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Abstract

We analyze the relationship between two compactifications of the moduli space of maps from curves to a Grassmannian: the Kontsevich moduli space of stable maps and the Marian–Oprea–Pandharipande moduli space of stable quotients. We construct a moduli space which dominates both the moduli space of stable maps to a Grassmannian and the moduli space of stable quotients, and equip our moduli space with a virtual fundamental class. We relate the virtual fundamental classes of all three moduli spaces using the virtual push-forward formula. This gives a new proof of a theorem of Marian–Oprea–Pandharipande: that enumerative invariants defined as intersection numbers in the stable quotient moduli space coincide with Gromov–Witten invariants.

Contents

1. Introduction

The Kontsevich moduli space of stable maps to Grassmannians and the moduli space of stable quotients of Marian–Oprea–Pandharipande are two compactifications of spaces of curves on Grassmannians. These moduli spaces come equipped with virtual classes in the sense of [BF97, LT96]. The purpose of this paper is to understand the relation between the two virtual fundamental classes and thus provide a new proof of a theorem in [MOP11]: that enumerative invariants defined as virtual intersection numbers in the two moduli spaces coincide. We do this by constructing a new moduli space of *map-quotients* which dominates both the moduli space of stable maps and the moduli space of stable quotients. We endow this space with a virtual class and determine its relation to the virtual classes of the moduli of stable maps and stable quotients using the virtual push-forward theorem [Man12b].

In the following we briefly review the main definitions and we outline the main constructions.

1.1 Stable maps and stable quotients

Stable maps to Grassmannians. Let $\mathbb{G}(k,r)$ be the Grassmannian of k-planes in the r-dimensional affine space. Let $(C, p_1, \ldots, p_n)$ be a nodal curve of genus g with n distinct markings

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which are different from the nodes. By the universal property of Grassmannians giving a degree d map from C to $\mathbb{G}(k,r)$ is equivalent to giving an exact sequence

$$0 \to S \to \mathcal{O}^{\oplus r} \to Q \to 0$$

where S is a rank k vector bundle of degree $-d$ and Q is a vector bundle. A map is called stable if its degree is positive on each unstable contracted component. It has been shown in [Tod11] that this is equivalent to

$$\omega_C(p_1 + \cdots + p_n) \otimes (\wedge^k S^\vee)^\epsilon$$

being ample on C for every $\epsilon \in \mathbb{Q}$ such that $\epsilon > 2$. The moduli space of degree d stable genus g maps with n marked points to $\mathbb{G}(k,r)$ will be denoted by $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$.

Stable quotients. Let $(\hat{C}, p_1, \ldots, p_n)$ be a nodal curve of genus g with n distinct markings which are different from the nodes. A quotient on $\hat{C}$

$$0 \to \hat{S} \to \mathcal{O}_{\hat{C}}^{\oplus r} \xrightarrow{q} \hat{Q} \to 0$$

is called *quasi-stable* if $\hat{Q}$ is locally free at nodes and markings. Let k be the rank of $\hat{S}$. A quotient $(\hat{C}, p_1, \ldots, p_n, q)$ is called stable if

$$\omega_{\hat{C}}(p_1 + \cdots + p_n) \otimes (\wedge^k \hat{S}^\vee)^\epsilon$$

is ample on $\hat{C}$ for every strictly positive $\epsilon \in \mathbb{Q}$. The moduli space of degree d stable genus g quotients with n marked points will be denoted by $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$.

The space $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ is another compactification of the space of genus g curves with n markings in the Grassmannian $\mathbb{G}(k,r)$.

Morphisms between moduli spaces of stable maps and moduli spaces of stable quotients. Marian–Oprea–Pandharipande showed in [MOP11] that if $k = 1$ then there exists a morphism of stacks

$$c : \overline{M}_{g,n}(\mathbb{G}(k,r),d) \to \overline{Q}_{g,n}(\mathbb{G}(k,r),d).$$

On points, c is obtained in the following way. Let C_i^0 be the rational tails without marked points of a stable map $(C, p_1, \ldots, p_n, f : C \to \mathbb{G}(k,r))$.

Let us suppose that the degree of f restricted to C_i^0 is d_i. Let $\hat{C}$ be the closure of $C \backslash C_i^0$ in C and let x_i be the intersection points of C_i with $\hat{C}$. Let $\hat{S} = S|_{\hat{C}}(-\sum d_i x_i)$. Then c associates to the stable map $0 \to S \to \mathcal{O}^{\oplus r} \to Q \to 0$ a stable quotient

$$0 \to \hat{S} \to \mathcal{O}_{\hat{C}}^{\oplus r} \xrightarrow{q} \hat{Q} \to 0.$$

We show that for $k > 1$ there is no such morphism; see Example 3.7.

1.2 Stable map-quotients

The purpose of the paper is to define a proper Deligne–Mumford (DM) stack $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ with the following properties:

(i) $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ admits natural morphisms

$$\begin{array}{ccc} & \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) & \\ {}^{c_1}\swarrow & & \searrow^{c_2} \\ \overline{M}_{g,n}(\mathbb{G}(k,r),d) & & \overline{Q}_{g,n}(\mathbb{G}(k,r),d) \end{array}$$

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(ii) $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ admits a dual relative obstruction theory $E^\bullet_{\overline{MQ}}$ (relative to some pure dimensional stack) which comes equipped with morphisms

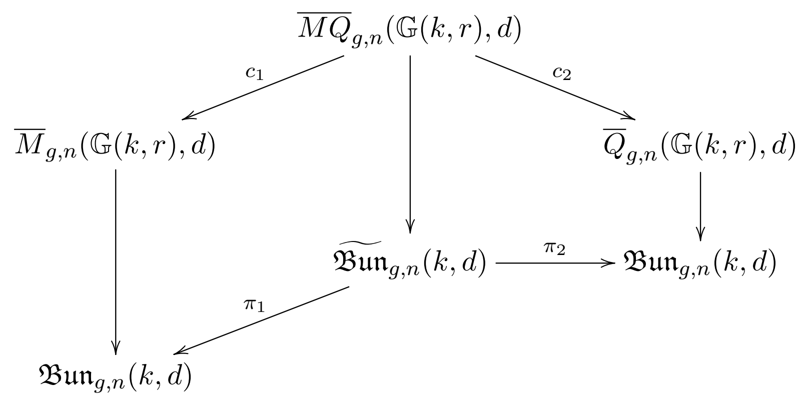
$$E^\bullet_{\overline{MQ}} \to c_1^* E^\bullet_{\overline{M}}, \tag{1}$$

$$E^\bullet_{\overline{MQ}} \to c_2^* E^\bullet_{\overline{Q}}. \tag{2}$$

Having constructed such a stack allows us to relate virtual classes by means of the virtual push-forward property [Man12b]. The idea behind the construction is explained in Remark 3.8. In the following we outline the structure of this paper.

In § 2 we review moduli spaces $\mathfrak{Bun}_{g,n}(k,d)$ of rank k, degree d vector bundles on genus g nodal curves with n marked points and introduce an auxiliary moduli space $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ which comes equipped with birational morphisms $\pi_1 : \widetilde{\mathfrak{Bun}}_{g,n}(k,d) \to \mathfrak{Bun}_{g,n}(k,d)$ and $\pi_2 : \widetilde{\mathfrak{Bun}}_{g,n}(k,d) \to \mathfrak{Bun}_{g,n}(k,d)$. In particular $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ has pure dimension.

In § 3 we construct a proper DM stack $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ which is a substack of $\overline{M}_{g,n}(\mathbb{G}(k,r), d) \times_{\mathfrak{M}^{\mathrm{rtf}}_{g,n}} \overline{Q}_{g,n}(\mathbb{G}(k,r),d)$, where $\mathfrak{M}^{\mathrm{rtf}}_{g,n}$ is the stack of prestable curves which do not have rational tails. This stack fits into the following commutative diagram.



Moreover, the rectangle on the left is cartesian and thus it gives rise to a perfect obstruction theory of $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ relative to $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$. This gives rise to a virtual class on $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$. In § 4 we use the virtual push-forward theorem to show that c_1 and c_2 satisfy the virtual push-forward property; see Theorem 4.3, which gives a new proof of the Marian–Oprea–Pandharipande theorem mentioned above.

One of the main technical difficulties is to make the constructions of $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ and $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ functorial. We will do slightly less, namely we will construct functorial spaces $\mathfrak{P}$ and $\bar{P}$ which contain $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ and $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ respectively.

Relation to other works. In the past years there have been many birational models of moduli spaces of stable maps constructed for particular targets. These include: moduli spaces of weighted stable maps introduced by Bayer and Manin [BM09], the moduli spaces defined by Mustaţă–Mustaţă [MAM07], moduli spaces of stable quotients of Marian, Oprea and Pandharipande [MOP11] with a more general version introduced by Toda [Tod11], moduli of stable toric quasi-maps [CK10] and finally, moduli spaces of stable quasi-maps to GITquotients [CKM14] introduced by Ciocan-Fontanine, Kim and Maulik which generalize [MOP11, Tod11] and [CK10]. These spaces are particularly interesting because they represent some natural functors and because they lead to invariants which are closely related to Gromov–Witten invariants. The stable quasi-map invariants (or variants of them) are also easier to compute in some cases (see [Giv98]).

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It is therefore interesting to compare quasi-map invariants to Gromov–Witten invariants. This has already been done for stable quotients in [MOP11] and [Tod11] by localization. Our approach is completely different and the main hope is that it will shed light on similar questions. More precisely, we first construct a (rather unnatural) auxiliary moduli space with a virtual class and then relate this virtual class to the virtual classes of the original spaces. We emphasize that this can be done with very little information on the auxiliary moduli space.

As regarding the birational geometry of moduli spaces of stable maps to GIT quotients little is known: divisors on moduli spaces of stable maps were studied mostly in genus zero and for homogeneous spaces (see, for example, [CHS09, CHS08, CS06]). The special feature in our case is that there is a morphism from an open dense set of the moduli space of maps to Grassmannians to the moduli space of stable quotients. This morphism does not extend in general. The proofs of these facts are essentially the same as the ones in [PR03].

As a final remark, one could hope that the techniques in this paper easily extend to the case of quasi-maps to GIT quotients. This is unfortunately not true as in general there is no map from an open dense set of the moduli space of stable maps to the moduli space of quasi-maps. This fact introduces extra challenges at the level of comparisons of virtual classes and it will be treated elsewhere.

Notation and conventions. We take the ground field to be $\mathbb{C}$.

Unless otherwise specified we will try to respect the following convention: we will denote curves over $\mathbb{C}$ by C, $\hat{C}$, etc., families of curves over some base scheme B by $\mathcal{C}$, $\hat{\mathcal{C}}$, etc. (with calligraphic fonts). We will apply the same convention to vector bundles: vector bundles over curves over $\mathbb{C}$ will be denoted by S, $\hat{S}$, etc. and families of vector bundles on curves over B will be denoted $\mathcal{S}$, $\hat{\mathcal{S}}$, etc. We will use normal fonts for DM stacks (e.g. $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$, $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$, $\bar{P}$, etc.). Artin stacks for which we know that they are not DM stacks will be generally denoted by gothic letters (e.g. $\mathfrak{M}_{g,n}$, $\mathfrak{Bun}_{g,n}(k,d)$, $\mathfrak{P}$, etc.).

By a commutative diagram of stacks we mean a 2-commutative diagram of stacks and by a cartesian diagram of stacks we mean a 2-cartesian diagram of stacks.

2. Moduli of bundles

2.1 Moduli of bundles over nodal curves

We review a few results concerning the existence and properties of stacks of vector bundles over prestable curves.

Moduli of prestable curves. Let us first fix notation. Let $g \geqslant 0$ and $n \geqslant 0$ be integers. We denote by $\mathfrak{M}_{g,n}$ the Artin stack of prestable curves of genus g with n marked points. We denote by $\mathfrak{C}_{g,n}$ the universal curve.

Definition 2.1. Let C be a nodal curve with marked points. A connected rational component (not necessarily irreducible) C^0 with no marked points such that C^0 intersects the rest of the curve in exactly one point is called a rational tail.

Definition 2.2. In the following we denote by $\mathfrak{M}^{\mathrm{rt}}_{g,n}$ the divisor of $\mathfrak{M}_{g,n}$ whose points are curves which have rational tails and by $\mathfrak{M}^{\mathrm{rtf}}_{g,n}$ the open substack of $\mathfrak{M}_{g,n}$, whose points are rational tail free curves.

