqrt{9} + \cdot\dot{9}\dot{9}$$
$$14 = 9 + \sqrt{9} + \cdot\dot{9} + \cdot\dot{9}$$
$$= 9 + 9 - \sqrt{9} - \cdot\dot{9}$$
$$15 = 9 + 9 - \frac{9}{\sqrt{9}}$$
$$= 9 + \frac{9+9}{\sqrt{9}}$$
$$16 = 9 + 9 - \sqrt{9} + \cdot\dot{9}$$
$$= 9 + 9 - \cdot\dot{9} - \cdot\dot{9}$$
$$= 9\cdot\dot{9} + \sqrt{9} + \sqrt{9}$$
$$= (9 - \cdot\dot{9})(\sqrt{9} - \cdot\dot{9})$$
$$17 = 9 + 9 - \frac{9}{9}$$
$$= 9 + 9 - \cdot\dot{9}\dot{9}$$
$$= 9\sqrt{9} - 9 - \cdot\dot{9}$$
$$= 9 + \sqrt{9} \times \sqrt{9} + \cdot\dot{9}$$
$$18 = 9 + 9 + 9 - 9$$
$$= 9 + \sqrt{9} + \sqrt{9} + \sqrt{9}$$
$$= \frac{(9 - \sqrt{9})9}{\sqrt{9}}$$
$$= 9\sqrt{9} \times \cdot\dot{9} - 9$$
$$= \frac{9\sqrt{9}}{\cdot\dot{9}} - 9$$
$$19 = 9 + 9 + \frac{9}{9}$$
$$= 9\sqrt{9} - 9 + \cdot\dot{9}$$
$$= 9 + \sqrt{9} \times \sqrt{9} + \cdot\dot{9}$$
$$20 = 9 + \frac{99}{9}$$
$$20 = 9 + 9 + \sqrt{9} - \cdot\dot{9}$$
$$= (9 + \cdot\dot{9})(\sqrt{9} - \cdot\dot{9})$$
$$21 = 9 + 9 + \frac{9}{\sqrt{9}}$$
$$= 9\sqrt{9} - 9 + \sqrt{9}$$
$$= 9 + 9 + \cdot\dot{9} \times \sqrt{9}$$
$$= 9 + 9 + \frac{\sqrt{9}}{\cdot\dot{9}}$$
$$22 = 9 + 9 + \sqrt{9} + \cdot\dot{9}$$
$$23 = 9\sqrt{9} - \sqrt{9} - \cdot\dot{9}$$
$$24 = 9 + 9 + 9 - \sqrt{9}$$
$$= 9 + 9 + \sqrt{9} + \sqrt{9}$$
$$= 9\sqrt{9} - \frac{9}{\sqrt{9}}$$
$$= \frac{99}{\sqrt{9}} - 9$$
$$25 = 9\sqrt{9} - \cdot\dot{9} - \cdot\dot{9}$$
$$= 9\sqrt{9} - \sqrt{9} + \cdot\dot{9}$$
$$26 = 9 + 9 + 9 - \cdot\dot{9}$$
$$= 9\sqrt{9} - \frac{9}{9}$$
$$= 9\sqrt{9} - \cdot\dot{9}\dot{9}$$
$$27 = 9 + 9 + \sqrt{9 \times 9}$$
$$28 = 9 + 9 + 9 + \cdot\dot{9}$$
$$= 9\sqrt{9} + \frac{9}{9}$$
$$= 9\sqrt{9} + \cdot\dot{9}\dot{9}$$
$$29 = 9\sqrt{9} + \cdot\dot{9} + \cdot\dot{9}$$
$$= 9\sqrt{9} + \sqrt{9} - \cdot\dot{9}$$
$$30 = 9 + 9 + 9 + \sqrt{9}$$
$$= \frac{99 - 9}{\sqrt{9}}$$
$$31 = 9\sqrt{9} + \sqrt{9} + \cdot\dot{9}$$
$$32 = \frac{99}{\sqrt{9}} - \cdot\dot{9}$$

$$33 = (9 + \sqrt{9})\sqrt{9} - \sqrt{9}$$
$$= \frac{99}{\sqrt{9}} \times \cdot\dot{9}$$
$$= \frac{99}{\sqrt{9}} \div \cdot\dot{9}$$
$$34 = \frac{99}{\sqrt{9}} + \cdot\dot{9}$$
$$35 = 9\sqrt{9} + 9 - \cdot\dot{9}$$
$$36 = 9 + 9 + 9 + 9$$
$$= \frac{99 + 9}{\sqrt{9}}$$

$$36 = 9\sqrt{9} + \sqrt{9 \times 9}$$
$$= 9\sqrt{9} + 9 \times \cdot\dot{9}$$
$$= 9\sqrt{9} + \frac{9}{\cdot\dot{9}}$$
$$= 9(\sqrt{9} + \cdot\dot{9}\dot{9})$$
$$37 = 9\sqrt{9} + 9 + \cdot\dot{9}$$
$$38 = 9\sqrt{9} + 9\cdot\dot{9} + \cdot\dot{9}$$

(Intractable)

$$39 = (9 + \sqrt{9})\sqrt{9} + \sqrt{9}$$
$$40 = (9 + \cdot\dot{9})(\sqrt{9} + \cdot\dot{9})$$

On the Solubility of Linear Algebraic Equations.—(*a*) *It is proved in treatises on Algebra that the equations (in* three variables for brevity)

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \\ a_3x + b_3y + c_3z + d_3 = 0, \end{aligned}\right\} \qquad \text{(1)}$$

have a unique solution given by

$$x \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} d_1, & b_1, & c_1 \\ d_2, & b_2, & c_2 \\ d_3, & b_3, & c_3 \end{vmatrix} = 0; \text{ etc.,} \qquad \text{(2)}$$

provided the determinant

$$\Delta \equiv \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \qquad \text{(3)}$$

does not vanish.

(*b*) It is also proved, from (2), that if the "degenerate" homogeneous system

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0, \end{aligned}\right\} \qquad \text{(4)}$$

has a *non-null* solution (*i.e.* a solution in which the variables are not all zero), then $\Delta = 0$.