Proposition 2.3. *Let $\mathfrak{S} \subset \mathfrak{M}_{g,n}$ be a substack of finite type of $\mathfrak{M}_{g,n}$. Then, there exists a morphism of stacks $p : \mathfrak{S} \to \mathfrak{M}^{\mathrm{rtf}}_{g,n}$ which contracts rational tails.*

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Proof. Let $\mathfrak{S}^{\mathrm{rtf}}$ be the open substack of $\mathfrak{S}$, whose points are rational tail free curves. Let $\mathfrak{S}'$ be the set of pairs $(C,\hat{C}) \in \mathfrak{S}\times\mathfrak{S}^{\mathrm{rtf}}$ such that there exists $p : C \to \hat{C}$ which contracts rational tails of C and is an isomorphism on the complement of such curves. In order to show that $\mathfrak{S}'$ is a substack of $\mathfrak{S}\times\mathfrak{S}^{\mathrm{rtf}}$ we show that we are under the hypothesis of Example 4.19 in [FGIKNV05]. It is clear that any cartesian diagram arrow whose target is in $\mathfrak{S}'$ is also in $\mathfrak{S}'$. Now let B_i be a covering of B, let $(\mathcal{C},\hat{\mathcal{C}})$ be an object in $\mathfrak{S}\times\mathfrak{S}^{\mathrm{rtf}}(B)$ and $(\mathcal{C}_i,\hat{\mathcal{C}}_i)$ the pull-backs of $(\mathcal{C},\hat{\mathcal{C}})$ to B_i. Suppose that $(\mathcal{C}_i,\hat{\mathcal{C}}_i)$ are objects in $\mathfrak{S}'(B_i)$. Then we need to show that $(\mathcal{C},\hat{\mathcal{C}})$ is an object in $\mathfrak{S}'(B)$. By our assumption we have morphisms $p_i : \mathcal{C}_i \to \hat{\mathcal{C}}_i$. As morphisms of schemes form a sheaf, we can glue the morphisms p_i to get a global $p : \mathcal{C} \to \hat{\mathcal{C}}$.

The algebraic stack $\mathfrak{S}'$ has a projection $\pi_1 : \mathfrak{S}' \to \mathfrak{S}$. We next prove that

$$\mathfrak{S}' \simeq \mathfrak{S}.$$

We claim that $\pi_1 : \mathfrak{S}' \to \mathfrak{S}$ is separated. As π_1 is also one to one we have that π_1 is an isomorphism. Let us sketch the proof of the claim. Let Δ be a nonsingular curve with a closed point $0 \in \Delta$ and let $\Delta^* = \Delta\backslash\{0\}$. It is enough to show that given a family of curves $\pi : \mathcal{C} \to \Delta$ and a projection $p : \mathcal{C} \to \hat{\mathcal{C}}$ over Δ^*, p extends uniquely over Δ. It is clear that any irreducible component of $\mathcal{C}$ whose fibers are rational tails can be contracted so we may assume that $\mathcal{C}$ is smooth. Let us add sections $p_{n+1},\dots,p_s$ of π such that each section intersects one unstable component in the fibers of π and no two sections intersect the same irreducible unstable component. Now take $\mathcal{C}' = \operatorname{Proj}\omega_\pi(\sum_{i=1}^s p_i)$. Replacing $\mathcal{C}$ with $\mathcal{C}'$ and repeating the process until there are no more chains of unstable rational tails in the fibers of π we obtain a contraction map $p : \mathcal{C} \to \hat{\mathcal{C}}$. The morphism p can be easily seen to be unique. □

Moduli of bundles over prestable curves. For fixed $g, n, d \geqslant 0$ let $\mathfrak{Bun}_{g,n}(k,d)(B)$ be the category whose objects are pairs $(\mathcal{C},\mathcal{S})$, where

(i) $\mathcal{C} \to B$ is a family of prestable curves of genus g with n sections
(ii) $\mathcal{S}$ is a vector bundle on $\mathcal{C}$ of rank k and degree $-d$.

Isomorphisms: An isomorphism

$$(\phi,\theta) : (\mathcal{C},\mathcal{S}) \to (\mathcal{C}',\mathcal{S}')$$

is an automorphism of curves

$$\phi : \mathcal{C} \to \mathcal{C}'$$

together with isomorphisms $\theta : \mathcal{S}^\vee \to \phi^*\mathcal{S}'^\vee$ such that $\phi(p_i) = p_i'$, for all i.

Let $Coh_{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ be the stack of coherent sheaves on $\mathfrak{C}_{g,n}$ relative to $\mathfrak{M}_{g,n}$. By [Lie06] we have that $Coh_{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ is an Artin stack. It can be easily seen that $\mathfrak{Bun}_{g,n}(k,d)$ is a substack of $Coh_{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ (see [CKM14] for more details and generalizations). Let $\mathcal{S}$ denote the universal bundle on the universal curve on $\mathfrak{Bun}_{g,n}(k,d)$. We will also consider moduli spaces of vector bundles on curves with stability conditions as follows.

Construction 2.4. Let $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ be the substack of $\mathfrak{Bun}_{g,n}(k,d)$ such that the line bundle

$$(\wedge^k S^\vee)^{\otimes\epsilon} \otimes \omega_C\left(\sum p_i\right) \tag{3}$$

is ample. As ampleness is an open condition $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ is an open substack of $\mathfrak{Bun}_{g,n}(k,d)$.

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Remark 2.5. Let

$$\psi : \mathfrak{Bun}_{g,n}(k,d) \to \mathfrak{M}_{g,n}$$

be the morphism which forgets the bundle. The morphism ψ is smooth as the relative obstruction in a point (C, S) is

$$Ext^2_C(S,S) = 0.$$

This shows that $\mathfrak{Bun}_{g,n}(k,d)$ is smooth of pure dimension $Ext^1(S,S) - Ext^0(S,S) + 3g - 3 + n = k^2(g-1) - \deg(S \otimes S^\vee) + 3g - 3 + n = k^2(g-1) + 3g - 3 + n$.

CONSTRUCTION 2.6. Consider the stack $\mathfrak{Bun}_{g,n}(k,d)^{\mathrm{rt}}$ defined by the following cartesian diagram.

$$\begin{array}{ccc} \mathfrak{Bun}_{g,n}(k,d)^{\mathrm{rt}} & \longrightarrow & \mathfrak{Bun}_{g,n}(k,d) \\ \downarrow & & \downarrow \psi \\ \mathfrak{M}^{\mathrm{rt}}_{g,n} & \xrightarrow{\ i\ } & \mathfrak{M}_{g,n} \end{array}$$

Let $\mathfrak{Bun}_{g,n}(k,d)^{\mathrm{rtf}}$ be the complement of $\mathfrak{Bun}_{g,n}(k,d)^{\mathrm{rt}}$ in $\mathfrak{Bun}_{g,n}(k,d)$.

Remark 2.7. We have that $\mathfrak{M}^{\mathrm{rt}}_{g,n}$ has codimension 1 in $\mathfrak{M}_{g,n}$. Remark 2.5 implies that $\mathfrak{Bun}_{g,n}(k,d)^{\mathrm{rt}}$ has codimension 1 in $\mathfrak{Bun}_{g,n}(k,d)$.

LEMMA 2.8. *Let $\pi : \mathcal{C} \to \mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ be the universal curve over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. Then there exists a rational tail free curve $\hat{\mathcal{C}}$ and a projection $p : \mathcal{C} \to \hat{\mathcal{C}}$ over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$.*

Proof. Let $\mathcal{S}$ be the tautological bundle on the tautological curve of $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. Without loss of generality we may assume that the divisor D on the tautological curve of $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ which consists of curves with rational tails is irreducible, otherwise we repeat the construction for each component. Let a be the degree of $\wedge^k \mathcal{S}^\vee$ restricted to the locus consisting of rational tails in the fibers of π. We have that

$$\mathcal{L} = (\wedge^k \mathcal{S}^\vee)^{\otimes \epsilon} \otimes \omega_\pi \left(\sum p_i\right)^{\otimes \epsilon a}$$

is trivial on the locus in $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ consisting of curves with one irreducible rational tail. As $\mathcal{C}$ is normal it follows that $\mathcal{L}$ is trivial on D. As $(\wedge^k \mathcal{S}^\vee)^{\otimes \epsilon} \otimes \omega_\pi(\sum p_i)$ is ample we have that $\mathcal{L}$ is π relatively ample on the complement of this locus. This shows that $\mathcal{L}^{\otimes m}$ is base point free for a sufficiently large m. Let

$$\hat{\mathcal{C}} = \mathrm{Proj} \oplus_l \mathcal{L}^{\otimes ml}.$$

As $\mathcal{L}^{\otimes m}$ is π-relatively base point free it determines a morphism $p : \mathcal{C} \to \hat{\mathcal{C}}$. It can be easily seen that $\hat{\mathcal{C}}$ is a family of genus g curves and flat over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. □

2.2 The most balanced locus

CONSTRUCTION 2.9. Let $\mathfrak{Bun}_{0,n}(k,d)^{\mathrm{bal}}$ be the substack of $\mathfrak{Bun}_{0,n}(k,d)$ such that on every component of a rational curve C the bundle S is the most balanced one. More precisely, if C_i is a rational component of C such that $S_i := S|_{C_i}$ has degree d_i, then

$$S_i = \mathcal{O}(a_i)^{\oplus k_1} \oplus \mathcal{O}(a_i - 1)^{\oplus k - k_1},$$

where a_i, k_1 are the unique integers such that $k_1 a_i + (k - k_1)(a_i - 1) = d_i$ and $1 \leqslant k_1 < k$. We call (C_i, S_i) as above balanced.

LEMMA 2.10. *$\mathfrak{Bun}_{0,n}(k,d)^{\mathrm{bal}}$ is an open substack of $\mathfrak{Bun}_{0,n}(k,d)$.*

Proof. We have that $\mathfrak{Bun}^{\mathrm{rt}}_{0,n}(k,d)$ is isomorphic to the union

$$\bigcup_{0\leqslant d_i\leqslant d} \mathfrak{Bun}_{0,n+1}(k,d-d_i)\times \mathfrak{Bun}_{0,1}(k,d_i).$$

Let $\mathcal{C}_i\to\mathfrak{Bun}_{0,1}(k,d_i)$ be the universal curve over $\mathfrak{Bun}_{0,1}(k,d_i)$, $\mathcal{S}_i$ the universal bundle on $\mathcal{C}_i$ and let $\pi_i:\mathcal{C}_i\times_{\mathfrak{Bun}_{0,1}(k,d_i)}\mathfrak{Bun}^{\mathrm{rt}}_{0,n}(k,d)\to\mathfrak{Bun}^{\mathrm{rt}}_{0,n}(k,d)$ be the pull-back of $\mathcal{C}_i$ to $\mathfrak{Bun}^{\mathrm{rt}}_{0,n}(k,d)$. If $i:\mathfrak{Bun}^{\mathrm{rt}}_{0,n}(k,d)\to\mathfrak{Bun}_{0,n}(k,d)$ denotes the inclusion, then the non-balanced locus of $\mathfrak{Bun}_{0,n}(k,d)$ is the support of the sheaf $\bigcup_i i_*\pi_{i*}Ext^1(\pi_i^*\mathcal{S},\pi_i^*\mathcal{S})$. □

Definition 2.11. Let $\mathfrak{Bun}_{g,n}(k,d)^{\mathrm{bal}}$ be the substack of $\mathfrak{Bun}_{g,n}(k,d)$ whose points are bundles on curves, which are balanced on the rational tails as explained above.

Proposition 2.12. *We have that $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)^{\mathrm{bal}}$ is an open substack of $\mathfrak{Bun}_{g,n}(k,d)$ and the complement has codimension at least 2.*

Proof. This follows just as before. In the notation of the above lemma let $\mathfrak{Z}$ be the support of the sheaf $\bigcup_i i_*\pi_{i*}Ext^1(\pi_i^*\mathcal{S},\pi_i^*\mathcal{S})$. Then $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)^{\mathrm{bal}}$ is the complement of $\mathfrak{Z}$ in $\mathfrak{Bun}_{g,n}(k,d)$ and since $Ext^1(\mathcal{S}_i,\mathcal{S}_i)=0$ on the balanced locus we have that $\mathfrak{Z}$ has codimension at least 2. □

We recall here a cohomology and base change lemma from [PR03] (Lemma 7.1 in [PR03]).

Lemma 2.13. *Let $\mathcal{C}\to B$ be a family of curves over a base scheme B and $p:\mathcal{C}\to\hat{\mathcal{C}}$ be a morphism over B contracting rational tails. If $\mathcal{S}$ is a vector bundle such that $R^1p_*\mathcal{S}=0$, then $p_*\mathcal{S}$ is a flat family of coherent sheaves over B and formation of $p_*\mathcal{S}$ commutes with arbitrary base change $B'\to B$.*

Proposition 2.14. *There exists a morphism*

$$r:\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)^{\mathrm{bal}}\to\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)^{\mathrm{rtf}}.$$

Proof. The proof follows essentially from [PR03]. Let D be the divisor on the tautological curve on $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ consisting of curves with rational tails. Let B be a scheme, $f:B\to\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$ a morphism and $\mathcal{C}$ the flat family of curves over B obtained by pulling back along f the universal curve over $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)$. As $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)^{\mathrm{rt}}$ has a finite number of irreducible components we have that the intersection of the divisor $\mathfrak{Bun}^{\epsilon}_{g,n}(k,d)^{\mathrm{rt}}$ with $\mathcal{C}$ has a finite number of components $D_1,\ldots,D_s$. Suppose that on the generic rational tail in the fiber of $D_i\to B$ we have that $\mathcal{S}$ splits as

$$\oplus^{k_1}\mathcal{O}(a_i)\oplus^{k-k_1}\mathcal{O}(a_i-1). \tag{4}$$

We have that the restriction of $\mathcal{S}(a_if^*D)$ to rational tails in the fiber of $D_i\to B$ splits as $\oplus^{k_1}\mathcal{O}\oplus^{k-k_1}\mathcal{O}(-1)$. Then the restriction of $\mathcal{S}(a_if^*D)$ rational tails in the fiber of $D_i\to B$ has no higher cohomology and as in [PR03], Proposition 7.2 we obtain that $\mathcal{R}^1p_*\mathcal{S}(a_if^*D)=0$. This implies that $\hat{\mathcal{S}}=p_*\sum_{i=1}^s\mathcal{S}(a_if^*D)$ is a vector bundle. The contraction of each rational tail on which $\mathcal{S}$ splits as in (4) increases the degree of $\hat{\mathcal{S}}^\vee$ with $-k_1a_i-(k-k_1)(a_i-1)$. This implies that $\hat{\mathcal{S}}$ is ϵ-stable if $\mathcal{S}$ is ϵ-stable. □

Example 2.15. Let C be a nodal curve consisting of a curve $\hat{C}$ glued in one point to a curve C^0 which is isomorphic to $\mathbb{P}^1$. Let p denote the node of C. Let S be a vector bundle on C and suppose that $S|_{C^0}\simeq\oplus^{k_1}\mathcal{O}(a)\oplus^{k-k_1}\mathcal{O}(a-1)$ with $a\leqslant 0$. We denote by S' the restriction of S to $\hat{C}$. If $\{s_1,\ldots,s_k\}$ is a local basis of S' at p compatible with the splitting of $S|_{C^0}$ and x a local coordinate at p on $\hat{C}$ then $\hat{S}$ above is the subsheaf of S' locally generated by $\{x^{-a}s_1,\ldots,x^{-a}s_{k_1},\ldots,x^{-a+1}s_k\}$. In particular, Proposition 2.14 shows that $\hat{S}$ does not depend on the choice of the local basis.

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Remark 2.16. In general one would naïvely expect to define a morphism which extends r in the following way. Given C as above and S such that $S|_{C^0} \simeq \bigoplus_{i=1}^{k} \mathcal{O}(a_i)$ with $a_i \leqslant 0$ we could associate to S a vector bundle $\hat{S}$ which is the subsheaf of S' locally generated by $\{x^{-a_1}s_1, \ldots, x^{-a_k}s_k\}$. However this will depend on the choice of a local basis (see [PR03] the remark after Proposition 7.3).

2.3 Construction of $\widetilde{\mathfrak{Bun}}_{g,n}(k, r, d)$

The goal of this section is to find a stack $\widetilde{\mathfrak{Bun}}_{g,n}(k, r, d)$ which surjects to $\mathfrak{Bun}_{g,n}(k, d)$ and a morphism $\widetilde{\mathfrak{Bun}}_{g,n}(k, r, d) \to \mathfrak{Bun}_{g,n}(k, d)^{\mathrm{rtf}}$ which extends the morphism r.

Notation 2.17. Let $\mathcal{C}$ be a family of nodal curves over a scheme B. We denote by $\mathcal{U}$ the complement of the locus of rational tails in $\mathcal{C}$. By Proposition 2.3 there exists a family of rational tail free curves $\hat{\mathcal{C}}$ and a morphism $p : \mathcal{C} \to \hat{\mathcal{C}}$. Note that $\mathcal{U}$ is canonically identified via p with an open subset of $\hat{\mathcal{C}}$. By abuse of notation we say that $\mathcal{U}$ is also an open subset of $\hat{\mathcal{C}}$.

Construction 2.18. Let $\mathfrak{P}(B)$ be the category whose objects are

$$(\mathcal{C} \to B, p : \mathcal{C} \to \hat{\mathcal{C}}, p_1, \ldots, p_n, \mathcal{S}, \hat{\mathcal{S}}, p_*(\mathcal{S}^\vee) \xrightarrow{\rho} \hat{\mathcal{S}}^\vee)$$

with $\mathcal{C}$, $\hat{\mathcal{C}}$ flat families of prestable curves of genus g with sections $p_1, \ldots, p_n$, $\mathcal{S}$ is a vector bundle of rank k and degree d on $\mathcal{C}$ and $\hat{\mathcal{S}}$ is a vector bundle of rank k and degree d on $\hat{\mathcal{C}}$, such that the following conditions hold:

(i) $p : \mathcal{C} \to \hat{\mathcal{C}}$ is the morphism of B-schemes from Lemma 2.8;

(ii) the restriction of ρ to $\mathcal{U}$ is an isomorphism;

(iii) $\omega_{\mathcal{C}/B}(\sum p_i) \otimes (\wedge^k \mathcal{S}^\vee)^{\otimes \epsilon}$ is ample for $\epsilon \in \mathbb{Q}$ such that $\epsilon > 2$;

(iv) in every fiber, for every morphism

$$p_*(S^\vee) \xrightarrow{\rho} \hat{S}^\vee \to \tau \to 0$$

and every $x_i \in \hat{C}$ such that the fiber of p over x_i is a one-dimensional rational curve C_i^0 we have that

$$\mathrm{length}(\tau_{x_i}) = \deg(S^\vee|_{C_i^0}),$$

where τ_{x_i} denotes the subsheaf of τ supported at x_i;

(v) the number of components of $\mathcal{C}$ is bounded by some N.

Isomorphisms. An isomorphism

$$\psi : (p : \mathcal{C} \to \hat{\mathcal{C}}, \mathcal{S}, \rho : p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee) \to (p' : \mathcal{C}' \to \hat{\mathcal{C}}', \mathcal{S}', \rho' : p_*(\mathcal{S}'^\vee) \to \hat{\mathcal{S}'}^\vee)$$

is an automorphism of curves

$$\phi : \mathcal{C} \to \mathcal{C}'$$
$$\hat{\phi} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}'$$

together with isomorphisms $\theta : \mathcal{S}^\vee \to \phi^* \mathcal{S}'^\vee$ and $\hat{\theta} : \hat{\mathcal{S}}^\vee \to \phi^* \hat{\mathcal{S}'}^\vee$ such that:

(i) $\phi(p_i) = p_i'$, for all i;

(ii) $p' \circ \phi = \hat{\phi} \circ p$;

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(iii) the following diagram commutes

$$\begin{array}{ccc} p_*(\mathcal{S}^\vee) & \xrightarrow{\rho} & \hat{\mathcal{S}}^\vee \\ {\scriptstyle \hat{\theta}_p}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\theta}} \\ \hat{\phi}^* p_*(\mathcal{S}'^\vee) & \xrightarrow{\hat{\phi}^*\rho'} & \hat{\phi}^*\hat{\mathcal{S}}'^\vee \end{array}$$

where $\hat{\theta}_p : p_*(\mathcal{S}^\vee) \to p_*(\phi^*\mathcal{S}'^\vee)$ is the isomorphism of sheaves induced by θ followed by the isomorphism $p_*(\phi^*\mathcal{S}'^\vee) \simeq \hat{\phi}^* p_*(\mathcal{S}'^\vee)$ from Lemma 2.13.

Remark 2.19. Although $\mathfrak{P}$ and subsequently $\widetilde{\mathfrak{Bun}}_{g,n}(k,r,d)$ depends on N we will omit N from the notation. In the next section N will be some large enough number (see Remark 3.2 for more details).

Lemma 2.20. *Let $\mathcal{S}$ and $\mathcal{S}'$ be locally free sheaves on a flat family of curves $\mathcal{C} \to B$ and $f : \mathcal{S} \to \mathcal{S}'$ an injective morphism of sheaves. If f is injective at the general point of every fiber of $\mathcal{C} \to B$, then the quotient*

$$0 \to \mathcal{S} \xrightarrow{f} \mathcal{S}' \to \mathcal{Q} \to 0 \tag{5}$$

is flat over B.

Proof. This follows from [PR03]. Let $\mathcal{I}$ be an ideal sheaf of a subscheme of B. Tensoring (5) with $\mathcal{O}_B/\mathcal{I}$ we obtain a sequence

$$0 \to Tor_1(\mathcal{O}_B/\mathcal{I}, \mathcal{Q}) \to \mathcal{S} \otimes \mathcal{O}_B/\mathcal{I} \to \mathcal{S}' \otimes \mathcal{O}_B/\mathcal{I} \to \mathcal{Q} \otimes \mathcal{O}_B/\mathcal{I} \to 0.$$

From the hypothesis we have that the map $\mathcal{S} \otimes \mathcal{O}_B/\mathcal{I} \to \mathcal{S}' \otimes \mathcal{O}_B/\mathcal{I}$ is injective and therefore $Tor_1(\mathcal{O}_B/\mathcal{I}, \mathcal{Q}) = 0$. □

Lemma 2.21. *Let $\varphi : B' \to B$ and let $(\mathcal{C}, \mathcal{S}, \hat{\mathcal{S}}, p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee)$ be an object of $\mathfrak{P}(B)$. Let $\mathcal{C}' := \mathcal{C} \times_B B'$, $\hat{\mathcal{C}}' := \hat{\mathcal{C}} \times_{B'} B$, $U' = U \times_B B'$,$\mathcal{S}' := \varphi^*\mathcal{S}$ and $\hat{\mathcal{S}}' := \hat{\varphi}^*\hat{\mathcal{S}}$. Then the natural morphism $\rho' : p'_*(\mathcal{S}'^\vee) \to \hat{\mathcal{S}}'^\vee$ is an isomorphism on U'.*

Proof. By the second condition in the definition of the stack $\mathfrak{P}$ we have that

$$\hat{\varphi}^*\rho : \hat{\varphi}^* p_*(\mathcal{S}^\vee) \to \hat{\varphi}^*\hat{\mathcal{S}}^\vee$$

is generically injective in every fiber of $\mathcal{C} \to B$. By Lemma 2.20 we have that the restriction of $\hat{\varphi}^*\rho$ is injective on U. As ρ is surjective on U, we have that $\hat{\varphi}^*\rho$ is surjective on U. This shows that $\hat{\varphi}^*\rho$ is an isomorphism on U. Let C^0 be the rational tail of C. As $H^1(C^0, \mathcal{S}^\vee) = 0$ for any fiber C, we have by Lemma 2.13 that cohomology commutes with base change. This gives

$$\hat{\varphi}^* p_*(\mathcal{S}^\vee) \simeq p'_* \varphi^*(\mathcal{S}^\vee).$$

Combining the two relations we obtain the conclusion. □

Proposition 2.22. *The functor $\mathfrak{P}$ is an Artin stack.*

Proof. Let $\mathfrak{S} \subset \mathfrak{M}_{g,n}$ be the substack of finite type of curves with at most N components and let $\pi : \mathcal{C} \to \mathfrak{S}$ be the universal curve. Fix $\mathcal{O}_{\mathcal{C}}(1)$ a π-relatively ample line bundle. Let $P(m) = d + k(1 - g + m \deg \mathcal{O}_{\mathcal{C}}(1))$ denote the Hilbert polynomial of vector bundles of rank k and degree d. Consider $\mathfrak{X}_l = Quot_P(\mathcal{O}_{\mathcal{C}}(-l)^{P(l)})$ the π-relative *Quot* scheme and let $\mathcal{G}$ be the

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universal quotient. Similarly, let $\hat{\mathfrak{S}} \subset \mathfrak{M}^{\mathrm{rtf}}_{g,n}$ be the image of $\mathfrak{S}$ via the morphism which contracts rational tails and $\hat{\pi} : \hat{\mathcal{C}} \to \hat{\mathfrak{S}}$ be the universal curve. Consider as before $\mathcal{O}_{\hat{\mathcal{C}}}(1)$ a $\hat{\pi}$-relatively ample line bundle and $\hat{\mathfrak{X}}_l = Quot_P(\mathcal{O}_{\hat{\mathcal{C}}}(-l)^{P(l)})$ the $\hat{\pi}$-relative $Quot$ scheme and let $\hat{\mathcal{F}}$ be the universal quotient. Similarly, let $\hat{\mathcal{F}}'$ be the universal quotient on $\hat{\mathfrak{X}}_m = Quot_P(\mathcal{O}_{\hat{\mathcal{C}}}(-m)^{P(m)})$. By [Lie06] we have that $Hom_{\hat{\mathfrak{X}}_l \times_{\hat{\mathfrak{S}}} \hat{\mathfrak{X}}_m}(\mathcal{O}(-l)^{P(l)}, \mathcal{O}(-m)^{P(m)})$ is an Artin stack. Let $\mathfrak{H}$ be the locally closed locus in

$$Hom_{\hat{\mathfrak{X}}_l \times_{\hat{\mathfrak{S}}} \hat{\mathfrak{X}}_m}(\mathcal{O}(-l)^{P(l)}, \mathcal{O}(-m)^{P(m)})$$

defined by the following:

(i) the universal quotient $\hat{\mathcal{F}}'$ on the second factor is locally free;
(ii) $\hat{\mathcal{F}}(l)$ and $\hat{\mathcal{F}}'(m)$ are generated by global sections and higher cohomology of $\hat{\mathcal{F}}(l)$ and $\hat{\mathcal{F}}'(m)$ vanishes;
(iii) the morphism $\mathcal{O}(-l)^{P(l)} \to \mathcal{O}(-m)^{P(m)}$ induces a morphism

$$\rho : \hat{\mathcal{F}} \to \hat{\mathcal{F}}'.$$

Let us explain condition (iii) above. Denote by $0 \xrightarrow{j} \mathcal{K} \to \mathcal{O}(-l)^{P(l)} \to \hat{\mathcal{F}} \to 0$ and $0 \to \mathcal{K} \to \mathcal{O}(-m)^{P(m)} \xrightarrow{q} \hat{\mathcal{F}}' \to 0$ the tautological sequences on $\hat{\mathfrak{X}}_l$, $\hat{\mathfrak{X}}_m$ respectively and $f : \mathcal{O}(-l)^{P(l)} \to \mathcal{O}(-m)^{P(m)}$ the tautological morphism. Then condition (iii) translates into $q \circ f \circ j = 0$. This is a closed condition. Then let $\mathfrak{A}_{l,m} = \mathfrak{X}_l \times_{\hat{\mathfrak{S}}} \mathfrak{H}$ where the morphism $\mathfrak{X}_l \to \hat{\mathfrak{S}}$ is the composition $\mathfrak{X}_l \to \mathfrak{S} \xrightarrow{p} \hat{\mathfrak{S}}$. Let $\mathfrak{B}$ be the locally closed locus in $\mathfrak{A}_{l,m}$ such that:

(i) the universal quotient $\mathcal{G}$ on $\mathfrak{X}_l$ is locally free;
(ii) $\omega_\pi(\sum p_i) \otimes \wedge^k \mathcal{G}^\epsilon$ is ample for $\epsilon > 2$;
(iii) the restriction of ρ to $\mathcal{U}$ is an isomorphism;
(iv) $p_* \mathcal{G} = \hat{\mathcal{F}}$;
(v) if C denotes a fiber of the universal curve $\mathcal{C} \to \mathfrak{X}_l$ and $S^\vee$ denotes the restriction of $\mathcal{G}$ to C, then for every morphism

$$\hat{\mathcal{F}} \xrightarrow{\rho} \hat{\mathcal{F}}' \to \tau \to 0$$

and every $x_i \in \hat{C}$ such that the fiber of p over x_i is a one dimensional rational curve C_i^0 we have that

$$\mathrm{length}(\tau_{x_i}) = \deg(S^\vee|_{C_i^0}).$$

We have that $G_{l,m} = \mathbf{Gl}_l \times \mathbf{Gl}_m$ acts on $\mathfrak{A}$. We would now like to take $\bigcup_{l,m} \mathfrak{A}_{l,m}/G_{l,m}$ but in general such a quotient would only define a 2-stack. Let us further sketch how to describe $\mathfrak{P}$ as a quotient of a scheme by a group. The construction is rather standard. Take $\mathcal{H}'$ as in [MOP11] Section 6.1. More precisely $\mathcal{H}'$ is a subscheme of $Hilb(\mathbb{P}(V)) \times \mathbb{P}(V)^n$, where $Hilb(\mathbb{P}(V))$ is the Hilbert scheme of genus g curves and degree $F = 1 - g + k(d+1)(2g-2+n) + kd$ for $k \geqslant 5$ in the projective space $\mathbb{P}(V)$ with $V \simeq \mathbb{C}^F$. Then $\mathcal{H}'$ comes equipped with a universal curve $\mathcal{C}'$, a contraction of rational tails $\mathcal{C}' \to \hat{\mathcal{C}}'$ and a $\mathbf{PGL}(V)$ action. If in the above construction we replace $\mathcal{C}$ by $\mathcal{C}'$ and $\hat{\mathcal{C}}$ by $\hat{\mathcal{C}}'$ we get a scheme $A_{l,m} = X_l \times_{\hat{\mathcal{C}}'} H$ with $X_m = Quot_P(\mathcal{O}_{\mathcal{C}'}(-m)^{P(m)})$, $\hat{X}_m = Quot_P(\mathcal{O}_{\hat{\mathcal{C}}'}(-m)^{P(m)})$ and H a subscheme of

$$Hom_{\hat{X}_l \times_{\hat{\mathcal{C}}'} \hat{X}_m}(\mathcal{O}(-l)^{P(l)}, \mathcal{O}(-m)^{P(m)}).$$

Then $\mathfrak{P}$ is the stack quotient $\bigcup_{l,m}[A_{l,m}/G_{l,m} \times \mathbf{PGL}(V)]$. This concludes the proof. □

Remark 2.23. The stability condition for stable maps implies that the restriction of $S^\vee$ to any rational tail has positive degree. This together with condition (iv) in Construction 2.18 shows that the restriction of $\hat{S}^\vee$ to any unstable component of $\hat{C}$ has positive degree. This means that $\omega_{\hat{C}}(\sum p_i) \otimes \wedge^k \hat{S}^\epsilon$ is ample for any $\epsilon > 0$.

Lemma 2.24. *There exists a morphism*

$$t : \mathfrak{Bun}_{g,n}(k,d)^{\mathrm{bal}} \to \mathfrak{P}.$$

Proof. This is essentially Proposition 7.2 in [PR03]. We use the notation in Proposition 2.14. Let

$$t(\mathcal{S}^\vee) = (\mathcal{S}^\vee, p_*(\mathcal{S}^\vee) \xrightarrow{\rho} p_*(\mathcal{S}^\vee((-a_i+1)f^*D)))$$

where ρ is the map induced by $\mathcal{S}^\vee \overset{f^*D^{-a_i+1}}{\to} \mathcal{S}^\vee((-a_i+1)f^*D)$. The map is well defined since $p_*(\mathcal{S}^\vee((-a_i+1)f^*D))$ is a vector bundle by Lemma 7.1 in [PR03]. □

Construction 2.25. Let $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ be the irreducible component of $\mathfrak{P}$ which contains the image of t.

Remark 2.26. Note that as d becomes large $\mathfrak{P}$ might not be irreducible.

Lemma 2.27. *The stack $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ has pure dimension equal to the dimension of $\mathfrak{Bun}_{g,n}(k,d)$.*

Proof. Let $\mathfrak{U}$ be the product $\widetilde{\mathfrak{Bun}}_{g,n}(k,d) \times_{\mathfrak{M}_{g,n}} \mathfrak{M}^{sm}_{g,n}$, where $\mathfrak{M}^{sm}_{g,n}$ is the locus in $\mathfrak{M}_{g,n}$ of smooth curves. Then $\mathfrak{U}$ is an open substack of $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$. As $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ is irreducible it is enough to show that $\mathfrak{U}$ has pure dimension equal to the dimension of $\mathfrak{Bun}_{g,n}(k,d)$. For this we show that $\mathfrak{U} \simeq \mathfrak{Bun}_{g,n}(k,d) \times_{\mathfrak{M}_{g,n}} \mathfrak{M}^{sm}_{g,n}$. Objects of $\mathfrak{U}$ are pairs $(\mathcal{S}^\vee, \mathcal{S}^\vee \xrightarrow{\rho} \mathcal{S}^\vee)$ with ρ an isomorphism and any such object is isomorphic to $(\mathcal{S}^\vee, \mathcal{S}^\vee \xrightarrow{\mathrm{id}} \mathcal{S}^\vee)$ by composing ρ with ρ^{-1}. It is clear that any isomorphism of $(\mathcal{S}^\vee, \mathcal{S}^\vee \xrightarrow{\mathrm{id}} \mathcal{S}^\vee)$ induces an isomorphism of $\mathcal{S}$. Vice versa for any automorphism θ of $\mathcal{S}$ there exists a unique automorphism $\hat{\theta} = \theta$ of $\mathcal{S}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\mathrm{id}} & \mathcal{S} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\hat{\theta}} \\ \mathcal{S} & \xrightarrow{\mathrm{id}} & \mathcal{S} \end{array}$$

This concludes the proof. □

Remark 2.28. We have a diagram

$$\begin{array}{ccccc} & & \widetilde{\mathfrak{Bun}}_{g,n}(k,d) & & \\ & {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} & \\ \mathfrak{Bun}_{g,n}(k,d) & & \overset{r}{\dashrightarrow} & & \mathfrak{Bun}^{\mathrm{rtf}}_{g,n}(k,d) \end{array}$$

such that $r \circ \pi_1 = \pi_2$.

Proof. We have a morphism

$$\begin{aligned} \mathfrak{P} &\to \mathfrak{Bun}_{g,n}(k,d) \times_{\mathfrak{M}_{g,n}} \mathfrak{Bun}_{g,n}(k,d) \\ (p : \mathcal{C} \to \hat{\mathcal{C}}, \mathcal{S}, p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee) &\mapsto (\mathcal{S}, \hat{\mathcal{S}}). \end{aligned}$$

We define π_i to be the composition of the natural projection of the product to the ith factor composed with the above map. □

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3. Stable map-quotients

3.1 Construction

In this section we construct a proper algebraic stack which surjects to both moduli spaces of stable maps to $\mathbb{G}(k,r)$ and stable quotients. We explain the definition at the end of this section.

Definition 3.1. Let $\bar{P}(B)$ be the category whose objects are

$$(\mathcal{C} \to B, p : \mathcal{C} \to \hat{\mathcal{C}}, p_1, \ldots, p_n, \mathcal{O}_{\mathcal{C}}^{\oplus r} \xrightarrow{s} \mathcal{S}^\vee, p_*(\mathcal{S}^\vee) \xrightarrow{\rho} \hat{\mathcal{S}}^\vee)$$

such that:

(i) $p : \mathcal{C} \to \hat{\mathcal{C}}$ is the morphism of Lemma 2.8 and $\mathcal{U}$ is as in Notation 2.17;

(ii) $\mathcal{O}_{\mathcal{C}}^{\oplus r} \xrightarrow{s} \mathcal{S}^\vee$, is a stable map of rank k and degree d;

(iii) $\hat{\mathcal{S}}$ is a rank k degree $-d$ locally free sheaf;

(iv) the morphism of sheaves $\rho : p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee$ is an isomorphism on $\mathcal{U}$;

(v) $\omega_{\hat{\mathcal{C}}/B}(\sum p_i) \otimes (\wedge^k \hat{\mathcal{S}}^\vee)^{\otimes \epsilon}$ is ample for any $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$;

(vi) in every fiber, for every morphism

$$p_*(S^\vee) \xrightarrow{\rho} \hat{S}^\vee \to \tau \to 0$$

and every $x_i \in \hat{C}$ such that the fiber of p over x_i is a one-dimensional rational curve C_i^0 we have that

$$\text{length}(\tau_{x_i}) = \deg(S^\vee|_{C_i^0}),$$

where τ_{x_i} denotes the subsheaf of τ supported at x_i.

Isomorphisms. An isomorphism

$$\psi : (\mathcal{C} \xrightarrow{p} \hat{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}^{\oplus r} \xrightarrow{s} \mathcal{S}^\vee, p_*(\mathcal{S}^\vee) \xrightarrow{\rho} \hat{\mathcal{S}}^\vee) \to (\mathcal{C}' \xrightarrow{p'} \hat{\mathcal{C}}', \mathcal{O}_{\mathcal{C}'}^{\oplus r} \xrightarrow{s'} \mathcal{S}'^\vee, p'_*(\mathcal{S}'^\vee) \xrightarrow{\rho'} \hat{\mathcal{S}}'^\vee)$$

is an automorphism of curves

$$\phi : \mathcal{C} \to \mathcal{C}'$$
$$\hat{\phi} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}'$$

together with isomorphisms $\theta : \mathcal{S}^\vee \to \phi^*\mathcal{S}'^\vee$ and $\hat{\theta} : \hat{\mathcal{S}}^\vee \to \phi^*\hat{\mathcal{S}}'^\vee$ such that:

(i) $\phi(p_i) = p'_i$, for all i;

(ii) $p' \circ \phi = \hat{\phi} \circ p$;

(iii) the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{C}}^{\oplus r} & \xrightarrow{s} & \mathcal{S}^\vee \\ \downarrow & & \downarrow \theta \\ \phi^*\mathcal{O}_{\mathcal{C}'}^{\oplus r} & \xrightarrow{\phi^* s'} & \phi^*\mathcal{S}'^\vee \end{array}$$

(iv) the following diagram commutes

$$\begin{array}{ccc} p_*(\mathcal{S}^\vee) & \xrightarrow{\rho} & \hat{\mathcal{S}}^\vee \\ \hat{\theta}_p \downarrow & & \downarrow \hat{\theta} \\ \hat{\phi}^* p_*(\mathcal{S}'^\vee) & \xrightarrow{\hat{\phi}^*\rho'} & \hat{\phi}^*\hat{\mathcal{S}}'^\vee \end{array}$$

where $\hat{\theta}_p : p_*(\mathcal{S}^\vee) \to \hat{\phi}^* p_*(\mathcal{S}'^\vee)$ is the isomorphism of sheaves induced by θ followed by the isomorphism $p_*(\phi^*\mathcal{S}'^\vee) \simeq \hat{\phi}^* p_*(\mathcal{S}'^\vee)$ from Lemma 2.13.

We call a point in $\bar{P}$ a map-quotient.

Remark 3.2. Here we do not need to impose that the number of components of C is bounded as for any stable map $(C, p_1, \dots, p_n, f)$ there exists an N depending on g, n and d such that C has at most N components.

Remark 3.3. Let us show that condition (ii) in Definition 3.1 is equivalent to the one in [MOP11]. Given a stable quotient $\hat{S} \to \mathcal{O}^{\oplus r}$ we get by dualizing a morphism $\mathcal{O}^{\oplus r} \to \hat{S}^\vee$ which is generically surjective in all fibers. Conversely, given a morphism $\mathcal{O}^{\oplus r} \to \hat{S}^\vee$ which is generically surjective in fibers we get an injective morphism $\hat{S} \to \mathcal{O}^{\oplus r}$ whose quotient is flat by Lemma 2.20.

Proposition 3.4. *We have an isomorphism of stacks*

$$\bar{P} \simeq \overline{M}_{g,n}(\mathbb{G}(k,r), d) \times_{\mathfrak{Bun}_{g,n}(k,d)} \mathfrak{P}$$

for some N in the definition of $\mathfrak{P}$ as in Remark 3.2. In particular, $\bar{P}$ is an Artin stack.

Proof. It follows easily from definitions. □

Lemma 3.5. *Let $\overline{M}_{g,n}(\mathbb{G}(k,r), d)^{\mathrm{bal}}$ denote the fiber product*

$$\overline{M}_{g,n}(\mathbb{G}(k,r), d) \times_{\mathfrak{Bun}_{g,n}(k,d)} \mathfrak{Bun}_{g,n}(k,d)^{\mathrm{bal}}.$$

Then we have a morphism of stacks

$$R : \overline{M}_{g,n}(\mathbb{G}(k,r), d)^{\mathrm{bal}} \to \bar{P}$$

over $\mathfrak{Bun}_{g,n}(k,d)$.

Proof. Let $(\mathcal{O}_{\mathcal{C}}^{\oplus r} \to \mathcal{S}^\vee) \in \overline{M}_{g,n}(\mathbb{G}(k,r), d)^{\mathrm{bal}}$. We define

$$R(\mathcal{O}^r \to \mathcal{S}^\vee) = (\mathcal{O}_{\mathcal{C}}^{\oplus r} \to \mathcal{S}^\vee, p_*(\mathcal{S}^\vee) \xrightarrow{\rho} (p_*\mathcal{S}^\vee((-a_i+1)f^*D_i))).$$

As before we have by Lemma 7.1 in [PR03] that $p_*(\mathcal{S}^\vee((-a_i+1)f^*D_i))$ is a vector bundle. The quotient stability for $\hat{\mathcal{S}}^\vee = p_*(\mathcal{S}^\vee((-a_i+1)f^*D_i))$ is immediate. □

Proposition 3.6. *There exists a morphism*

$$c : \overline{M}_{g,n}(\mathbb{G}(k,r), d)^{\mathrm{bal}} \to \overline{Q}_{g,n}(\mathbb{G}(k,r), d).$$

Proof. Let $(\mathcal{O}_{\mathcal{C}}^{\oplus r} \to \mathcal{S}^\vee) \in \overline{M}_{g,n}(\mathbb{G}(k,r), d)^{\mathrm{bal}}$ and $p : \mathcal{C} \to \hat{\mathcal{C}}$ be the contraction of rational tails. Then we have a generically surjective morphism $\mathcal{O}_{\hat{\mathcal{C}}}^{\oplus r} \to p_*(\mathcal{S}^\vee)$. Composing this with the morphism ρ above we obtain a stable quotient $\mathcal{O}_{\hat{\mathcal{C}}}^{\oplus r} \to \hat{\mathcal{S}}^\vee$ with $\hat{\mathcal{S}}^\vee = p_*(\mathcal{S}^\vee((-a_i+1)f^*D_i))$. □

Example 3.7. Let us prove that in general R and c do not extend. The proof for the two morphisms is the same. We repeat the argument in [PR03], Theorem 7.4. Let us consider a one-dimensional constant family of stable genus zero maps $f^1 : \mathcal{C}_B^1 \to \mathbb{G}(k,r)$ with a constant section $s^1 : B \to \mathcal{C}^1$, $s^1(b) = P$, for any point $b \in B$. Let $f^0 : \mathcal{C}_B^0 \to \mathbb{G}(k,r)$ be a family of genus zero maps, such that on the general fiber $C_b \to \mathbb{G}(k,r)$ the pull-back of the tautological subbundle is isomorphic to $\mathcal{O}(-1) \oplus \dots \oplus \mathcal{O}(-1)$ and on the special fiber $C_0^0 \to \mathbb{G}(k,r)$ the pull-back of the tautological subbundle is isomorphic to $\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \dots \oplus \mathcal{O}(-1)$. Let us consider a section $s^0 : B \to \mathcal{C}^0$ such that $f^1(s^1(B)) = f^0(s^0(B))$. By identifying $f^1(s^1(B))$ with $f^0(s^0(B))$ we obtain a family of stable maps $f : \mathcal{C} \to \mathbb{G}(k,n)$. Let $f_0 : C_0 \to \mathbb{G}(k,r)$ be the special fiber and p the gluing point of C_0^1 with C_0^0. Let $\{s_1, \ldots, s_k\}$ be a local basis for $S|_{C_0^1}$ at p. If x is a local coordinate around p, then $\{xs_1, \ldots, xs_k\}$ is a basis of $\hat{S}$.

Let us now consider a second family of stable maps. As $\overline{M}_{0,n}(\mathbb{G}(k,r),d)$ is irreducible we can find a family of stable maps B' whose special fiber is $f_0 : C_0 \to \mathbb{G}(k,r)$ and general fiber with smooth domain. By Proposition 7.3 in [PR03] the limiting stable quotient over B' is $\hat{S}$ with $\hat{S}$ around p generated by $\{s_1, x^2 s_2, xs_3, \ldots, xs_k\}$. As the two quotients are different R and c do not extend.

Remark 3.8. Let us explain the idea behind the definition of $\bar{P}$. For this let us understand better why c does not extend. In the following it is a little easier to use the equivalent description of stable maps to Grassmannians and stable quotients explained in Remark 3.3 and work with the dual definition. In the spirit of Remark 2.16 we could try to extend c in the following way. Let C be a curve with a rational tail C^0 and a rational tail free part $\hat{C}$ as in Remark 2.16. Given a stable map

$$0 \to S \to \mathcal{O}_C^{\oplus r} \to Q \to 0$$

such that $S|_{C^0} \simeq \bigoplus_{i=1}^k \mathcal{O}(a_i)$ with $a_i \leqslant 0$ we could associate to S a vector bundle $\hat{S}$ which is the subsheaf of $S' = S|_{\hat{C}}$ locally generated by sections $\{x^{-a_1}s_1, \ldots, x^{-a_k}s_k\}$. Such a vector bundle comes by construction with a morphism to $\mathcal{O}_{\hat{C}}^{\oplus r}$ and by Lemma 2.20 the quotient of the injective morphism $\hat{S} \to \mathcal{O}_{\hat{C}}^{\oplus r}$ is flat. It can also be seen that the stability for stable quotients is satisfied. However, as shown in Example 3.7 this association does not define a morphism. The reason why c does not extend is that in general there is no one to one relation between the subsheaf S' of S and $S|_{C^0}$ (see Example 3.7 and Example 7.1 in [PR03]). Equivalently, there is no one to one relation between the torsion subsheaf of $\hat{Q}$ in the definition and the splitting of $S|_{C^0}$. The idea in the definition of $\bar{P}$ is to take all possible pairs

$$(0 \to S \to \mathcal{O}_C^{\oplus r} \to Q \to 0, 0 \to \hat{S} \to \mathcal{O}_{\hat{C}}^{\oplus r} \to \hat{Q} \to 0)$$

such that:

1. C has a rational tail free part isomorphic to $\hat{C}$ and rational tails $C_1^0, \ldots, C_p^0$ attached to $\hat{C}$ in $x_1, \ldots, x_p$;
2. $S' = S|_{\hat{C}}$ is the saturation of $\hat{S}$ in $\mathcal{O}_{\hat{C}}^{\oplus r}$;
3. the restriction of S to the rational tails C_i^0 has degree equal to the degree of the torsion part of $\hat{Q}$ which is supported on x_i, for all $i \in \{1, \ldots, p\}$.

Let us rephrase the above in more geometric terms. A stable quotient gives a rational application $\hat{C} \dashrightarrow \mathbb{G}(k,r)$ which extends to a morphism $\bar{f}$. Then condition 2 translates into:

2′. $\hat{C} \dashrightarrow \mathbb{G}(k,r)$ is not defined at $x_1, \ldots, x_p$ and the restriction of the stable map to $\hat{C}$ is $\bar{f}$.

Conditions 1 and 2 (or equivalently 2′) explain condition (iv) in Definition 3.1 while condition 3 explains condition (vi). Let us also note that if we have just one rational tail condition (vi) follows from condition (iv), but not in general.

Construction 3.9. Let

$$\overline{MQ}_{g,n}(\mathbb{G}(k,r),d) = \overline{M}_{g,n}(\mathbb{G}(k,r),d) \times_{\mathfrak{Bun}_{g,n}(k,d)} \widetilde{\mathfrak{Bun}}_{g,n}(k,d).$$

Then by construction $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ is a substack of $\bar{P}$ and it contains the image of $\bar{R}$.

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3.2 Properties

We first show that $\bar{P}$ is proper.

Separatedness

Proposition 3.10. *We have a morphisms of stacks*

$$i : \bar{P} \to \overline{M}_{g,n}(\mathbb{G}(k,r),d) \times_{\mathfrak{M}^{\mathrm{rtf}}_{g,n}} \overline{Q}_{g,n}(\mathbb{G}(k,r),d)$$

which is an immersion of DM stacks. In particular, we have morphisms of stacks

$$c_1 : \bar{P} \to \overline{M}_{g,n}(\mathbb{G}(k,r),d),$$
$$c_2 : \bar{P} \to \overline{Q}_{g,n}(\mathbb{G}(k,r),d).$$

Proof. The exact sequence $0 \to \mathcal{Q}^\vee \to \mathcal{O}^{\oplus r} \to \mathcal{S}^\vee \to 0$ induces an exact sequence

$$0 \to p_*(\mathcal{Q}^\vee) \to \mathcal{O}^{\oplus r} \to p_*(\mathcal{S}^\vee) \to R^1 p_*(\mathcal{Q}^\vee).$$

Let us show that $R^1p_*(\mathcal{Q}^\vee)$ is supported on the complement of U in $\hat{\mathcal{C}}$. We have that $R^1p_*(\mathcal{Q}^\vee)|_U = R^1p_*(\mathcal{Q}^\vee|_U)$ and since p is the identity on U we obtain that $R^1p_*(\mathcal{Q}^\vee)|_U = 0$. Composing the generically surjective morphism $\mathcal{O}^{\oplus r} \to p_*(\mathcal{S}^\vee)$ with $p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee$ we obtain an element in $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$.

Let us now show that given a stable map $m = (C, \mathcal{O}^{\oplus r} \xrightarrow{s} \mathcal{S}^\vee \to 0)$ and a stable quotient $q = (\hat{C}, \mathcal{O}^{\oplus r} \xrightarrow{\hat{s}} \hat{\mathcal{S}}^\vee)$ there exists at most one $mq = (p : C \to \hat{C}, \mathcal{O}_C^{\oplus r} \xrightarrow{s} \mathcal{S}^\vee, p_*(\mathcal{S}^\vee) \xrightarrow{\rho} \hat{\mathcal{S}}^\vee)$ such that $i(mq) = (m,q)$. As the map $p_*s : p_*\mathcal{O}^{\oplus r} \to p_*(\mathcal{S}^\vee)$ induced by s is surjective away from torsion and zero on torsion we obtain that a map $\rho : p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee$ such that $\rho \circ p_*s = \hat{s}$ must be unique. □

Corollary 3.11. *We have that $\bar{P}$ is separated.*

Properness

Lemma 3.12. *Let B be a curve and let $\mathcal{C} \to B$ be a flat family of curves over B. Let us assume that $B, \mathcal{C}$ are smooth and let D be a (-1) curve on $\mathcal{C}$. Then we have that*

$$Tor_1(\mathcal{O}_D(-a), \mathcal{O}_D) \simeq \mathcal{O}_D(-a+1)$$

for any $a > 0$.

Proof. We have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

Let $a > 0$. Tensoring it with $\mathcal{O}(aD)$ we obtain the following free resolution of $\mathcal{O}_D(-a)$

$$0 \to \mathcal{O}((a-1)D) \to \mathcal{O}(aD) \to \mathcal{O}_D(-a) \to 0.$$

Tensoring the resolution with $\mathcal{O}_D$ we obtain a sequence

$$0 \to Tor_1(\mathcal{O}_D(-a), \mathcal{O}_D) \to \mathcal{O}_D((-a+1)D) \to \mathcal{O}_D(-a) \to \mathcal{O}_D(-a) \to 0.$$

This shows that $Tor_1(\mathcal{O}_D(-a), \mathcal{O}_D) = \mathcal{O}_D(-a+1)$. □

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Lemma 3.13. *Let $\mathcal{C} \to B$ be a family of curves as before, D be a (-1)-curve and*

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathcal{C}}^{\oplus r} \xrightarrow{q} Q \to 0$$

a flat quotient on $\mathcal{C}$. Let $p' : \mathcal{C} \to \mathcal{C}'$ be the contraction of D. Then there exists a locally free sheaf $\mathcal{S}'$ on $\mathcal{C}'$ and morphism of sheaves $\mathcal{S}' \to p'_*\mathcal{S}$ on $\mathcal{C}'$ which is an isomorphism on $\mathcal{C}\backslash D$.

Proof. Let $\mathcal{S}|_D \simeq \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_k)$ with $a_i \geqslant 0$ and let us assume that $a_1 \leqslant \cdots \leqslant a_k$. Let us prove that there exists a vector bundle $\mathcal{T}$ on $\mathcal{C}$ such that the following conditions are fulfilled:

(i) we have a morphism $\mathcal{T} \to \mathcal{S}$ which is an isomorphism on $\mathcal{C}\backslash D$;
(ii) $\mathcal{T}|_D \simeq \mathcal{O}^{\oplus k}$.

Let $\mathcal{T}_1$ be the kernel of the natural projection $\mathcal{S} \to \mathcal{O}_D(-a_k)$ so that we have an exact sequence

$$0 \to \mathcal{T}_1 \to \mathcal{S} \to \mathcal{O}_D(-a_k) \to 0.$$

Restricting the above sequence to D we have an exact sequence

$$0 \to Tor_1(\mathcal{O}_D(-a_k), \mathcal{O}_D) \to \mathcal{T}_1|_D \to \mathcal{S}|_D \to \mathcal{O}_D(-a_k) \to 0. \tag{6}$$

By Lemma 3.12 we have that $Tor_1(\mathcal{O}_D(-a_k), \mathcal{O}_D) \simeq \mathcal{O}_D(-a_k+1)$ which shows that we have an exact sequence

$$0 \to \mathcal{O}_D(-a_k+1) \to \mathcal{T}_1|_D \to \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_{k-1}) \to 0. \tag{7}$$

Let us assume that $\mathcal{T}_1|_D \simeq \mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_k)$ with $b_1 \leqslant \cdots \leqslant b_k$. By sequence (7) we have that $b_i \geqslant 0$ and $\sum_{i=1}^k b_i = \sum_{i=1}^k a_i - 1$ which means that $\sum_{i=1}^k b_i < \sum_{i=1}^k a_i$. Repeating the above procedure for $\mathcal{T}_1$ instead of $\mathcal{S}$ we obtain $\mathcal{T}$ as in the above claim.

Let $\mathcal{S}' = p'_*\mathcal{T}$. We have that $\mathcal{S}'$ is a vector bundle over $\mathcal{C}'$. By construction $p'_*\mathcal{T} \to p'_*\mathcal{S}$ is an isomorphism on U. □

Remark 3.14. Let

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathcal{C}}^{\oplus n} \xrightarrow{q} Q \to 0$$

be a flat quotient on $\mathcal{C}$. If $\mathcal{S}'$ is the vector bundle constructed in the above lemma, then we obtain a flat family of quotients over $\mathcal{C}'$. Indeed we have that the composition $\mathcal{S}' \to p'_*\mathcal{S} \to \mathcal{O}^{\oplus r}$ in injective on all fibers of $\mathcal{C}' \to B$. By Lemma 2.20 we have that $\mathcal{S}' \to \mathcal{O}^{\oplus r} \to \mathcal{Q}'$ is a flat family of quotients over $\mathcal{C}'$.

Proof of properness. Let Δ be a nonsingular curve with a closed point $0 \in \Delta$ and let $\Delta^* = \Delta\backslash\{0\}$. Let

$$(p : \mathcal{C}^* \to \hat{\mathcal{C}}^*, \mathcal{S}^* \xrightarrow{j^*} \mathcal{O}_{\mathcal{C}^*}^{\oplus r}, p_*(\mathcal{S}^{*\vee}) \xrightarrow{\rho^*} (\hat{\mathcal{S}}^*)^\vee)$$

be a family of map-quotients over Δ^*. By normalizing and possibly restricting Δ we may assume that $\hat{\mathcal{C}}^*$ is smooth. By the properness of the moduli space of stable maps we can extend $0 \to \mathcal{S}^* \xrightarrow{j^*} \mathcal{O}_{\mathcal{C}^*}^{\oplus r} \to Q^* \to 0$ to

$$0 \to \mathcal{S} \xrightarrow{j} \mathcal{O}_{\mathcal{C}}^{\oplus r} \to Q \to 0$$

with Q flat over Δ. Let $\mathcal{C}_0$ be the (reducible) two-dimensional component of $\mathcal{C}$ contracted by p. Let $\mathcal{C}'$ be the closure of the complement of $\mathcal{C}_0$ in $\mathcal{C}$, $p' : \mathcal{C} \to \mathcal{C}'$ be the projection to $\mathcal{C}'$ and $\mathcal{S}'$ the restriction of $\mathcal{S}$ to $\mathcal{C}'$. By the semistable reduction theorem and by possibly blowing up the nodes of the central fiber we may assume that $\mathcal{C}'$ is smooth. Without loss of generality we may

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assume that $\mathcal{C}'$ does not have unstable fibers over $\Delta^\star$. By the properness of the moduli space of stable maps we have a family $0 \to \mathcal{S} \to \mathcal{O}_{\mathcal{C}}^{\oplus r} \to \mathcal{Q} \to 0$ over Δ. Restricting to $\mathcal{C}'$ we obtain a family of stable maps

$$0 \to \mathcal{S}' \to \mathcal{O}_{\mathcal{C}'}^{\oplus r} \to \mathcal{Q}' \to 0.$$

We have the exact sequence

$$0 \to T \xrightarrow{i} p'_*(\mathcal{S}^\vee) \xrightarrow{q} \mathcal{S}'^\vee \to 0 \tag{8}$$

where T is the torsion subsheaf of $p'_*(\mathcal{S}^\vee)$. By our assumption $p' = p$ over Δ^*. This gives that over Δ^* we have a morphism

$$p'_*(\mathcal{S}^\vee) \xrightarrow{\rho} \hat{\mathcal{S}}^\vee.$$

As $\rho \circ i(T) = 0$, the universal property of quotients gives that ρ factors (uniquely) through $\mathcal{S}'^\vee$. This gives a morphism of locally free sheaves

$$0 \to \mathcal{S}'^\vee \xrightarrow{\rho'} \hat{\mathcal{S}}^\vee \tag{9}$$

over Δ^* which is an isomorphism on U^*. Dualizing (9) and applying Lemma 2.20 we have that $\hat{\mathcal{S}} \to \mathcal{S}'$ gives a flat family of quotients over Δ^* and by the properness of the *Quot* scheme we have

$$0 \to \hat{\mathcal{S}}' \to \mathcal{S}' \tag{10}$$

over $\mathcal{C}'$. Dualizing again we have that

$$0 \to \mathcal{S}'^\vee \xrightarrow{\rho'} \hat{\mathcal{S}}'^\vee \to \tau \to 0 \tag{11}$$

is a flat family of quotients which extends (9). As $\mathcal{C}'$ is smooth and $\hat{\mathcal{S}}'^\vee$ reflexive we obtain that $\hat{\mathcal{S}}'^\vee$ is locally free. By (8) and (11) we have an exact sequence

$$p'_*\mathcal{S}^\vee \to \hat{\mathcal{S}}'^\vee \to \tau \to 0 \tag{12}$$

over $\mathcal{C}'$ which satisfies the conditions (i)–(iv) from Construction 3.1. Let us prove that it also satisfies condition (v). Let $\tau_1, \ldots, \tau_m$ be the irreducible components of τ and $\mathcal{C}_{0,1}, \ldots, \mathcal{C}_{0,m}$ be the irreducible components of $\mathcal{C}_0$. As $length(\tau_i) = \deg(\mathcal{S}^\vee|_{\mathcal{C}_{0,i}})$ over Δ^* for all i we obtain that $length(\tau_i) = \deg(\mathcal{S}^\vee|_{\mathcal{C}_{0,i}})$ over Δ. This implies that condition (v) in Definition 3.1 is true for the points $x \in \mathcal{C}' \cap \mathcal{C}_0$.

We now contract the remaining unstable curves in the central fiber. Let $p'' : \mathcal{C}' \to \mathcal{C}''$ be the contraction of *one* unstable rational tail C^0, x the attachment point with the rest of the curve and $-d^0 = \deg(\hat{\mathcal{S}}'|_{C^0})$. Let us show that we can find a vector bundle $\mathcal{S}''$ and a morphism of sheaves

$$p''_*(\hat{\mathcal{S}}'^\vee) \xrightarrow{\rho''} \mathcal{S}''^\vee \tag{13}$$

on $\mathcal{C}'$ which satisfies the conditions (i)–(iv) from Definition 3.1 and such that if τ' denotes the quotient of ρ'' we have that $\text{length}(\tau'_x) = d^0$. By Lemma 3.13 we have a bundle $\mathcal{T}$ with a morphism $\mathcal{T} \to \hat{\mathcal{S}}'$ which is an isomorphism away from C^0. Dualizing and pushing forward we obtain a morphism

$$p''_*(\hat{\mathcal{S}}'^\vee) \to p''_*(\mathcal{T}^\vee)$$

which is an isomorphism away from x. By construction $\mathcal{T}|_D$ is trivial which implies that $p''_*(\mathcal{T}^\vee)$ is a vector bundle. Let us take $\mathcal{S}'' = p''_*(\mathcal{T}^\vee)$. The fact that ρ'' is an isomorphism away from x and $\mathcal{S}''$ is a vector bundle implies that $\text{length}(\tau'_x) = d^0$. Repeating this procedure for all unstable

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rational tails we may assume that (13) holds for $p'' : \mathcal{C}' \to \hat{\mathcal{C}}$, which is the contraction of *all* unstable rational tails. This means that $p = p'' \circ p'$. Take $\hat{\mathcal{S}} = \mathcal{S}''$ and by (12) we get a morphism

$$\rho : p_*(\mathcal{S}^\vee) \to \hat{\mathcal{S}}^\vee$$

on $\hat{\mathcal{C}}$ as in construction (3.1). In the end we contract -2 curves on which $\mathcal{S}$ and $\hat{\mathcal{S}}$ are trivial.

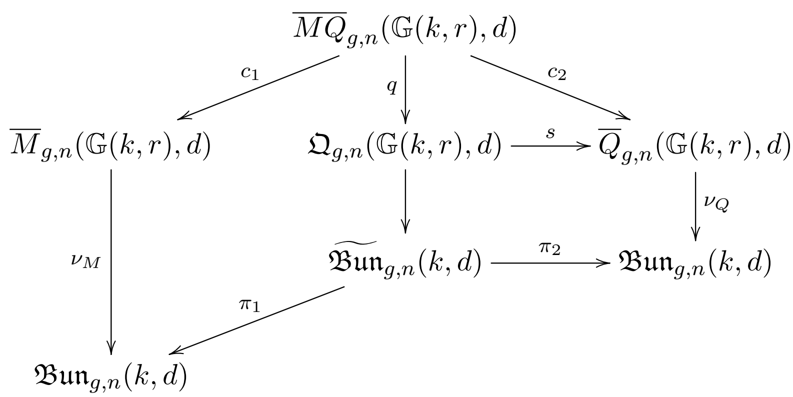
Corollary 3.15. *The stack $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ is a proper DM stack.*

Proof. By definition $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ is a closed substack of $\bar{P}$. As $\bar{P}$ is proper, we obtain that $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ is proper. □

Proposition 3.16. *Let us consider*

$$\mathfrak{Q}_{g,n}(\mathbb{G}(k,r),d) := \overline{Q}_{g,n}(\mathbb{G}(k,r),d) \times_{\mathfrak{Bun}_{g,n}(k,d)} \widetilde{\mathfrak{Bun}}_{g,n}(k,d).$$

Then we have the following commutative diagram.



Proof. The definitions imply that the following diagram is commutative.

$$\begin{CD}
\overline{MQ}_{g,n}(\mathbb{G}(k,r),d) @>>> \overline{Q}_{g,n}(\mathbb{G}(k,r),d)\\
@VVV @VVV\\
\widetilde{\mathfrak{Bun}}_{g,n}(k,d) @>>> \mathfrak{Bun}_{g,n}(k,d)
\end{CD}$$

By the universal property of cartesian products we obtain a map

$$q : \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{Q}_{g,n}(\mathbb{G}(k,r),d).$$ □

Remark 3.17. Let C be nodal curve, C_0 be its rational tail and $\hat{C}$ the rational tail free part of C. For simplicity we assume that C_0 is irreducible. Let moreover

$$0 \to S \to \mathcal{O}_C^{\oplus r}$$

be a stable map from C to $\mathbb{G}(k,r)$. Then the map q forgets the map

$$S|_{C_0} \to \mathcal{O}_{C_0}^{\oplus r}.$$

More precisely, we have that the fiber of q over the point

$$(p : C \to \hat{C}, S, \mathcal{O}_{\hat{C}}^{\oplus r} \to p_*(S^\vee) \xrightarrow{\rho} \hat{S}^\vee) \in \mathfrak{Q}_{g,n}(\mathbb{G}(k,r),d)$$

is included in the subspace of the space of sections of $S|_{C_0}$ which agree with the sections $\mathcal{O}_{\hat{C}}^{\oplus r} \to (S')^\vee$ at the node. Here we denoted by S' the vector bundle $S|_{\hat{C}}$. Note that $(S')^\vee$ is isomorphic to the torsion free part of $p_*(S^\vee)$.

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Let us finally remark that as we do not have a good description of $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ we also do not have a good description for $\mathfrak{Q}_{g,n}(\mathbb{G}(k,r),d)$ or for the fibers of q.

3.3 Obstruction theories

Let us briefly recall a few basic facts about obstruction theories of moduli spaces of stable maps to $\mathbb{G}(k,r)$ and stable quotients. Let

$$\epsilon_M : \overline{M}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{M}_{g,n}$$

be the morphism that forgets the map (and does not stabilize the pointed curve) and

$$\pi_M : \mathcal{C}_M \to \overline{M}_{g,n}(\mathbb{G}(k,r),d)$$

the universal curve over $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$ and let ev denote the evaluation map $ev : \overline{M}_{g,n+1} \times (\mathbb{G}(k,r),d) \to \mathbb{G}(k,r)$ (see [Beh97]). Let

$$0 \to \mathcal{S}_M \to \mathcal{O}_{\mathcal{C}_M}^{\oplus r} \to \mathcal{Q}_M \to 0$$

be the universal sequence on $\mathcal{C}_M$. We have that $ev^*T_{\mathbb{G}(k,r)} \simeq \mathcal{Q}_M \otimes \mathcal{S}_M^\vee$. This shows that

$$E^\bullet_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{M}_{g,n}} = \mathcal{R}^\bullet \pi_{M*} \mathcal{Q}_M \otimes \mathcal{S}_M^\vee$$

is a dual obstruction theory for the morphism ϵ_M (see [BF97]). We call

$$[\overline{M}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} = {\epsilon_M}^! [\mathfrak{M}_{g,n}]$$

the virtual class of $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$. Here ${\epsilon_M}^!$ is the virtual pull-back associated to the obstruction theory $E^\bullet_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{M}_{g,n}}$ defined in [Man12a].

As the moduli space of stable maps, the moduli space of stable quotients $\bar{Q}_{g,n}(\mathbb{G}(k,r),d)$ has a morphism $\epsilon_Q : \bar{Q}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{M}_{g,n}$to the Artin stack of nodal curves. Let $\pi_Q : \hat{\mathcal{C}}_Q \to \bar{Q}_{g,n}(\mathbb{G}(k,r),d)$ be the universal curve over $\bar{Q}_{g,n}(\mathbb{G}(k,r),d)$ and let

$$0 \to \hat{\mathcal{S}}_Q \to \mathcal{O}_{\hat{\mathcal{C}}_Q}^{\oplus r} \to \hat{\mathcal{Q}}_Q \to 0$$

be the universal sequence on $\hat{\mathcal{C}}_Q$. Then the complex

$$E^\bullet_{\overline{Q}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{M}} = R\pi_{Q*} RHom(\hat{\mathcal{S}}_Q, \hat{\mathcal{Q}}_Q)$$

is a dual obstruction theory relative to ϵ_Q. We call

$$[\overline{Q}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} = {\epsilon_Q}^! [\mathfrak{M}_{g,n}]$$

the virtual class of $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$.

Obstruction theories relative to moduli spaces of bundles. In the following we define obstruction theories relative to $\mathfrak{Bun}_{g,n}(k,d)$. The map

$$\nu_M : \overline{M}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{Bun}_{g,n}(k,d)$$

induces a morphism between cotangent complexes and thus we obtain a distinguished triangle

$$\nu_M^* L_{\mathfrak{Bun}_{g,n}(k,d)} \to L_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)} \to L_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{Bun}_{g,n}(k,d)}. \tag{14}$$

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Tensoring the tautological sequence on the universal curve over $\overline{M}_{g,n}(\mathbb{G}(k,r),d)$ with $\mathcal{S}_M^\vee$ we obtain an exact sequence

$$0 \to \mathcal{S}_M \otimes \mathcal{S}_M^\vee \to (\mathcal{S}_M^\vee)^{\oplus r} \to \mathcal{Q}_M \otimes \mathcal{S}_M^\vee \to 0$$

which induces a distinguished triangle

$$R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} \to R^\bullet \pi_{M*}\mathcal{Q}_M \otimes \mathcal{S}_M^\vee \to R^\bullet \pi_{M*}\mathcal{S}_M \otimes \mathcal{S}_M^\vee[1].$$

If $\pi : \mathcal{C} \to \mathfrak{Bun}_{g,n}(k,d)$ is the universal curve and $\mathcal{S}$ denotes the tautological bundle on $\mathcal{C}$, then the tangent complex $T_{\mathfrak{Bun}_{g,n}(k,d)}$ is isomorphic to $R^\bullet\pi_*\mathcal{S} \otimes \mathcal{S}^\vee[1]$. By the cohomology and base change theorem we obtain that $\nu_M^* T_{\mathfrak{Bun}_{g,n}(k,d)} = R^\bullet \pi_{M*}\mathcal{S}_M \otimes \mathcal{S}_M^\vee[1]$. This shows that we have the following commutative diagram with the first row the dual of the exact triangle (14).

$$\begin{array}{ccccc}
T_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)/\mathfrak{Bun}_{g,n}(k,d)} & \longrightarrow & T_{\overline{M}_{g,n}(\mathbb{G}(k,r),d)} & \longrightarrow & \nu_M^* T_{\mathfrak{Bun}_{g,n}(k,d)} \\
\downarrow & & \downarrow & & \| \\
R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} & \longrightarrow & R^\bullet \pi_{M*}\mathcal{Q}_M \otimes \mathcal{S}_M^\vee & \longrightarrow & R^\bullet \pi_{M*}\mathcal{S}_M \otimes \mathcal{S}_M^\vee[1]
\end{array}$$

This shows that $R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r}$ is a dual relative obstruction theory for ν_M.

In a completely analogous manner we obtain that $R^\bullet \pi_{M*}(\hat{\mathcal{S}}_Q^\vee)^{\oplus r}$ is a dual relative obstruction theory for ν_Q.

Construction 3.18. From the cartesian diagram in Proposition 3.16 we obtain that

$$E^\bullet_{\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)/\widetilde{\mathfrak{Bun}}_{g,n}(k,d)} = c_1^* R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r}$$

is a dual perfect obstruction for the map

$$\nu_{MQ} : \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) \to \widetilde{\mathfrak{Bun}}_{g,n}(k,d).$$

By Lemma 2.27 $\widetilde{\mathfrak{Bun}}_{g,n}(k,d)$ has pure dimension, which means that the obstruction theory $E^\bullet_{\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)/\widetilde{\mathfrak{Bun}}_{g,n}(k,d)}$ gives rise to a virtual class

$$[\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} = (\nu_{MQ})^! [\widetilde{\mathfrak{Bun}}_{g,n}(k,d)].$$

4. Comparison of virtual fundamental classes

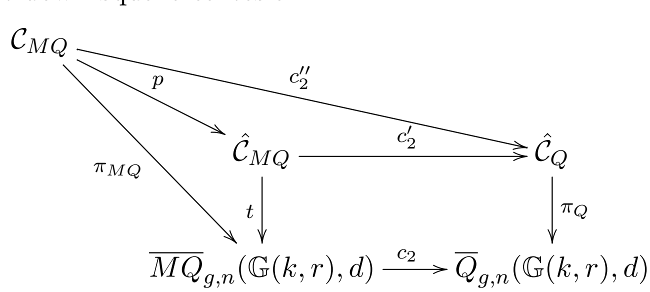
Proposition 4.1. *The tautological morphism*

$$\rho : p_*(\mathcal{S}_{MQ}^\vee) \to \hat{\mathcal{S}}_{MQ}^\vee$$

on $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ *induces a morphism*

$$c_1^* R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} \to c_2^* R^\bullet \pi_{Q*}(\hat{\mathcal{S}}_Q^\vee)^{\oplus r}.$$

Proof. By the construction of $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ we see that we have the following commutative diagram with the right-down square cartesian.



(15)

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By cohomology and base change in the diagram

$$\begin{array}{ccc} \mathcal{C}_{MQ} & \longrightarrow & \mathcal{C}_M \\ {\scriptstyle \pi_{MQ}}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) & \xrightarrow{c_1} & \overline{M}_{g,n}(\mathbb{G}(k,r),d) \end{array}$$

we have that $c_1^* R^\bullet \pi_{M*}\mathcal{S}_M^\vee \simeq R^\bullet \pi_{MQ_*} c_1^* \mathcal{S}_M^\vee$ and by construction we have that $c_1^*\mathcal{S}_M^\vee \simeq \mathcal{S}_{MQ}^\vee$. Combining the two relations we obtain a canonical isomorphism

$$c_1^* R^\bullet \pi_{M*}\mathcal{S}_M^\vee \simeq R^\bullet \pi_{MQ_*}\mathcal{S}_{MQ}^\vee. \tag{16}$$

From the commutativity of diagram (15) we have that

$$R^\bullet \pi_{MQ_*}\mathcal{S}_{MQ}^\vee \simeq R^\bullet (t\circ p)_* \mathcal{S}_{MQ}^\vee \tag{17}$$

and by the construction of $\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)$ we have $\hat{\mathcal{S}}_{MQ}^\vee \simeq {c_2'}^* \hat{\mathcal{S}}_Q^\vee$. Now using cohomology and base change in diagram (15) we obtain that

$$c_2^* R^\bullet \pi_{Q_*}\hat{\mathcal{S}}_Q^\vee \simeq R^\bullet t_* {c_2'}^* \hat{\mathcal{S}}_Q^\vee. \tag{18}$$

By (16)–(18) we see that $\rho : p_*(\mathcal{S}_{MQ}^\vee) \to \hat{\mathcal{S}}_{MQ}$ induces a morphism

$$c_1^* R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} \to c_2^* R^\bullet \pi_{Q_*}(\hat{\mathcal{S}}_Q^\vee)^{\oplus r}.$$ □

Lemma 4.2. *Let F be the cone of the morphism*

$$c_1^* R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} \to c_2^* R^\bullet \pi_{Q_*}(\hat{\mathcal{S}}_Q^\vee)^{\oplus r}.$$

Then, F is a perfect complex.

Proof. Let us consider

$$(C, p_1, \ldots, p_n, \mathcal{O}_C^{\oplus r} \to S^\vee)$$

a stable map, $p : C \to \widehat{C}$ the morphism contracting the rational tails and let $x_1, \ldots, x_p$ be the gluing points of the rational tails C_i^0 with the rest of the curve. Then we need to show that the morphism

$$H^1(C, S^\vee) \to H^1(\hat{C}, \hat{S}^\vee)$$

is surjective. Since

$$H^1(C, S^\vee) \simeq H^1(\hat{C}, p_*(S^\vee))$$

we need to show that

$$H^1(\widehat{C}, p_*(S^\vee)) \to H^1(\hat{C}, \hat{S}^\vee)$$

is surjective. As the quotient of the morphism $p_*(S^\vee) \to \hat{S}^\vee$ is supported on points, it has no higher cohomology. This shows that the above morphism is surjective. □

Theorem 4.3. *Let $\gamma_1, \ldots, \gamma_n \in A^*(\mathbb{G}(k,r))$. Then we have that*

$$ev_1^*\gamma_1 \cdots ev_n^*\gamma_n \cdot [\overline{M}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} = ev_1^*\gamma_1 \cdots ev_n^*\gamma_n \cdot [\overline{Q}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}}.$$

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Proof. From Costello's push-forward formula [Cos06] applied to the cartesian diagram

$$\begin{array}{ccc} \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) & \xrightarrow{c_1} & \overline{M}_{g,n}(\mathbb{G}(k,r),d) \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{Bun}}_{g,n}(k,d) & \longrightarrow & \mathfrak{Bun}_{g,n}(k,d) \end{array}$$

we obtain that

$$c_{1*}[\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} = [\overline{M}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}}, \tag{19}$$

where $[\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}}$ is the class from Construction 3.18.

Let us now show that c_2 satisfies the virtual push-forward property. For this we analyze the commutative diagram in which the lower rectangle is a cartesian diagram.

$$\begin{array}{ccc} \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) & & \\ \downarrow q & \searrow^{c_2} & \\ \mathfrak{Q}_{g,n}(\mathbb{G}(k,r),d) & \xrightarrow{s} & \overline{Q}_{g,n}(\mathbb{G}(k,r),d) \\ \downarrow \nu_{\mathfrak{Q}} & & \downarrow \nu_Q \\ \widetilde{\mathfrak{Bun}}_{g,n}(k,d) & \xrightarrow{\pi_2} & \mathfrak{Bun}_{g,n}(k,d) \end{array} \tag{20}$$

Let us moreover remark that we have a commutative diagram

$$\begin{array}{ccc} \overline{MQ}_{g,n}(\mathbb{G}(k,r),d) & & \\ \downarrow q^\circ & \searrow^{c_2} & \\ \mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d) & \xrightarrow{s^\circ} & \overline{Q}_{g,n}(\mathbb{G}(k,r),d) \end{array}$$

where $\mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d)$ is the image of q and q°, s° are the maps induced by q and s. In the following we show that s° is unobstructed. For this let us understand better the points of $\mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d)$. Let us note that to give a point $(\hat{C}, p_1, \ldots, p_n, \hat{S}, \mathcal{O}_{\hat{C}}^{\oplus r} \to \hat{S}^\vee) \in \overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ means to give a rational map $f : \hat{C} \dashrightarrow \mathbb{G}(k,r)$ with base points $x_1, \ldots, x_p$. As f is defined around the nodes of C we see that f extends to a morphism $\bar{f} : \hat{C} \to \mathbb{G}(k,r)$. Then $\bar{f}$ determines a vector bundle S' on $\hat{C}$ obtained by pulling back the tautological sequence on $\mathbb{G}(k,r)$ via $\bar{f}$. Let us show that S' is the saturation of $\hat{S}$ in $\mathcal{O}_{\hat{C}}^{\oplus r}$. By definition, the saturation of $\hat{S}$ in $\mathcal{O}_{\hat{C}}^{\oplus r}$ is defined by a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & & & & & \tau & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \hat{S} & \longrightarrow & \mathcal{O}_{\hat{C}}^{\oplus r} & \longrightarrow & \hat{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S' & \longrightarrow & \mathcal{O}_{\hat{C}}^{\oplus r} & \longrightarrow & Q' & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \tau & & & & & & \end{array}$$

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where τ is the torsion subsheaf of $\hat{Q}$. As the morphisms $\hat{S} \to \mathcal{O}_{\hat{C}}^{\oplus r}$ and $S' \to \mathcal{O}_{\hat{C}}^{\oplus r}$ agree away from $x_1, \dots, x_p$ and Q' is torsion free we have that $S' \to \mathcal{O}_{\hat{C}}^{\oplus r}$ determines the unique morphism $\bar{f} : \hat{C} \to \mathbb{G}(k,r)$ which extends $f : \hat{C} \dashrightarrow \mathbb{G}(k,r)$. Summarizing, we have that any stable quotient $\mathcal{O}_{\hat{C}}^{\oplus r} \to \hat{S}^\vee$ determines uniquely a vector bundle S' on $\hat{C}$ together with a morphism $\hat{S} \to S'$ such that the composition

$$\mathcal{O}_{\hat{C}}^{\oplus r} \to S'^\vee \to \hat{S}^\vee$$

gives the stable quotient we started with. This shows that to give a point in $\mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d)$ means to give a stable quotient $(\hat{C}, p_1, \dots, p_n, \hat{S}, \mathcal{O}_{\hat{C}}^{\oplus r} \to \hat{S}^\vee)$, a curve C with rational tail free part $\hat{C}$ and rational tails $C_1^0, \dots, C_p^0$ and a vector bundle S on C such that the restriction of S to $\hat{C}$ is isomorphic to S'. More precisely, a point of $\mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d)$ is a stable quotient on $\hat{C}$ with additional markings $x_1, \dots, x_p$ and p vector bundles $S_1^0, \dots, S_p^0$ on the (possibly reducible) rational 1-pointed curves $C_1^0, \dots, C_p^0$ with markings $y_1, \dots, y_p$ together with identifications $S|_{x_i} \simeq S_i^0|_{y_i}$ for all $i \in \{1, \dots, p\}$.

Now let L be the complex on $\mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d)$ whose fibers are

$$Ext^\bullet_C(S,S) \to Ext^\bullet_{\hat{C}}(S',S').$$

Let $\mathfrak{S} \subset \mathfrak{M}_{g,n}$ be the image of the morphism $\overline{M}_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{M}_{g,n}$. Let $L_{\mathfrak{S}/\mathfrak{M}^{\mathrm{rtf}}_{g,n}}$ be the cotangent complex of the morphism $p : \mathfrak{S} \to \mathfrak{M}^{\mathrm{rtf}}_{g,n}$ which contracts rational tails (see Proposition 2.3), L_{s° the relative cotangent complexes of s°, and let $\epsilon : \mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d) \to \mathfrak{S}$ be the forgetful morphism which sends a point $(C, S, \mathcal{O}_{\hat{C}}^{\oplus r} \to \hat{S}^\vee) \in \mathfrak{Q}^\circ_{g,n}(\mathbb{G}(k,r),d)$ to C. The fiber of s° over a point $(\hat{C}, p_1, \dots, p_n, \hat{S}, \mathcal{O}_{\hat{C}}^{\oplus r} \to \hat{S}^\vee)$ in $\overline{Q}_{g,n}(\mathbb{G}(k,r),d)$ is isomorphic to the fiber of the morphism $\mathfrak{Bun}_{g,n}(k,d) \to \mathfrak{Bun}_{g,n}(k,d)$ which sends (C,S) to $(\hat{C}, S')$. This shows that we have a distinguished triangle

$$L \to L_{s^\circ} \to \epsilon^* L_{\mathfrak{S}/\mathfrak{M}^{\mathrm{rtf}}_{g,n}}.$$

Tensoring the normalization sequence

$$0 \to S \to S' \oplus \bigoplus_{i=1}^p S_i^0 \to \mathbb{C}^p \to 0$$

with $S^\vee$ and taking cohomology we obtain

$$0 \to H^0(C, S \otimes S^\vee) \to H^0(\hat{C}, S' \otimes (S')^\vee) \oplus \bigoplus_{i=1}^p H^0(C_i^0, S_i^0 \otimes (S_i^0)^\vee) \to \mathbb{C}^p \otimes \mathbb{C}^k$$
$$\to H^1(C, S \otimes S^\vee) \to H^1(\hat{C}, S' \otimes (S')^\vee) \oplus \bigoplus_{i=1}^p H^1(C_i^0, S_i^0 \otimes (S_i^0)^\vee) \to 0.$$

This shows that $L^\vee$ has fibers

$$\bigoplus_{i=1}^p H^0(C_i^0, S_i^0 \otimes (S_i^0)^\vee) \to \bigoplus_{i=1}^p \mathbb{C}^k \oplus H^1(C_i^0, S_i^0 \otimes (S_i^0)^\vee)$$

concentrated in $[-1,0]$. In particular s° is unobstructed and the normal cone of s° is a vector bundle stack of rank 0. In the notation of Lemma 4.2 let F be the cone of the morphism $c_1^* R^\bullet \pi_{M*}(\mathcal{S}_M^\vee)^{\oplus r} \to c_2^* R^\bullet \pi_{Q*}(\hat{\mathcal{S}}_Q^\vee)^{\oplus r}$. By Lemma 4.2 we have that F is a perfect complex. The

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fact that s° is unobstructed together with the fact that the moduli space of stable quotients is connected ([KP01, Tod11]) shows that we are under the hypothesis of Proposition 3.14 in [Man12b] which implies that

$$c_{2*}[\overline{MQ}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}} = [\overline{Q}_{g,n}(\mathbb{G}(k,r),d)]^{\mathrm{virt}}. \tag{21}$$

The conclusion follows from (19), (21) and the projection formula. □

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